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On weighted Calderón-Zygmund singular integrals and applications

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Abstract. This paper studies some weighted norm inequalities related to some Calderon-Zygmund singular integrals. Applications to the Sobolev-Gagliardo-Nirenberg inequality, differential forms, and the potential equation $du = f$ are given.

Keywords: Singular integrals, Sobolev-Gagliardo-Nirenberg inequality, potential equation.

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1 Introduction

In the present work we study a class of Calderón-Zygmund operators on weighted Besov-Triebel-Lizorkin spaces .
Next, we recall some needed results.

We say that w is a weight if w is an a.e. positive locally integrable function on $\mathbb{R}^n$. Let w to be a weight and $0 < p < \infty$, we say that $f \in L^p(w)$ if and only if

$$||f||_{p,w} = (\left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} < \infty.$$

A nonnegative locally integrable function w is said to be in the Muckenhoupt classes A_p, $1 < p < \infty$ if there exists a constant $C_p > 0$ such that for all cube Q,

$$\frac{1}{|Q|} \int_Q w dy \left(\frac{1}{|Q|} \int_Q w^{1-p'} dy \right)^{p-1} \leq C_p$$

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when $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, well for $p = 1$,

$$\frac{1}{|Q|}\int_{Q} w dy \leq C_1 w(x),$$

for a.e. $x \in Q$. The A_∞- Muckenhoupt class is defined by $A_\infty = \cup_{p\geq 1} A_p$. For more details concerning the Muckunhoupt class we refer to [14,31,37].

Let $\mathcal{S}(\mathbb{R}^n)$ to be the space of all Schwartz functions on $\mathbb{R}^n$ with the classical topology generated by the family of semi-norms

$$||\nu||_{k,N} = \sup_{x\in\mathbb{R}^n}\sup_{|\beta|\leq N} (1+|x|)^k\,|\partial^\beta \nu(x)| \quad k, N \in \mathbb{N}_0, \quad \nu \in \mathcal{S}(\mathbb{R}^n).$$

The topological dual space, $\mathcal{S}'(\mathbb{R}^n)$ of $\mathcal{S}(\mathbb{R}^n)$ is the set of all continuous linear functional $\mathcal{S}(R^n) \longrightarrow \mathbb{C}$ endowed with the weak $\star$-topology. We denote by $\mathcal{S}_\infty(\mathbb{R}^n)$, the topological subspace of functions in $\mathcal{S}(\mathbb{R}^n)$ having all vanishing moments :

$$\mathcal{S}_\infty(\mathbb{R}^n) = \left\{\nu \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} x^\beta \nu(x)dx = 0, \qquad \forall \beta \in \mathbb{N}^n\right\}.$$

$\mathcal{S}'_\infty(\mathbb{R}^n)$ denotes the topological dual space of $\mathcal{S}_\infty(\mathbb{R}^n)$, namely, the set of all continuous linear functional on $\mathcal{S}'_\infty(\mathbb{R}^n)$. The space $\mathcal{S}'_\infty(\mathbb{R}^n)$ is also endowed with the weak $\star$-topology. It is well known that $\mathcal{S}'_\infty(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ as topological spaces, where $\mathcal{P}(\mathbb{R}^n)$ denotes the set of all polynomials on $\mathbb{R}^n$; see, for example, [36, Proposition 8.1]. The Fourier transform, $\mathcal{F}\nu = \hat{\nu}$, of Schwartz function ν is defined by

$$\hat{\nu}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi . x}\nu(x)dy$$

and the convolution of two function $\nu, \mu \in \mathcal{S}(\mathbb{R}^n)$ is defined by

$$\nu \star \mu(x) = \int_{\mathbb{R}^n} \nu(x-y)\mu(y)dy$$

and still belongs to $\mathcal{S}(\mathbb{R}^n)$. The convolution operator can be extended to $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n)$ via $\nu \star f(x) = \langle f, \mu(x - .)\rangle$. It makes sense pointwise and is a C^∞ function on $\mathbb{R}^n$ of at most polynomial growth.

To simplifying notation, we write often $\nu f = \nu \star f$. In some other situations, to avoid confusion, we keep the notation $\nu \star f$. Throughout the paper, for all $t > 0$ and $x \in \mathbb{R}^n$, we put $\nu_t(x) = t^{-n}\nu(\frac{x}{t})$.

Definition 1.1. *Let ν in the Schwartz space with $supp\,\hat{\nu}$ contained in an annulus about the origin and*

$$\sum_{j\in\mathbb{Z}} \hat{\nu}(2^{-j}\xi) = 1 \;\; for \;\; all \;\; \xi \neq 0.$$

Let $w \in A_\infty$, $0 < p, q \leq \infty$ and $\gamma \in \mathbb{R}$, the homogeneous Triebel-Lizorkin space $\dot{F}^{\gamma,q}_{p,w}$ is the set of all distribution f in $\mathcal{S}'_\infty(\mathbb{R}^n)$ such that:

$$||f||_{\dot{F}^{\gamma,q}_{p,w}} = \left|\left|\left(\sum_{j\in\mathbb{Z}} 2^{j\gamma q}|\nu_{2^{-j}} f|^q\right)^{\frac{1}{q}}\right|\right|_{p,w} < \infty;\; 0 < p, q < \infty$$

and

$$||f||_{\dot{F}^{\gamma,q}_{\infty,w}} = \sup_Q \Big\{ \frac{1}{w(Q)} \int_Q \sum_{j=-log_2 l(Q)}^{\infty} 2^{j\gamma q} |\nu_{2^{-j}} f|^q w(x) dx \Big\}^{\frac{1}{q}} < \infty;\ 0 < q \leq \infty$$

with the interpretation that when $q = \infty$,

$$||f||_{\dot{F}^{\gamma,\infty}_{\infty,w}} = \sup_Q \sup_{j \geq -log_2 l(Q)} \frac{1}{w(Q)} \int_Q 2^{j\gamma} |\nu_{2^{-j}} f| w(x) dx < \infty.$$

The homogeneous Besov-Lipshitz space $\dot{B}^{\gamma,q}_p$ is the set of all distribution f in $\mathcal{S}'_\infty(\mathbb{R}^n)$ such that:

$$||f||_{\dot{B}^{\gamma,q}_{p,w}} = \Big(\sum_{j\in\mathbb{Z}} 2^{j\gamma q} ||\nu_{2^{-j}} f||^q_{p,w} \Big)^{\frac{1}{q}} < \infty;\ 0 < p, q < \infty.$$

The supremum is taken over all dyadic cubes Q and $l(Q))$ denotes the length of sides of the cube Q.
Let μ be a Schwartz function given by

$$\hat{\mu}(\xi) + \sum_{j\geq 1} \hat{\nu}(2^j \xi) = 1,\ \ \forall \xi\ \in \mathbb{R}^n.$$

Definition 1.2. *The nonhomogeneous Triebel-Lizorkin space $F^{\gamma,q}_p$ is the set of all distribution f in $\mathcal{S}'_\infty(\mathbb{R}^n)$ such that:*

$$||f||_{F^{\gamma,q}_{p,w}} = ||\mu \star f||_{p,w} + \Big|\Big| \Big(\sum_{j\geq 1} 2^{j\gamma q} |\nu_{2^{-j}} f|^q \Big)^{\frac{1}{q}} \Big|\Big|_{p,w} < \infty.$$

The nonhomogeneous Besov-Lipshitz space $B^{\gamma,q}_p$ is the set of all distribution f in $\mathcal{S}'_\infty(\mathbb{R}^n)$ such that:

$$||f||_{B^{\gamma,q}_{p,w}} = ||\mu \star f||_{p,w} + \Big(\sum_{j\geq 1} 2^{j\gamma q} ||\nu_{2^{-j}} f||^q_{p,w} \Big)^{\frac{1}{q}} < \infty;\ 0 < p, q < \infty.$$

With standard modifications when $p = \infty$ or $q = \infty$.
It is well known that the Besov-Lipshitz space and Triebel-Lizorkin spaces are independent of the choices of ν, see, for example [3–5, 11–13, 40].

It has long been known that many classical smoothness spaces are covered by the Besov and Triebel-Lizorkin spaces. We recall some examples,

1. $\dot{F}^{0,2}_{p,w} = H_{p,w},\ 0 < p < \infty,$
2. $F^{0,2}_{p,w} = h_{p,w},\ 0 < p < \infty,,$
   where $H_{p,w}$ denotes the weighted Hardy spaces of $f \in \mathcal{S}'$ for which
   $$||f||_{H_{p,w}} = ||\sup_{t>0} \mu_t \star f||_{p,w} < \infty,$$
   and $h_{p,w}$ is the local weighted Hardy space, the space of $f \in \mathcal{S}'$ for which
   $$||f||_{h_{p,w}} = ||\sup_{0<t<1} \mu_t \star f||_{p,w} < \infty,$$
   where μ is a fixed function in $\mathcal{S}$ with $\int_{\mathbb{R}^n} \mu(x)dx \neq 0$. By the fundamental work of C. Feferman and E. Stein [10] adapted to the weighted case, $H_{p,w}$ or $h_{p,w}$ does not depend on the choices of μ in its definition. In particular
   $$\dot{F}^{0,2}_{p,w} = F^{0,2}_{p,w} = L_{p,w},\quad 1 < p < \infty.$$

3. $\dot{F}^{\gamma,2}_{p,w} = H^{\gamma}_{p,w}$, $1 < p < \infty$, where $H^{\gamma}_{p,w}$ denotes the weighted Bessel potential space defined by
$$||f||_{H^{\gamma}_{p,w}} = ||\mathcal{F}^{-1}(1+|\xi|^2)^{\gamma/2}\mathcal{F}f||_{L_{p,w}}.$$
In particular, when the exponent is a natural number, say $\gamma = N \in \mathbb{N}$, then the weighted Bessel potential space can be identified with the classical Sobolev space
$$W^N_{p,w} = \{f \in L_{p,w} : ||\sum_{|\sigma|\leq N} \partial^{\sigma} f||_{L_{p,w}} < \infty\}, \quad 1 < p < \infty.$$
Where all identities have to be understood in the sense of equivalent quasi-norms.

In the present paper, the following results will be used. The first one is the continuous characterisation of the Treibel-Lizorkin spaces, the second is the Calderón reproducing formula and the third one is the characterization of the weighted Hardy inequality.

Theorem 1.1. *[4, 18]. Let $\gamma \in \mathbb{R}$, $0 < p, q < \infty$, $0 < \delta < min(p,q)$ and $w \in A_{p/\delta}$. Assume $\nu \in S_\infty$ satisfying the Tauberian condition, i.e.,for each $\xi \neq 0$ there exists a $t > 0$ s.t $\hat{\nu}(t\xi) \neq 0$. Then*
$$\left|\left|\left(\int_0^\infty t^{-\gamma q}(\nu_t^\star f)^q \frac{dt}{t}\right)^{\frac{1}{q}}\right|\right|_{p,w} \approx ||f||_{\dot{F}^{\gamma,q}_{p,w}} \approx \| g^\star_{\lambda,q} f \|_{p,w}$$
for all $f \in \dot{F}^{\gamma,q}_{p,w}$. where $\nu_t^\star f(x)$ is the Peetre maximal function and $g^\star_{\lambda,\gamma,q}$ is the Littlewood-Paley g function defined respectively by
$$\nu_t^\star f(x) = \sup_{y\in\mathbb{R}^n} \frac{|\nu_t \star f(y)|}{(1+\frac{|x-y|}{t})^\lambda}$$
$$g^\star_{\lambda,\gamma,q} f(x) = \Big(\int_0^\infty \int_{\mathbb{R}^n} s^{-\gamma q}|\nu_s f(y)|^q (1+\frac{|x-y|}{s})^{-\lambda q}\frac{dyds}{s^{n+1}}\Big)^{1/q}$$
for some large λ
and
$$||f||_{\dot{F}^{\gamma,q}_{\infty,w}} \simeq \Big|\Big|N_{\gamma,q} f\Big|\Big|_{\infty,w} \simeq \Big|\Big|N^\star_{\gamma,q} f\Big|\Big|_{\infty,w}, \ 0 < q < \infty,$$
where
$$N_{\gamma,q} f(x) = \sup_{R>0}\left(\frac{1}{w\,(B(x,R))}\int_{B(x,R)}\int_0^R t^{-\gamma q}|\nu_t f(y)|^q w(y)\frac{dtdy}{t}\right)^{\frac{1}{q}}$$
$$N^\star_{\gamma,q} f(x) = \sup_{R>0}\left(\frac{1}{w\,(B(x,R))}\int_{B(x,R)}\int_0^R t^{-\gamma q}(\nu_t^\star f(y))^q w(y)\frac{dtdy}{t}\right)^{\frac{1}{q}}.$$
When $\gamma = 0$ we put $g^\star_{\lambda,0,q} = g^\star_{\lambda,q}$.

Remark 1.1. *Note also that*

1. *if I_α denotes the Riesz potential defined by $\widehat{I_\alpha f}(\xi) = |\xi|^{-\alpha}\hat{f}(\xi)$ then*
$$||I_\alpha f||_{\dot{F}^{\alpha+\gamma,q}_{p,w}} \approx ||f||_{\dot{F}^{\gamma,q}_{p,w}},$$

2. *if R_j, $j = 1, \ldots, n$ denotes the Riesz transform defined by $\widehat{R_j f}(\xi) = i\xi_j|\xi|^{-1}\hat{f}(\xi)$ then*
$$||R_j f||_{\dot{F}^{\gamma,q}_{p,w}} \approx ||f||_{\dot{F}^{\gamma,q}_{p,w}},$$
and

3. *if N is a positive integer then $f \in \dot{F}^{N,q}_{p,w}$ if and only if $\partial^\sigma f \in \dot{F}^{0,q}_{p,w}$ for all σ such that $|\sigma| = N$ and*
$$||f||_{\dot{F}^{N,q}_{p,w}} \simeq \sum_{|\sigma|=N} ||\partial^\sigma f||_{\dot{F}^{0,q}_{p,w}}.$$

In fact, let ν as in Theorem 1.1. Then $I_\alpha \nu$ behaves as ν, $\nu_t(I_\alpha f) = t^\alpha (I_\alpha \nu)_t f$ and $I_\alpha I_{-\alpha} = id$. So we have the first assertion.
The second assertion can obtained from the identity $\nu_t(R_j f) = (R_j \nu)_t f$. And finally, the third assertion can obtained by iteration from the identity $\partial_j = -R_j \circ I_1$. In fact, we have $\partial_\sigma = (-1)^{|\sigma|} R^\sigma \circ I_{-|\sigma|}$ with $R = (R_1, \ldots, R_n)$.

Theorem 1.2. *[4]. Let $\gamma \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, $a > 0$, $0 < \delta < min(p,q)$ and $w \in A_{p/\delta}$. Assume μ, $\nu \in S$ satisfying the Tauberian condition , $\Phi \in S$ and that φ satisfies the strong Tauberian condition $\hat{\varphi}(0) \neq 0$, then there exists a positive constant b for which*
$$||\Phi^\star f||_{p,w} + \Big|\Big| \left(\int_0^a t^{-\gamma q} (\mu_t^\star f)^q \frac{dt}{t}\right)^{\frac{1}{q}} \Big|\Big|_{p,w} \leq ||f||_{F^{\gamma,q}_{p,w}}$$
$$\leq ||\varphi^\star f||_{p,w} + \Big|\Big| \left(\int_0^b t^{-\gamma q} (\nu_t^\star f)^q \frac{dt}{t}\right)^{\frac{1}{q}} \Big|\Big|_{p,w}$$
for all $f \in F^{\gamma,q}_{p,w}$ and for large λ.

Lemma 1.1. *[6,7,20] (Calderón reproducing formula).*
Let $\nu \in S$ satisfying the Tauberian condition and having all vanishing moments, then there exists $\zeta \in S$ with supp $\hat{\zeta}$ contained in an annulus about the origin such that
$$\int_0^\infty \hat{\nu}(s\xi)\hat{\zeta}(s\xi)\frac{ds}{s} = 1 \;\; for \;\; all \; \xi \neq 0.$$

Lemma 1.2. *[4] (Sub-mean value property).*
Let $\mu \in S$ satisfying the Tauberian condition. Assume that $\hat{\mu}$ is supported in an annulus about the origin. Then for every $r > 0$ and $N > 0$, there exists $C > 0$ for wich
$$|\mu_s f(x)|^r \leq C \int_0^\infty \int_{\mathbb{R}^n} |\mu_\tau f(y)|^r (1 + \frac{|x-y|}{\tau})^{-Nr} (min(\frac{\tau}{s}, \frac{s}{\tau}))^{Nr} \frac{dy d\tau}{\tau^{n+1}}$$
for all $g \in S'$, $x \in \mathbb{R}^n$ and $s > 0$.

The next lemma gives a characterization of the weighted Hardy inequality
$$\left(\int_a^b \left(\int_a^s f(t)dt\right)^q u(s)ds\right)^{1/q} \leq C \left(\int_b^a f^p(s)v(s)\right)^{1/p} \tag{1.1}$$
where $-\infty \leq a < b \leq \infty$, $1 < p, q < \infty$ and u; v are measurable functions and positive a.e. in (a, b).

Lemma 1.3. *1. If $1 \leq p \leq q < \infty$; then the inequality 1.1 holds for all measurable functions $f(x) \geq 0$ on (a,b) if and only if*

$$\sup_{a<s<b} \left(\int_s^b u(t)dt\right)^{1/q} \left(\int_a^s v^{1-p'}(t)dt\right)^{1/p'} < \infty.$$

2. If $1 < p < q < \infty$; then the inequality 1.1 holds for all measurable functions $f(x) \geq 0$ on (a,b) if and only if

$$\int_a^b \left(\int_s^b u(t)dt\right)^{\varepsilon/q} \left(\int_a^s v^{1-p'}(t)dt\right)^{\epsilon/q'} v^{1-p'}(s)ds < \infty,$$

where $1/\epsilon = 1/q - 1/p$.

There is extensive literature devoted to the different kinds of the Hardy inequalities, see for instance [?,2,19,21–27].

2 Statements of results

Let $z \longmapsto K(x,z)$ belongs to $\in C^{\infty}(\mathbb{R}^n \backslash \{0\})$ for $a.e.x \in \mathbb{R}^n$ such that for some $0 \leq \alpha \leq n$

$$K(x, \lambda z) = \lambda^{-n-\alpha} K(x,z),\ \forall \lambda > 0,\ a.e\ x,\ z \in \mathbb{R}^n,\ z$$

and satisfies for some $R > 0$ sufficiently large,

$$\sup_{x \in \mathbb{R}^n} \sup_{y \in S^{n-1}} |\partial_x^{\sigma} \partial_y^{\beta} K(x,y)| = M < +\infty,$$

for all multi-indices β s.t $|\beta| + |\sigma| < R$. We also assume for $\alpha = 0$ that

$$\int_{S^{n-1}} K(x,z) d\sigma_z = 0.$$

We denote by TF the operator, initially defined on $S(\mathbb{R}^n)$, by

$$Tf(x) = \int_{\mathbb{R}^n} K(x, x-y) f(y) dy,$$

(with the integral taken in the principal value sense when $\alpha = 0$).

In this work, we shall prove the following results.

Theorem 2.1. *Let $\gamma \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$ and $w \in A_{\infty}$ then*

$$||Tf||_{F_{p,w}^{\alpha+\gamma,q}} \leq ||f||_{F_{p,w}^{\gamma,q}}.$$

If in addition K is a convolution kernel, then

$$||Tf||_{\dot{F}_{p,w}^{\alpha+\gamma,q}} \leq C||f||_{\dot{F}_{p,w}^{\gamma,q}}.$$

Theorem 2.2. *Given $\beta \in \mathbb{R}$, $0 < p,d < \infty$, $0 < q,r \leq \infty$ and R is large enough. Let $\gamma \in \mathbb{R}$ with $0 < \alpha - \gamma + \beta$ and $w \in A_{\infty}$ and let $p_{\star}$ be such that*

$$\frac{1}{p_{\star}} = \frac{1}{p} - \frac{\alpha - \gamma + \beta}{d}.$$

If $w(B(x,t)) \geq Ct^d$ *for all* $0 < t < 1$ *and all* x*, then*

$$||Tf||_{F^{\gamma,q}_{p_\star,w}} \leq C||f||_{F^{\beta,r}_{p,w}}. \tag{2.1}$$

If in addition K *is a convolution kernel and* $w(B(x,t)) \geq Ct^d$ *for some* $d > 0$ *and all* x*, then*

$$||Tf||_{\dot{F}^{\gamma,q}_{p_\star,w}} \leq C||f||_{\dot{F}^{\beta,r}_{p,w}}. \tag{2.2}$$

Theorem 2.3. *Given any reals* $\alpha,\ \gamma$ *s.t* $0 < \alpha - \gamma$*,* $0 < p < \infty$*,* $0 < q, r \leq \infty$*, and* $d > 0$*. Assume* $w \in A_\infty$ *and let* $0 < p_\star \leq \infty$ *determined by*

$$\frac{1}{p_\star} = \frac{1}{p} - \frac{\alpha - \gamma}{d}.$$

If $w(B(x,t)) \geq Ct^d$ *for all* $0 < t < \infty$ *and all* x*, then*

$$\dot{F}^{\alpha,q}_{p,w} \subseteq \bigcap_{q>0} \dot{F}^{\gamma,r}_{p_\star,w},$$

and if $w(B(x,t)) \geq Ct^d$ *for all* $0 < t < 1$ *and all* x*, then*

$$F^{\alpha,q}_{p,w} \subseteq \bigcap_{q>0} F^{\gamma,r}_{p_\star,w}$$

with the continuous imbedding $\dot{F}^{\alpha,q}_{p,w} \hookrightarrow \dot{F}^{\gamma,r}_{p_\star,w}$*,* $F^{\alpha,q}_{p,w} \hookrightarrow F^{\gamma,r}_{p_\star,w}$ *for each* $0 < r \leq \infty$*.*

Remark 2.1. *We note that Theorem 2.3 is a generalization of Theorem 2.6 in [3] and includes the particular case* $p_\star = \infty$ *for which we do not know if it is known yet in the weight case. In the unweight and homogenous case, Theorem 2.3 is proved by Young-Kum Cho [42].*

We note also that some similar results can be found in [15–17].

The proof of our results is based essentially on the expansion of the kernel K into spherical harmonics. A good reference is [32].

Denote by Π_m the set of all real polynomials in $x \in \mathbb{R}^n$, $n \geq 2$ which are homogeneous of degree m. It is well known that Π_m is a finite dimensional vector space of dimension $g_m = C^{n-1}_{m+n-1}$.

Solid harmonics of degree m are polynomials $P \in \Pi_m$ which satisfy $\Delta P = 0$.

The set of all solid harmonics of degree m, denoted by S_m is a subspace of Π_m of dimension

$$d_m = g_m - g_{m-2} = C^{n-1}_{m+n-1} - C^{n-1}_{m+n-3}.$$

The restriction of solid harmonics to unit sphere is called spherical harmonics of degree m and we denote by Q_m the set of all spherical harmonics of degree m.

The vector space Q_m can be seen as a linear subspace of the Hilbert space $L_2(S^{n-1})$, with inner product

$$\langle f, g \rangle = \int_{S^{n-1}} fg d\sigma.$$

With respect this inner product, we can construct in each Q_m an orthonormal basis $Y_{km},\ k = 1, \ldots, d(m)$. Moreover, we have

Theorem 2.4. *The collection* $\{Y_{km}(z)\}$*,* $k \in \{1, \ldots, d_m\}$*,* $m \in \mathbb{N}$ *is a complete orthonormal system of spherical harmonics on* $L^2\left(S^{n-1}\right)$*.*

On the other hand, if we denote by L the operator defined by $Lf = |x|^2 \Delta f$ then

Lemma 2.1. *[32]*

a) $d_m \leq c(n) m^{n-2}$

b) $| \left(\frac{\partial}{\partial x}\right)^{\alpha} \left[|z|^m Y_{km}(\frac{z}{|z|})\right] | \leq C(\alpha, n) m^{(n-2)/2+|\alpha|} |z|^{m-|\alpha|}$

c) $L^r Y_{km} = (-m)^r (m+n-2)^r Y_{km}$, $\forall r \in \mathbb{N}$,

d) *if $f, g \in C^{2r}\left(R^n \backslash \{0\}\right)$ and are homogeneous of degree zero, then*

$$\int_{S^{n-1}} f L^r g d\sigma = \int_{S^{n-1}} f L^r g d\sigma.$$

Denote by $z' = \frac{z}{|z|}$. Then we can write [32] Ch III, IV)

$$K(x,z) = |z|^{-n+\alpha} K(x, \frac{z}{|z|}) = \sum_{m=1}^{+\infty} \sum_{k=1}^{d_m} a_{km}(x) \frac{Y_{km}(z')}{|z|^{n-m-\alpha}} \tag{2.3}$$

where

$$a_{km}(x) = \int_{S^{n-1}} K(x,z) Y_{km}(z) d\sigma_z.$$

Lemma 2.2. *Let $R > 0$ and assume*

$$\sup_{x \in \mathbb{R}^n} \sup_{y \in S^{n-1}} |\partial_x^{\sigma} \partial_y^{\beta} K(x,y)| = M < +\infty$$

for all multi-indices β, σ s.t $|\beta| + |\sigma| < r$. Then

$$||\partial^{\sigma} a_{km}||_{L^\infty} \leq C(n) M_r m^{-2r}.$$

Proof. It follows from Lemma 2.1

$$\begin{aligned} a_{km}(x) &= (-m)^{-r}(m+n-2)^{-r} \int_{S^{n-1}} K(x,z) L^r Y_{km} d\sigma_z \\ &= (-m)^{-r}(m+n-2)^{-r} \int_{S^{n-1}} L^r K(x,z) Y_{km} d\sigma_z. \end{aligned}$$

Using Hölder's inequality and the hypothesis of the lemma we get

$$||\partial^{\sigma} a_{km}||_{L^\infty} \leq C(n) M_r m^{-2r} \left(\int_{S^{n-1}} |Y_{km}|^2 d\sigma_z \right)^{\frac{1}{2}} \leq C(n) M_r m^{-2r}.$$

$\square$

Lemma 2.3. *([38] p.73) Let* $K_{km\alpha}(x) = \gamma_{m,\alpha}^{-1}|x|^{-n+m+\alpha}Y_{km}(x)$ *with*

$$\gamma_{m,\alpha} = i^m \pi^{n/2-\alpha} \frac{\Gamma\left(\frac{m+\alpha}{2}\right)}{\Gamma\left(\frac{m+n-\alpha}{2}\right)}$$

and define

$$b_{km\alpha}(z) = \hat{K}_{km\alpha}(z),$$

then

$$b_{km\alpha}(z) = |z|^{-m-\alpha}Y_{km}(z)$$

provided either $0 < \alpha < n$ *and* $j \in \mathbb{N}_0$*, or* $\alpha \in \{0, n\}$ *and* $j \in \mathbb{N}_0$*.*

Lemma 2.4. *Let* $\delta > 0$*. Then, on* $1/2\delta \le |z| \le 2\delta$ *we have*

$$|(\partial^\sigma b_{km\alpha})(z)| \le C(n)\delta^{-\alpha-|\sigma|}m^{n/2-1+|\sigma|}$$

$\forall \sigma = (\sigma_1, ..., \sigma_n) \in \mathbb{N}^n$.

Proof. Define $C \equiv |\xi_i - z_i| = \epsilon\delta, i = 1, ..., n$ with $\epsilon > 0$ sufficiently small. Since $|z|^{-m-\alpha}$ is real analytic on $\mathbb{R}^n \setminus \{0\}$ then the contour integral representation of $|z|^{-m-\alpha}$ along C implies

$$|(\partial^\sigma |z|^{-m-\alpha})(z)| \le C(n)\sigma!\ \delta^{-|\sigma|}\max_{\xi\in C}|\xi|^{-m-\alpha} \le C(n)\sigma!\ \delta^{-m-\alpha-|\sigma|}.$$

Using Leibnitz's rule to obtain,

$$\begin{aligned}
|(\partial^\sigma b_{km\alpha})(z)| &= |\sum_{\gamma\le\sigma} C_\sigma^\gamma(\partial^\gamma|z|^{-m-\alpha})(z)(\partial^{\sigma-\gamma}Y_{km})(z)| \\
&\le C\sum_{\gamma\le\sigma} C_\sigma^\gamma \gamma!\ \delta^{-m-\alpha-|\gamma|}m^{n/2-1+|\sigma-\gamma|}|z|^{m-|\sigma-\gamma|} \\
&\le C\delta^{-\alpha-|\sigma|}m^{n/2-1+|\sigma|}\sum_{\gamma\le\sigma} C_\sigma^\gamma\gamma!\ m^{-\gamma} \\
&\le C\delta^{-\alpha-|\sigma|}m^{n/2-1+|\sigma|}.
\end{aligned}$$

□

Lemma 2.5. *Let* ν *a be a Schwartz function and assume that* $\hat{\nu}$ *is supported in* $1/2 \le |\xi| \le 2$*. Then, for every* $r > 0$ *and* $\lambda > 0$ *we have*

$$\nu_s^\star f(x) \le C g_{\lambda,r}^\star f(x)$$

for all $f \in S'$*,* $x \in \mathbb{R}^n$ *and* $s > 0$*.*

Proof. Let $x,\ y,\ z \in \mathbb{R}^n$ then if $s > \tau > 0$ we have

$$\begin{aligned}
(1 + |y-z|/\tau)^{-\lambda r} &\le (1 + |y-z|/s)^{-\lambda r} \\
&\le (1 + |x-z|/s)^{\lambda r}(1 + |y-x|/s)^{-\lambda r} \\
&\le s^{\lambda r}(1 + |x-z|/s)^{\lambda r}(s + |y-x|)^{-\lambda r} \\
&\le s^{\lambda r}(1 + |x-z|/s)^{\lambda r}(\tau + |y-x|)^{-\lambda r} \\
&\le (\frac{s}{\tau})^{\lambda r}(1 + |x-z|/s)^{\lambda r}(1 + |y-x|/\tau)^{-\lambda r}
\end{aligned}$$

if $\tau > s > 0$ then we have

$$\begin{aligned}(1+|y-z|/\tau)^{-\lambda r} &\leq (1+|x-z|/\tau)^{\lambda r}(1+|y-x|/\tau)^{-\lambda r}\\ &\leq (1+|x-z|/s)^{\lambda r}(1+|y-x|/\tau)^{-\lambda r}\\ &\leq (\frac{\tau}{s})^{\lambda r}(1+|x-z|/s)^{\lambda r}(1+|y-x|/\tau)^{-\lambda r}.\end{aligned}$$

Hence

$$(1+|y-z|/\tau)^{-\lambda r} \leq max(\frac{s}{\tau},\frac{\tau}{s})^{\lambda r}(1+|x-z|/s)^{\lambda r}(1+|y-x|/\tau)^{-\lambda r}.$$

The last estimate and Lemma 1.2 lead to

$$\begin{aligned}|\nu_s f(z)|^r &\leq C\int_0^\infty \int_{\mathbb{R}^n} |\nu_\tau f(y)|^r (1+|y-z|/\tau)^{-\lambda r}(min(\tau/s, s/\tau)^{Nr}\frac{dyd\tau}{\tau^{n+1}}\\ &\leq C(1+|x-z|/s)^{\lambda r}\int_0^\infty\\ &\times \int_{\mathbb{R}^n} |\nu_\tau f(y)|^r(1+|y-x|/\tau)^{-\lambda r}(max(\frac{s}{\tau},\frac{\tau}{s}))^{\lambda r}(min(\frac{s}{\tau},\frac{\tau}{s}))^{\lambda r}\frac{dyd\tau}{\tau^{n+1}}\\ &\leq C(1+|x-z|/s)^{\lambda r}\int_0^\infty\int_{\mathbb{R}^n}|\nu_\tau f(y)|^r(1+|y-x|/\tau)^{-\lambda r}\frac{dyd\tau}{\tau^{n+1}}\\ &\leq C(1+|x-z|/s)^{\lambda r} g^\star_{\lambda,r}f(x).\end{aligned}$$

□

Lemma 2.6. *Let ν and μ be as in Lemma 2.5. For a tempered distribution f, set $T_{km\alpha}f(x) = K_{km\alpha}\star f(x)$. Given $\alpha \in \mathbb{R}$ and $2 \leq r < \infty$.*

1) *If $\Phi = \mu \star \nu$, then for all $x,\ y\ \in \mathbb{R}^n$,*

$$|\Phi_t(T_{km\alpha}f)(y)| \leq Ct^\alpha m^{n/2-1+N}\left(1+\frac{|y-x|}{t}\right)^N \mu_t^\star f(x) \tag{2.4}$$

2) *if $|x-y| < t$, then*

$$|\nu_t(T_{km\alpha}f)(y)| \leq Ct^\alpha m^{n/2-1+N} g^\star_{N,r}f(x) \tag{2.5}$$

3) *if $|x-y| < t$, then*

$$|\nu_t(T_{km\alpha}f)(y)| \leq Ct^\alpha m^{n/2-1+N} N_\lambda f(x) \tag{2.6}$$

with

$$N_\lambda f(x) = \sup_{t>0, y\in\mathbb{R}^n}\frac{|\mu_t\star f(y)|}{(1+\frac{|y-x|}{t})^\lambda}.$$

Corollary 2.1. *Let $K(x,y)$ as at the beginning of this section satisfying 2 with $\sigma = 0$ and assume in addition that K is a convolution kernel,i.e, $K(x,y) = K(x-y)$. For a tempered distribution f, set $Tf(x) = K\star f(x)$.*

1) *If $\Phi = \mu \star \nu$, then for all $x,\ y\ \in \mathbb{R}^n$,*

$$|\Phi_t(Tf)(y)| \leq Ct^{\alpha}\left(1+\frac{|y-x|}{t}\right)^N \mu_t^{\star} f(x) \tag{2.7}$$

2) *if $|x-y|<t$, then*

$$|\nu_t(Tf)(y)| \leq Ct^{\alpha} g^{\star}_{N,r} f(x) \tag{2.8}$$

3) *if $|x-y|<t$, then*

$$|\nu_t(Tf)(y)| \leq Ct^{\alpha} N_{\lambda} f(x). \tag{2.9}$$

Here ν, μ, Φ, α and r are as in Lemma 2.6.

Proof. We prove only the first estimate of Corollary 2.1. The proof of the rest of estimates is similar.

Since T is a convolution operator, we have $|\Phi_t(Tf)(y)| = |T(\Phi_t \star f)(y)|$, then using the expansion 2.3 and Lemma 2.6 to obtain

$$\begin{aligned}|\Phi_t(Tf)(y)| &\leq \sum_{m=1}^{+\infty}\sum_{k=1}^{d_m} |\gamma_{m,\alpha} a_{km}(y)||\Phi_t(T_{km}f)(y)| \\ &\leq t^{\alpha}\left(1+\frac{|y-x|}{t}\right)^N \mu_t^{\star} f(x) \sum_{m=1}^{+\infty}\sum_{k=1}^{d_m} |\gamma_{m,\alpha} a_{km}(y)| m^{n/2-1+N}.\end{aligned}$$

Using the estimates in Lemma 2.2, Lemma 2.1 together with the estimate $\gamma_{m,\alpha} \leq C(\alpha, n) m^{\alpha-n/2}$ to obtain

$$\begin{aligned}|\Phi_t(Tf)(y)| &\leq t^{\alpha}\left(1+\frac{|y-x|}{t}\right)^N \mu_t^{\star} f(x) \sum_{m=1}^{+\infty} m^{-2r+n-2+\alpha-n/2+n/2-1+N} \\ &\leq t^{\alpha}\left(1+\frac{|y-x|}{t}\right)^N \mu_t^{\star} f(x) \sum_{m=1}^{+\infty} m^{-2r+n+\alpha+N-3} \\ &\leq t^{\alpha}\left(1+\frac{|y-x|}{t}\right)^N \mu_t^{\star} f(x)\end{aligned}$$

if we choose r large so that $-2r+n+\alpha+N-3<0$.

□

Theorem 2.5. *Let $\gamma \in \mathbb{R},\ 0<p\leq\infty,\ 0<q\leq\infty, 0<\delta<min(p,q)$ and $w \in A_{p/\delta}$ if N is large enough then*

$$||T_{km\alpha} f||_{\dot{F}^{\alpha+\gamma,q}_{p,w}} \leq Cm^{n/2-1+N}||f||_{\dot{F}^{\gamma,q}_{p,w}}$$

and

$$||T_{km\alpha} f||_{F^{\alpha+\gamma,q}_{p,w}} \leq Cm^{n/2-1+N}||f||_{F^{\gamma,q}_{p,w}}.$$

Proof. The first inequality of Theorem 2.5 is an immediate consequence of Theorem 1.1, Lemma 2.6 and the inequality 2.4. The second inequality is also an immediate consequence of Theorem 1.2 and the inequality 2.4. □

As consequence of Theorem 2.5 is the well known lifting property of the Riesz potential $I_\alpha f$ and the Bessel potential $J_\alpha f$ defined by $\widehat{J_\alpha f}(\xi) = (1+|\xi|^2)^{-\alpha/2}\hat{f}(\xi)$.

Proposition 2.1. *Lifting property. Let $\gamma \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty, 0 < \delta < min(p,q)$ and $w \in A_{p/\delta}$ then*

$$||I_\alpha f||_{\dot{F}^{\alpha+\gamma,q}_{p,w}} \approx ||f||_{\dot{F}^{\gamma,q}_{p,w}}$$
and
$$||J_\alpha f||_{F^{\alpha+\gamma,q}_{p,w}} \approx ||f||_{F^{\gamma,q}_{p,w}}.$$

Theorem 2.6. *Let $\gamma \in \mathbb{R}$, $0 < p < \infty$, $0 < q, r \leq \infty$, N is large enough and $w \in A_\infty$. If $w(B(x,t)) \geq Ct^d$ for all $t > 0$ and all x, then*

$$||T_{km\alpha} f||_{\dot{F}^{\gamma,q}_{p_\star,w}} \leq Cm^{n/2-1+N}||f||_{\dot{F}^{0,r}_{p,w}} \tag{2.10}$$

and if $w(B(x,t)) \geq Ct^d$ for all $0 < t < 1$ and all x, then

$$||T_{km\alpha} f||_{F^{\gamma,q}_{p_\star,w}} \leq Cm^{n/2-1+N}||f||_{F^{0,r}_{p,w}} \tag{2.11}$$

with $p_\star$ is given by

$$\frac{1}{p_\star} = \frac{1}{p} - \frac{\alpha-\gamma}{d}. \tag{2.12}$$

Proof. First we assume $0 < q, r < \infty$. We will consider two cases
Case 1. $0 < \alpha - \gamma < \frac{d}{p}$.
From Lemma 2.6 if $x, y \in \mathbb{R}^n, t > 0$ s.t $|x - y| < t$,

$$|\nu_t(T_{km\alpha} f)(y)| \leq Ct^\alpha m^{n/2-1+N} g^\star_{N,r} f(x). \tag{2.13}$$

Fix y, rise the last inequality to the pth power and integrate over the ball $B(y,t)$ with respect $w(x)dx$ to obtain

$$\begin{aligned}|\nu_t(T_{km\alpha} f)(y)| &\leq Ct^\alpha m^{n/2-1+N}\frac{1}{[w(B(y,t))]^{1/p}}||g^\star_{N,r}f||_{p,w}\\ &\leq Ct^{\alpha-\frac{d}{p}}m^{n/2-1+N}||f||_{\dot{F}^{0,r}_{p,w}}\end{aligned}$$

since $w(B(x,t)) \geq Ct^d$ and $||g^\star_{N,r}f||_{p,w} \simeq ||f||_{\dot{F}^{0,r}_{p,w}}$ by Theorem 1.1. It follows that

$$|\nu_t(T_{km\alpha} f)(y)| \leq Cm^{n/2-1+N} min\left(t^\alpha g^\star_{N,r}f(y), t^{\alpha-\frac{d}{p}}||f||_{\dot{F}^{0,r}_{p,w}}\right) \tag{2.14}$$

for all $y \in \mathbb{R}^n$ and $t > 0$. Let $\delta > 0$ to be choose later and using the estimate 2.14 to get

$$\begin{aligned}&\left(\int_0^\infty t^{-\gamma q}|\nu_t(T_{km\alpha} f)(y)|^q\frac{dt}{t}\right)^{1/q}\\ &\leq Cm^{n/2-1+N}\left(\int_0^\delta t^{(\alpha-\gamma)q}\left(g^\star_{N,r}f(y)\right)^q\frac{dt}{t} + \int_\delta^\infty t^{(\alpha-\gamma-\frac{d}{p})q}\left(||f||_{\dot{F}^{0,r}_{p,w}}\right)^q\frac{dt}{t}\right)^{1/q}\\ &\leq Cm^{n/2-1+N}\left(\delta^{(\alpha-\gamma)q}\left(g^\star_{N,r}f(y)\right)^q + \delta^{(\alpha-\gamma-\frac{d}{p})q}\left(||f||_{\dot{F}^{0,r}_{p,w}}\right)^q\right)^{1/q}\end{aligned}$$

since $0 < \alpha - \gamma < \frac{d}{p}$.
Choose $\delta > 0$ so that

$$\delta^{\alpha-\gamma}g^\star_{N,r}f(y) = \delta^{\alpha-\gamma-\frac{d}{p}}||f||_{\dot{F}^{0,r}_{p,w}}$$

to obtain

$$\left(\int_0^\infty t^{-\gamma q}|\nu_t(T_{km\alpha}f)(y)|^q\frac{dt}{t}\right)^{1/q} \leq Cm^{n/2-1+N}\left(g^\star_{N,r}f(y)\right)^{\frac{p}{p_\star}}\left(||f||_{\dot{F}^{0,r}_{p,w}}\right)^{1-\frac{p}{p_\star}},$$

hence, by Theorem 1.1 we have

$$\begin{aligned}||T_{km\alpha}f||_{\dot{F}^{\gamma,q}_{p_\star,w}} &\simeq \left(\int_{\mathbb{R}^n}\left(\int_0^\infty t^{-\gamma q}|\nu_t(T_{km\alpha}f)(y)|^q\frac{dt}{t}\right)^{p_\star/q}w(y)dy\right)^{1/p_\star}\\ &\leq Cm^{n/2-1+N}\left(\int_{\mathbb{R}^n}\left(g^\star_{N,r}f(y)\right)^p w(y)dy\right)^{\frac{1}{p_\star}}\left(||f||_{\dot{F}^{0,r}_{p,w}}\right)^{1-\frac{p}{p_\star}}\\ &\simeq m^{n/2-1+N}||f||_{\dot{F}^{0,r}_{p,w}}.\end{aligned}$$

Case 2. $\alpha-\gamma=\frac{d}{p}$.
The estimate 2.13 shows that for $|x-y|<\delta$,

$$w(x)\left(\int_0^\delta t^{-\gamma q}|\nu_t(T_{km\alpha}f)(y)|^q\frac{dt}{t}\right)^{p/q} \leq Cm^{(n/2-1+N)p}\delta^{(\alpha-\gamma)p}\left(g^\star_{N,r}f(x)\right)^p w(x).$$

Integrating over the ball $B(y,\delta)$ with respect x to get

$$\begin{aligned}\left(\int_0^\delta t^{-\gamma q}|\nu_t(T_{km\alpha}f)(y)|^q\frac{dt}{t}\right)^{1/q} &\leq Cm^{n/2-1+N}\frac{1}{[w\left(B(y,t)\right)]^{1/p}}\delta^{\alpha-\gamma}||f||_{\dot{F}^{0,r}_{p,w}}\\ &\leq Cm^{n/2-1+N}\delta^{\alpha-\gamma-\frac{d}{p}}||f||_{\dot{F}^{0,r}_{p,w}} = Cm^{n/2-1+N}||f||_{\dot{F}^{0,r}_{p,w}}.\end{aligned}$$

It follows that for all $\delta>0$ and all $z\in\mathbb{R}^n$,

$$\left(\frac{1}{w\left(B(z,\delta)\right)}\int_{B(z,\delta)}\int_0^\delta t^{-\gamma q}|\nu_t f(y)|^q w(y)\frac{dtdy}{t}\right)^{\frac{1}{q}} \leq Cm^{n/2-1+N}||f||_{\dot{F}^{0,r}_{p,w}}.$$

Theorem 1.1 and the last estimate lead to

$$||T_{km\alpha}f||_{\dot{F}^{\gamma,q}_{\infty,w}} \leq Cm^{n/2-1+N}||f||_{\dot{F}^{0,r}_{p,w}}.$$

Now suppose $r=\infty$ and $0<q\leq\infty$. Then, using 2.6 and arguing as before to get

$$||T_{km\alpha}f||_{\dot{F}^{\gamma,q}_{p_\star,w}} \leq Cm^{n/2-1+N}||N_\lambda f||_{p,w} \simeq m^{n/2-1+N}||f||_{\dot{F}^{0,\infty}_{p,w}},$$

whenever $0<\alpha-\gamma<\frac{d}{p}$ and

$$||T_{km\alpha}f||_{\dot{F}^{\gamma,q}_{\infty,w}} \leq Cm^{n/2-1+N}||f||_{\dot{F}^{0,\infty}_{p,w}}$$

whenever $\alpha-\gamma=\frac{d}{p}$.
To prove the second imbedding of Theorem 2.6, we note that the inequality 2.4 implies, for all $x,\ y\ \in\mathbb{R}^n$,

$$|\Phi^\star(T_{km\alpha}f)(y)| \leq Cm^{n/2-1+N}\mu^\star f(x).$$

Fix y , rise the last inequality to the pth power and integrate over the ball $B(y,1)$ with respect $w(x)dx$ and using 1.2 to obtain

$$|\Phi^\star(T_{km\alpha}f)(y)| \leq Cm^{n/2-1+N}||f||_{F^{0,r}_{p,w}}.$$

Put $C_m = Cm^{n/2-1+N}$ and $A = C_m||f||_{F^{0,r}_{p,w}}$, then for any $0 < p < \infty$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} |\Phi^\star(T_{km\alpha}f)(y)|^{p_\star}w(y) &= \int_0^A \tau^{p_\star-1}w\{\Phi^\star T_{km\alpha}f)(y) > \tau\}d\tau \\ &\le \int_0^A \tau^{p_\star-1}w\{\mu^\star f)(y) > C_m^{-1}\tau\}d\tau \\ &\le C_m^{p_\star}\int_0^{C_m^{-1}A} \tau^{p_\star-1}w\{\mu^\star f)(y) > \tau\}d\tau.\end{aligned}$$

Let $0 < \epsilon < p/r_0$, where $r_0 = inf\{s : w \in A_s\}$, so that in particular, $w \in A_{p/\epsilon}$. Then we have

$$\begin{aligned}w\{\mu^\star f)(y) > \tau\} \le w\{M(\mu^\star f)^\epsilon)(y) > \tau^\epsilon\} &\le \tau^{-p}\int_{\mathbb{R}^n}(\mu^\star f)^p(y)w(y)dy \\ &\le ||f||^p_{F^{0,r}_{p,w}}.\end{aligned}$$

It follows,

$$\begin{aligned}\int_{\mathbb{R}^n} |\Phi^\star(T_{km\alpha}f)(y)|^{p_\star}w(y) &\le C_m^{p_\star}||f||^p_{F^{0,r}_{p,w}}\int_0^{C_m^{-1}A}\tau^{p_\star-p-1}d\tau \\ &\le C_m^{p_\star}||f||^p_{F^{0,r}_{p,w}}(C_m^{-1}A)^{p_\star-p} \simeq C_m^{p_\star}||f||^{p_\star}_{F^{0,r}_{p,w}}.\end{aligned} \tag{2.15}$$

On the other hand by [39, Theorem 1, Chapter V] we have for $\mu,\ \nu,\ \psi\ \in S$ such that $\hat{\mu}(0) \neq 0$, for any $0 < \delta < \infty$, $N > 0$ large enough and $x, z \in \mathbb{R}^n$, there exists $C = C(\mu, \nu, \psi, r, n)$ such that

$$|\psi_t f(x)| \le C\left(\int_0^t\int_{R^n} |\Phi_s f)(y)|^\delta(1+\frac{|x-y|}{s})^{-N\delta}(s/t)^{N\delta}\frac{dyds}{s^{n+1}}\right)^{1/\delta}. \tag{2.16}$$

Using 2.4 and 2.16 to get for $\lambda >$ and $N > 0$ large enough,

$$\begin{aligned}|C_m^{-1}\psi_t(T_{km\alpha}f)(x)|^\delta &\le \int_0^t\int_{\mathbb{R}^n}|\Phi_s(T_{km\alpha}f)(y)|^\delta\left(1+\frac{|x-y|}{s}\right)^{-N\delta}(s/t)^{N\delta}\frac{dyds}{s^{n+1}} \\ &\le \int_0^t\int_{\mathbb{R}^n} s^{\delta\alpha}|\mu_s^\star f(x)|^\delta\left(1+\frac{|x-y|}{s}\right)^{-N\delta+\lambda\delta}(s/t)^{N\delta}\frac{dyds}{s^{n+1}} \\ &\le \int_0^t\int_{\mathbb{R}^n} t^{\delta\gamma}(s/t)^{\delta\gamma}s^{\delta\alpha-\delta\gamma}|\mu_s^\star f(x)|^\delta \\ &\times\left(1+\frac{|z-y|}{s}\right)^{-N\delta+\lambda\delta}\left(1+\frac{|x-z|}{s}\right)^{N\delta-\lambda\delta}(s/t)^{N\delta}\frac{dyds}{s^{n+1}} \\ &\le t^{\delta\gamma}\left(1+\frac{|z-x|}{t}\right)^{-N\delta+\lambda\delta} \\ &\times\int_0^t\int_{\mathbb{R}^n} s^{\delta\alpha-\delta\gamma}|\mu_s^\star f(x)|^\delta\left(1+\frac{|z-y|}{s}\right)^{-N\delta+\lambda\delta}(s/t)^{\lambda\delta}\left(1+\frac{|x-z|}{s}\right)^{N\delta-\lambda\delta}\frac{dyds}{s^{n+1}} \\ &\le t^{\delta\gamma}\left(1+\frac{|x-z|}{t}\right)^{N\delta-\lambda\delta}\int_0^t\int_{\mathbb{R}^n} s^{\delta(\alpha-\beta)}|\mu_s^\star f(x)|^\delta\left(1+\frac{|z-y|}{s}\right)^{-N\delta+\lambda\delta}(s/t)^{\lambda\delta}\frac{dyds}{s^{n+1}} \\ &\le t^{\delta\gamma}\left(1+\frac{|x-z|}{t}\right)^{N\delta-\lambda\delta}\int_0^t s^{\delta(\alpha-\gamma)}|\mu_s^\star f(x)|^\delta(s/t)^{\lambda\delta}\frac{ds}{s}.\end{aligned}$$

We conclude that for all $t > 0$

$$|\psi_t^\star(T_{km\alpha}f)(x)| \le C_m t^\gamma \left(\int_0^t s^{\delta(\alpha-\gamma)}|\mu_s^\star f(x)|^\delta (s/t)^{\lambda\delta}\frac{ds}{s}\right)^{1/\delta},$$

hence

$$\int_0^1 t^{-\gamma q}|\psi_t^\star(T_{km\alpha}f)(x)|^q\frac{dt}{t} \le C_m^q \int_0^1 t^{-\lambda q-1}\left(\int_0^t s^{\delta(\alpha-\gamma)}|\mu_s^\star f(x)|^\delta (s/t)^{\lambda\delta}\frac{ds}{s}\right)^{q/\delta} dt. \tag{2.17}$$

We will now consider two cases to finish the proof.

Case 1. Assume $0 < r \le q < \infty$ and fix $0 \le \delta \le r$. Put $u(t) = t^a$, $v(t) = t^b$ with $a = -\lambda q - 1$ and $b = -\lambda r + r/\delta - 1$. Then we have for $q_1 = q/\delta$ and $r_1 = r/\delta$

$$\begin{aligned}\sup_{0<s<1}\left(\int_s^1 u(t)dt\right)^{1/q_1}\left(\int_0^s v^{1-r_1'}(t)dt\right)^{1/r_1'} &\le \sup_{0<s<1}(s^{a+1}-1)^{1/q_1}s^{\frac{b(1-r_1')+1}{r_1'}}\\ &\le \sup_{0<s<1}(1-s^{-a-1})^{1/q_1}s^{\frac{a+1}{q_1}-\frac{b+1}{r_1}+1}\\ &\le \sup_{0<s<1}(1-s^{\lambda q_1})^{1/q_1} \le 1;\end{aligned}$$

since $\frac{a+1}{q_1} - \frac{b+1}{r_1} + 1 = -\delta\lambda + \delta(\lambda - 1/\delta) + 1 = 0$. If $1 < r/\delta < q/\delta$ then by Lemma 1.3 the Hardy inequality 1.1 holds. It follows

$$\begin{aligned}&\left(\int_0^1 t^{-\lambda q-1}\left(\int_0^t s^{\delta(\alpha-\gamma)}|\mu_s^\star f(x)|^\delta (s/t)^{\lambda\delta}\frac{ds}{s}\right)^{q/\delta} dt\right)^{1/q}\\ &\le \left(\int_0^1 s^{r(\alpha-\gamma)}|\mu_s^\star f(x)|^r s^{\lambda r - r/\delta}v(s)ds\right)^{1/r} = \left(\int_0^1 s^{r(\alpha-\gamma)}|\mu_s^\star f(x)|^r\frac{ds}{s}\right)^{1/r}.\end{aligned}$$

The last estimate and 2.17 lead to

$$\left|\left|\left(\int_0^1 t^{-\gamma q}|\psi_t^\star(T_{km\alpha}f)(x)|^q\frac{dt}{t}\right)^{1/q}\right|\right|_{p_\star,w} \tag{2.18}$$

$$\le C_m\left|\left|\left(\int_0^1 s^{r(\alpha-\gamma)}|\mu_s^\star f(x)|^r\frac{ds}{s}\right)^{1/r}\right|\right|_{p_\star,w}$$

$$\le C_m||f||_{F^{-\alpha+\gamma,r}_{p_\star,w}} \le C_m||f||_{F^{0,r}_{p,w}}. \tag{2.19}$$

Where in the last step we have used the continuous inclusion $F^{0,r}_{p,w} \subset F^{-\alpha+\gamma,r}_{p_\star,w}$, since $-\alpha+\gamma = \frac{d}{p_\star} - \frac{d}{p}$, see [3, Theorem 2.6]. Combining the estimate 2.15, the estimate 2.19 and applying Theorem 1.2 to obtain

$$||T_{km\alpha}f||_{F^{\gamma,r}_{p_\star,w}} \le Cm^{n/2-1+N}||f||_{F^{0,r}_{p,w}}.$$

Case 2. Assume $0 < q \le r < \infty$ and fix $0 \le \delta \le q < r$. Let $u(t)$, $v(t)$, q_1 and r_1 as in the first case. Then on can check that

$$\begin{aligned}&\int_0^1\left(\int_s^1 u(t)dt\right)^{\varepsilon/q_1}\left(\int_0^s v^{1-r_1'}(t)dt\right)^{\epsilon/q_1'} v^{1-r_1'}(s)ds\\ &\simeq \int_0^1(1-s^{-a-1})^{\varepsilon/q_1}s^{\epsilon(\frac{a+1}{q_1}-\frac{b+1}{r_1}+1)},\end{aligned}$$

where $1/\varepsilon = 1/q_1 - 1/r_1$. Using Hardy inequality 1.1 in the case $1 < q/\delta < r/\delta$ and arguing as in Case 1 to conclude. □

Corollary 2.2. *Let $\gamma \in \mathbb{R}$, $0 < p < \infty$, $0 < q, r \leq \infty$ and N is large enough. Assume $w \in A_\infty$ satisfying $w(B(x,t)) \geq Ct^d$ for some $d > 0$ and all x. If $0 < \alpha - \gamma + \beta \leq \frac{d}{p}$ then*

$$||T_{km\alpha} f||_{\dot{F}^{\gamma,q}_{p_\star,w}} \leq Cm^{n/2-1+N} ||f||_{\dot{F}^{\beta,r}_{p,w}}. \tag{2.20}$$

If $w(B(x,t)) \geq Ct^d$ for all $0 < t < 1$ and all x, then

$$||T_{km\alpha} f||_{F^{\gamma,q}_{p_\star,w}} \leq Cm^{n/2-1+N} ||f||_{F^{\beta,r}_{p,w}}. \tag{2.21}$$

Here $p_\star$ is given by

$$\frac{1}{p_\star} = \frac{1}{p} - \frac{\alpha - \gamma + \beta}{d}.$$

Proof. Using the lifting property and Theorem 2.6 to obtain

$$\begin{aligned} ||T_{km\alpha} f||_{\dot{F}^{\gamma,q}_{p_\star,w}} &\simeq ||I_\beta \Big\{ T_{km\alpha}(I_{-\beta} f) \Big\}||_{\dot{F}^{\gamma,q}_{p_\star,w}} \\ &\simeq ||T_{km\alpha}(I_{-\beta} f)||_{\dot{F}^{\gamma-\beta,q}_{p_\star,w}} \\ &\leq Cm^{n/2-1+N} ||I_{-\beta} f||_{\dot{F}^{0,r}_{p,w}} \simeq m^{n/2-1+N} ||f||_{\dot{F}^{\beta,r}_{p,w}}. \end{aligned}$$

Hence 2.20 is proved. A similar argument leads to 2.21 by using the lifting property of the Bessel potential. □

Proof of Theorem 2.1 and 2.2. The proof of Theorem 2.1 and Theorem 2.2 in the case homogeneous space is the same as the proof of Theorem 2.6 and 2.5. The proof Theorem 2.1 and Theorem 2.2 in the case nonhomogeneous space is an immediate consequence of Theorem 2.6, Theorem 2.5 and the following result due to Rychkov, S.V., see the estimate 2.52 in [33].

Lemma 2.7. *(Boundedness of the multiplication operator). Let $a \in C^N(\mathbb{R}^n)$ with N is large enough and $w \in A_\infty$. Assume that*

$$\| \partial^\sigma a \|_\infty \leq C_N, \quad \forall \ | \sigma | \leq N$$

then we have

$$||af||_{F^{\beta,r}_{p,w}} \leq C_N ||f||_{F^{\beta,r}_{p,w}}, \quad \forall\, f \in F^{\beta,r}_{p,w}.$$

□

Proof of Theorem 2.3. Using the lifting property and apply Theorem 2.6 to obtain

$$\begin{aligned} ||f||_{\dot{F}^{\gamma,q}_{p_\star,w}} &= ||I_\alpha(I_{-\alpha} f)||_{\dot{F}^{\gamma,q}_{p_\star,w}} \\ &\leq C||(I_{-\alpha} f||_{\dot{F}^{0,r}_{p,w}} \simeq ||f||_{\dot{F}^{\alpha,r}_{p,w}}, \end{aligned}$$

and

$$\begin{aligned} ||f||_{F^{\gamma,q}_{p_\star,w}} &= ||J_\alpha(J_{-\alpha} f)||_{F^{\gamma,q}_{p_\star,w}} \\ &\leq C||(J_{-\alpha} f||_{F^{0,r}_{p,w}} \simeq ||f||_{F^{\alpha,r}_{p,w}}. \end{aligned}$$

□

Proposition 2.2. *(General Sobolev-Gagliardo-Nirenberg inequalities). Let d, p, q, r and d as in Theorem 2.2 and let $p_\star$ be such that*

$$\frac{1}{p_\star} = \frac{1}{p} + \frac{\gamma - \beta}{d}.$$

Then

$$||f||_{\dot{F}^{\gamma+1,q}_{p_\star,w}} \leq C \sum_{j=1}^{n} ||\partial_j f||_{\dot{F}^{\beta,r}_{p,w}}.$$

In particular, if $\beta = 0, \gamma = -1$ and $r = q = 2$, then we have the following more general classical Sobolev-Gagliardo-Nirenberg inequalities,

$$||f||_{H_{p_\star,w}} \leq C \sum_{j=1}^{n} ||\partial_j f||_{H_{p,w}}$$

with

$$\frac{1}{p_\star} = \frac{1}{p} - \frac{1}{d}.$$

Proof. The proof is immediate by using the precedent results and the identity

$$id = -\sum_{j=1}^{n} I_1 \circ R_j \circ \partial_j.$$

□

Similar results hold in the homogeneous Besov spaces.

Remark 2.2. *The above result can be found in [41] in the unweight Lebesgues space.*

3 Weighted Sobolev-Gagliardo-Nirenberg inequality

3.1 Extention of Sobolev-Gagliardo-Nirenberg inequality to differential forms.

The space of differential forms with coefficients in $\dot{F}^{\gamma,q}_{p,w}$ is denoted by $\dot{F}^{\gamma,q}_{p,w,\Lambda}$ and the space of differential forms in $\dot{F}^{\gamma,q}_{p,w,\Lambda}$ of order l is denoted by $\dot{F}^{\gamma,q}_{p,w,\Lambda^l}$. We define by the similar way the space $\dot{B}^{\gamma,q}_{p,w,\Lambda}$ and $F^{\gamma,q}_{p,w,\Lambda^l}$.

The exterior derivative for forms is denoted by d and δ denotes its adjoint., with the convention, $du = 0$ whenever u is an n-form. Recall also that $d^2 = 0$.

To simplify notation we denote by $\dot{A}^{\gamma,q}_{p,w}$ the space $\dot{F}^{\gamma,q}_{p,w,\Lambda^l}$ or $\dot{B}^{\gamma,q}_{p,w,\Lambda}$.

Theorem 3.1. *Let $0 < p < \infty$ and $w \in A_\infty$. Then we have*

$$||\partial_j f||_{\dot{A}^{\gamma,q}_{p,w}} \leq ||df||_{\dot{A}^{\gamma,q}_{p,w}} + ||\delta f||_{\dot{A}^{\gamma,q}_{p,w}}.$$

Proof. We have

$$\partial_j f \star \nu_t \simeq t^{-1} f \star (\partial_j \nu)_t$$

so that in the case of 0-forms

$$||\partial_j f||_{\dot{A}^{\gamma,q}_{p,w}} \simeq ||f||_{\dot{A}^{\gamma+1,q}_{p,w}}. \tag{3.1}$$

Define the Riesz transform in $S'_\infty(\mathbb{R}^n, \Lambda)$ by

$$\mathcal{R} = d \circ I^1 = I^1 \circ d$$

and its adjoint by

$$\mathcal{R}^* = \delta \circ I^1 = I^1 \circ \delta.$$

It follows from 3.1 and the lifting property of I^1 that $\mathcal{R}$ and $\mathcal{R}^*$ are bounded on $\dot{A}^{\gamma+1,q}_{p,w}$. Now the following identity

$$\partial_j = R_j \circ \mathcal{R} \circ \delta + R_j \circ \mathcal{R}^* \circ d,$$

implies what we want to prove. □

Theorem 3.2. *Let $0 < p < \infty$, $0 < q, r \leq \infty$, and $w \in A_\infty$. Then we have*

$$||f||_{\dot{A}^{\gamma+1,q}_{p_\star,w}} \leq ||df||_{\dot{A}^{\beta,r}_{p,w}} + ||\delta f||_{\dot{A}^{\beta,r}_{p,w}}$$

with

$$\frac{1}{p_\star} = \frac{1}{p} + \frac{\gamma-\beta}{d}.$$

Proof. The proof is immediate from the precedent results by using the identity

$$f = I^1 \circ \mathcal{R}(df) + I^1 \circ \mathcal{R}^*(\delta f).$$

□

Remark 3.1. *In the unweighted case and whenever $\frac{1}{p_\star} = \frac{1}{p} - \frac{1}{n}$ a version of the above Theorem can be found*

1. *in [34] in the context of Homogeneous Triebel-Lizorkin spaces,*

2. *in [41] in the context of L_p spaces.*

Remark 3.2. *When $w = 1$ then*

$$||f||_{\dot{A}^{\gamma+1,q}_{p_\star,w}} \leq ||df||_{\dot{A}^{\beta,r}_{p,w}} + ||\delta f||_{\dot{A}^{\beta,r}_{p,w}}$$

holds if and only if

$$\frac{1}{p_\star} = \frac{1}{p} + \frac{\gamma-\beta}{n}.$$

In fact, let h_τ to be the 1-parameter group of linear dilatations given in $\mathbb{R}^n$ by $h_\tau(x) = \tau x$. Then one can check that for any $f \in \dot{A}^{\alpha,q}_{p_\star,w,\Lambda^k}$, we have

$$||h^*{}_\tau f||_{\dot{A}^{\alpha,q}_{p}} = \tau^{k-\frac{n}{p}+\alpha}||f||_{\dot{A}^{\alpha,q}_{p}}.$$

On the other hand we have from the definition of d, δ and $h_\tau(x)$ that $dh_\tau f = h_\tau df$ is a $(k+1)$-form and $\delta h_\tau f = \tau^2 h_\tau df$ is a $(k-1)$-form . Hence

$$||h^*{}_\tau f||_{\dot{A}^{\gamma+1,q}_{p_\star,w}} \leq ||dh^*{}_\tau f||_{\dot{A}^{\beta,r}_{p,w}} + ||\delta h^*{}_\tau f||_{\dot{A}^{\beta,r}_{p,w}}$$

holds if and only if

$$\tau^{k-\frac{n}{p_\star}+\gamma+1}||f||_{\dot{A}^{\gamma+1,q}_{p}} \leq \tau^{k+1-\frac{n}{p}+\beta}\left(||df||_{\dot{A}^{\beta,r}_{p}} + |\delta f||_{\dot{A}^{\beta,r}_{p}}\right).$$

Thus, we must have

$$\frac{1}{p_\star} = \frac{1}{p} + \frac{\gamma-\beta}{n}.$$

3.2 Some special limiting cases

Proposition 3.1. *Let $1 < p < \infty$, $0 < q, r \leq \infty$, and $w \in A_\infty$ with $w(B(x,t)) > t^d$ for all $t > 0$. Then we have*

$$||f||_{\dot{F}^{\gamma+1,q}_{p_\star,w}} \leq ||df||_{H_{p,w}} + ||\delta f||_{H_{p,w}}$$

with

$$\frac{1}{p_\star} = \frac{1}{p} + \frac{\gamma}{d}.$$

Proof. Using Theorem 3.2 by taking $\beta = 0, r = 2$ to get the result. □

In particular, we have

Proposition 3.2. *Let $0 < p < \infty$, $0 < q \leq \infty$, and $w \in A_\infty$ with $w(B(x,t)) > t^d$ for all $t > 0$. Then we have*

$$||f||_{\dot{F}^{\gamma+1,q}_{p_\star,w}} \leq ||df||_{H_{1,w}} + ||\delta f||_{H_{1,w}}.$$

with

$$\frac{1}{p_\star} = 1 + \frac{\gamma}{d}.$$

Note that $H_{1,w}$ is a good substitute of the space $L_{1,w}$. See [1], [28], [34] and [30] for comparison.

Proposition 3.3. *For every $-n \leq \gamma < 0$, $1 < p_\star \leq \infty$, with $\frac{1}{p_\star} = 1 + \frac{\gamma}{n}$, $0 < q \leq \infty$, $1 \leq l \leq n-1$ and $w \in A_\infty$ with $w(B(x,t)) > t^n$ for all $t > 0$, there exists $C > 0$ such that for every $f \in C^\infty_c(\mathbb{R}^n, \Lambda^l)$, one has*

$$||f||_{\dot{F}^{\gamma+1,q}_{p_\star}} \leq ||df||_{L_{1,w}} + ||\delta f||_{L_{1,w}}.$$

To prove Proposition 3.3 we need the following additional result due to J.V. Schafttingen. [34]

Lemma 3.1. *For every $0 < s < 1$, $1 < p < \infty$,with $\frac{1}{p} = 1 - \frac{s}{n}$, $1 < q \leq \infty$ and $1 \leq l \leq n-1$, there exists $C > 0$ such that for every $f \in C^\infty_c(\mathbb{R}^n, \Lambda^l)$ with $df = 0$, one has*

$$||f||_{\dot{F}^{-s,p}_{p}} \leq ||f||_{L_1}.$$

Corollary 3.1. *For every $0 < s < 1$, $1 < p < \infty$,with $\frac{1}{p} = 1 - \frac{s}{n}$, $1 < q \leq \infty$, $1 \leq l \leq n-1$ and $w \in A_\infty$ with $w(B(x,t)) > t^n$ for all $t > 0$ and all x , there exists $C > 0$ such that for every $f \in C^\infty_c(\mathbb{R}^n, \Lambda^l)$ with $df = 0$, one has*

$$||f||_{\dot{F}^{-s,q}_{p}} \leq ||f||_{L_{1,w}}.$$

Proof. Since $\frac{1}{|B(x,t|} \int_{B(x,t)} w(y)dy \succeq 1$ for all $t > 0$ and all x, we have by Lebesgue's differentiation theorem that $w(x) \succeq 1$ for a.e x. Now Corollary 3.1 follows directly from Lemma 3.1. □

Proof of Proposition 3.3. Using 3.2 and Lemma 3.1 by taking $1 < p < \infty$, $0 < q \leq \infty$, $r > 1$, $-1 < \beta = -s < 0$, $\frac{1}{p} = 1 - \frac{s}{n} = 1 + \frac{\beta}{n}$ and $\frac{1}{p_\star} = 1 + \frac{\gamma}{n}$ with $0 < -\gamma - s \leq \frac{n}{p} = n - s$ to conclude that

$$||f||_{\dot{A}^{\gamma+1,q}_{p_\star}} \leq ||df||_{\dot{A}^{\beta,r}_{p}} + ||\delta f||_{\dot{A}^{\beta,r}_{p}} \leq ||df||_{L_{1,w}} + ||\delta f||_{L_{1,w}},$$

whenever $-n \leq \gamma < 0$. □

Remark 3.3. *In particular, we have by taking $\gamma = -1$*

$$||f||_{L_{\frac{n}{n-1}}} \leq ||df||_{L_{1,w}} + ||\delta f||_{L_{1,w}}.$$

In the unweighted case, this estimate has obtained by J. Bourgain and H. Brezis [1], and L. Lanzani and E. Stein [28]. See also [29] for comparison.

4 On the equation du = f

In this section, we discuss briefly the equation $du = f$, whereas before d denotes the exterior derivative operator and where f is an exact l-form with coefficients are in a suitable function space.

Denote the full exterior algebra on $\mathbb{R}^n$ by Λ and the interior product (or contraction) by $\lrcorner$ with the convention that $f \lrcorner v = 0$ whenever f is a 0-form and v is a 1-form. The space of differential forms with coefficients in $\dot{F}^{\gamma,q}_{p,w}$ is denoted by $\dot{F}^{\gamma,q}_{p,w,\Lambda}$. We define by the similar way the space $F^{\gamma,q}_{p,w,\Lambda}$.

In their work [9] the authors construct a linear operator T to solve the potential equation $du = f$ where f is a given exact form. The potential u is given by $u = Tf$. Their construction is based essentially on the following Lemma due to Chang, Krantz and Stein [8, Lemma 3.4, Lemma 3.5].

Lemma 4.1. *Given an open O of $\mathbb{R}^n$ whose closure $\bar{O}$ is contained in some open cone $K \subset \mathbb{R}^n$. Then there exists a C^∞ function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ with the following properties:*

1. *supp* $\Phi \subset \bar{O}$,
2. $\int_{\mathbb{R}^n} \Phi(x)dx = 1$
3. $\int_{\mathbb{R}^n} \Phi(x)x_j dx = 0$ *whenever* $1 \leq j \leq n$, *and*
4. *if δ denotes the Dirac distribution, then*

$$\delta = \sum_{j=1}^{n} \lim_{N \to +\infty} \int_{1/N}^{N} (\partial_j \Phi)_t \star (\Psi_j)_t \frac{dt}{t}, \tag{4.1}$$

where $\Psi_j(x) = 2\Phi(x)x_j$ and the limit is taken in S'_∞.

Define the family of kernels $K^N(x)$ by

$$K^N(x) = \int_{1/N}^{N} (\Phi)_t \star (\Psi)_t dt$$

and for each $f \in S'_\infty$ define T by

$$Tf = \lim_{N \to +\infty} K^N \star \lrcorner f, \tag{4.2}$$

where Φ and Ψ are as in Lemma 4.1 and the limit is taken in S'_∞. Formally we write

$$Tf = \int_0^\infty (\Phi)_t \star (\Psi)_t \star \lrcorner f dt.$$

Lemma 4.2. *[9] The operator T defined by 4.2 has the following properties:*

$$T : S'_{\infty,\Lambda} \longrightarrow S'_{\infty,\Lambda},$$

$$dTf + Tdf = f \ \ whenever \ \ u \in S'_{\infty,\Lambda},$$

moreover the fourier transform b of its kernel is in $C^\infty(\mathbb{R}^n \backslash \{0\})$ and satisfying

$$|\partial^\sigma b(z)| \leq C|z|^{-1+|\sigma|}$$

$\forall \sigma = (\sigma_1, ..., \sigma_n) \in \mathbb{N}^n$.

As a consequence we have the following conclusions.

Theorem 4.1. *Let $\gamma \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$ and $w \in A_\infty$. Then*

$$||Tf||_{\dot{F}^{1+\gamma,q}_{p,w,\Lambda}} \leq C||f||_{\dot{F}^{\gamma,q}_{p,w,\Lambda}},$$

and

$$||Tf||_{F^{1+\gamma,q}_{p,w,\Lambda}} \leq C||f||_{F^{\gamma,q}_{p,w,\Lambda}}.$$

Theorem 4.2. *Let $\gamma \in \mathbb{R}$, $0 < p, d < \infty$, $0 < q, r \leq \infty$. Assume $0 < \beta - \gamma \leq \frac{d}{p}$ and $w \in A_\infty$. If $w(B(x,t)) \geq Ct^d$ for all $0 < t < 1$ and all $x \in \mathbb{R}^n$, then*

$$||Tf||_{F^{\gamma+1,q}_{p_\star,w,\Lambda}} \leq C||f||_{F^{\beta,r}_{p,w,\Lambda}}. \tag{4.3}$$

If $w(B(x,t)) \geq Ct^d$ for some $d > 0$ and all $x \in \mathbb{R}^n$ and $t > 0$, then

$$||Tf||_{\dot{F}^{\gamma+1,q}_{p_\star,w,\Lambda}} \leq C||f||_{\dot{F}^{\beta,r}_{p,w,\Lambda}}, \tag{4.4}$$

where $p_\star$ is given by

$$\frac{1}{p_\star} = \frac{1}{p} + \frac{\gamma - \beta}{d}.$$

Denote by $\dot{A}^\gamma_w(\mathbb{R}^n, \Lambda)$ the space $\dot{F}^{\gamma,q}_{p_\star,w,\Lambda}$ and by $A^\gamma_w(\mathbb{R}^n, \Lambda)$ the space $F^{\gamma,q}_{p_\star,w,\Lambda}$.

Corollary 4.1. *Given f in $\dot{A}^\gamma_w(\mathbb{R}^n, \Lambda)$, then there exists $g \in \dot{A}^{\gamma+1}_w(\mathbb{R}^n, \Lambda)$*

$$||g||_{\dot{A}^{\gamma+1}_w(\mathbb{R}^n,\Lambda)} \leq c||f||_{\dot{A}^\gamma_w(\mathbb{R}^n,\Lambda)} \leq c\left(||dg||_{\dot{A}^\gamma_w(\mathbb{R}^n,\Lambda)} + ||df||_{\dot{A}^{\gamma-1}_w(\mathbb{R}^n,\Lambda)}\right). \tag{4.5}$$

The similar result holds if we take $f \in B^\gamma_w(\mathbb{R}^n, \Lambda)$.

Corollary 4.2. *Given f in $\dot{A}^\gamma_w(\mathbb{R}^n, \Lambda)$ or in $A^\gamma_w(\mathbb{R}^n, \Lambda)$ such that $df = 0$, then there exists $g \in \dot{A}^{\gamma+1}_w(\mathbb{R}^n, \Lambda)$ such that $dg = f$. Moreover, there is a constant c independent of f such that*

$$||g||_{\dot{A}^{\gamma+1}_w(\mathbb{R}^n,\Lambda)} \leq c||f||_{\dot{A}^\gamma_w(\mathbb{R}^n,\Lambda)} \tag{4.6}$$

and

$$||g||_{A^{\gamma+1}_w(\mathbb{R}^n,\Lambda)} \leq c||f||_{A^\gamma_w(\mathbb{R}^n,\Lambda)}.$$

Consequently, the de Rham complex

$$0 \to \dot{A}^\gamma_w(\mathbb{R}^n, \Lambda^0) \xrightarrow{d} \dot{A}^{\gamma-1}_w(\mathbb{R}^n, \Lambda^1) \xrightarrow{d} \dot{A}^{\gamma-2}_w(\mathbb{R}^n, \Lambda^2) ... \xrightarrow{d} \dot{A}^{\gamma-n}_w(\mathbb{R}^n, \Lambda^n) \to 0$$

and

$$0 \to B_w^{\gamma}(\mathbb{R}^n,\Lambda^0) \xrightarrow{d} B_w^{\gamma-1}(\mathbb{R}^n,\Lambda^1) \xrightarrow{d} B_w^{\gamma-2}(\mathbb{R}^n,\Lambda^2)... \xrightarrow{d} B_w^{\gamma-n}(\mathbb{R}^n,\Lambda^n) \to 0$$

are exact, and each space has a direct sum decomposition

$$\dot{A}_w^{\gamma}(\mathbb{R}^n,\Lambda^k) = d\dot{A}_w^{\gamma+1}(\mathbb{R}^n,\Lambda^{k-1}) \oplus Td\dot{A}_w^{\gamma}(\mathbb{R}^n,\Lambda^k)$$

$$B_w^{\gamma}(\mathbb{R}^n,\Lambda^k) = dB_w^{\gamma+1}(\mathbb{R}^n,\Lambda^{k-1}) \oplus TdB_w^{\gamma}(\mathbb{R}^n,\Lambda^k)$$

with bounded projections dT and Td.

Proof. The proof is as in the proof of Corollary 4.2 in [9]. We give some indications. If $df = 0$ then $dTf = f$. So set $g = Tf$ and using Theorem 4.1 or Theorem 4.2 to obtain 4.6 and 4.5. On the other hand one can check that dT and Td are bounded projections and that $dT\dot{A}_w^{\gamma}(\mathbb{R}^n,\Lambda^k) = d\dot{A}_w^{\gamma+1}(\mathbb{R}^n,\Lambda^{k-1})$ and $dTA_w^{\gamma}(\mathbb{R}^n,\Lambda^k) = dA_w^{\gamma+1}(\mathbb{R}^n,\Lambda^{k-1})$. □

Remark 4.1. *Similar results hold in the nonhomogeneous Besov or Treibel-Lizorkin spaces.*

4.1 On the divergence equation

As before we define formally Tf by

$$Tf = \int_0^\infty (\Phi)_t \star (\Psi)_t \star f dt, \tag{4.7}$$

where Φ and Ψ are as in the precedent section. Then one can check easily that

$$div\ Tf = f. \tag{4.8}$$

Arguing as before to obtain the following results.

Theorem 4.3. *Let $\gamma \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$ and $w \in A_\infty$. Given $f \in \dot{F}^{\gamma,q}_{p,w,\Lambda}$. Then there exits $g \in \dot{F}^{1+\gamma,q}_{p,w,\Lambda}$ such that $div\ g = f$ and*

$$||g||_{\dot{F}^{1+\gamma,q}_{p,w}} \leq C||f||_{\dot{F}^{\gamma,q}_{p,w}}.$$

Theorem 4.4. *Let $\gamma \in \mathbb{R}$, $0 < p, d < \infty$, $0 < q, r \leq \infty$. Assume $0 < 1 - \gamma + \beta$ and $w \in A_\infty$. If $w(B(x,t)) \geq Ct^d$ for all $0 < t < 1$ and all x, then*

$$||g||_{F^{1+\gamma,q}_{p_\star,w}} \leq C||f||_{F^{\beta,r}_{p,w}}. \tag{4.9}$$

If $w(B(x,t)) \geq Ct^d$ for some $d > 0$ and all $t > 0$ and all x, then

$$||g||_{\dot{F}^{\gamma+1,q}_{p_\star,w}} \leq C||f||_{\dot{F}^{\beta,r}_{p,w}}, \tag{4.10}$$

where $p_\star$ is given by

$$\frac{1}{p_\star} = \frac{1}{p} + \frac{\gamma-\beta}{d}.$$

4.2 Some remarks on the divergence equation

For example, when $f \in L_p, 1 < p < \infty$ one can find a solution $g \in W_p^1$ of the equation $div\ g = f$. In the limiting case where $f \in L_1$, it is not always possible to pick $g \in W_1^1$ as shown by Bourgain and Brezis in [1]. However, if f is in the Hardy space H_1 which is a good substitute for L_1 one can find a solution $g \in F_{1,w}^{1,2}$ which is a good substitute for W_1^1.

More precisely we have the following results.

Proposition 4.1. *Let $f \in H_{1,w}$ and w as in Theorem 4.4 with $d = n$. Then there exists a solution $g \in \dot{F}_{1,w}^{1,2}$ of the equation $div\ g = f$ such that*

$$||g||_{\dot{F}_{1,w}^{1,2}} \leq C||f||_{H_{1,w}}. \tag{4.11}$$

Proof. Using Theorem 4.3 by taking $p = 1, ; q = 2$ and $\gamma = 0$ to get,

$$||g||_{\dot{F}_{1,w}^{1,2}} \leq ||f||_{H_{1,w}}.$$

□

In the similar way we can deduce also the following result.

Proposition 4.2. *Let $f \in h_{1,w}$ and w as in Theorem 4.4 with $d = n$. Then there exists a solution $g \in F_{1,w}^{0,2}$ of the equation $div\ g = f$ such that*

$$||g||_{F_{1,w}^{0,2}} \leq C||f||_{h_{1,w}}.$$

Another interesting special cases are the following.

Proposition 4.3. *Let $f \in L_{n,w}$ and w as in Theorem 4.4 with $d = n$. Then there exists a solution $g \in \dot{F}_{\infty,w}^{0,q}$ of the equation $div\ g = f$ such that*

$$||g||_{\dot{F}_{\infty,w}^{0,q}} \leq C||f||_{L_{n,w}}.$$

Proposition 4.4. *Let $f \in F_{n,w}^{s,q}$ with $s > 0$ and w as in Theorem 4.4 with $d = n$. Then there exists a bounded linear operator $T : F_{n,w}^{s,q} \longrightarrow L_{\infty,w}$ such that $div\ Tf = f$ and*

$$||Tf||_{L_{\infty,w}} \leq C||f||_{F_{n,w}^{s,q}}.$$

The last result is false in general whenever $s = 0$ (see Bourgain and Bresis [1]).

Proof of Proposition 4.3. Using Theorem 4.4 by taking $p_\star = \infty$, $p = n$, $r = 2$, $\beta = 0$ and $\gamma = -1$ to obtain,

$$||g||_{\dot{F}_{\infty,w}^{0,q}} \leq C||f||_{L_{n,w}}.$$

□

Proof of Proposition 4.4. The proof is an immediate consequence of Theorem 4.4 and the well known result:

$$||g||_{L_{\infty,w}} \leq C||g||_{F_{p,w}^{s+1,q}}$$

holds if and only if $s + 1 > \frac{n}{p}$ or $s + 1 = \frac{n}{p}$ and $0 < p \leq 1$. (See for instance [35]). □

In the context of Bessel potential spaces, we have the following result.

Proposition 4.5. *Let β, $\gamma \in \mathbb{R}$ $1 < p_\star \leq \infty$ and $1 < p < \infty$ and w as in Theorem 4.4. If $f \in H_{p,w}^{\beta}$, then there exists a solution $g \in H_{p_\star,w}^{\gamma+1}$ of the equation $div\ g = f$ such that*

$$||g||_{H_{p_\star,w}^{\gamma+1}} \leq C||f||_{H_{p,w}^{\beta}}.$$

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Conflict of Interest

The authors have no conflicts of interest to declare.

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