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# On dynamics and solution expressions of a three dimensional nonlinear difference equations system

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**Abstract.** In this paper, we construct the solution expressions of fourth order nonlinear difference systems

$$\Phi_{n+1} = \frac{\Phi_{n-2}\Psi_{n-3}}{\alpha + \Psi_n\left(\pm 1 \pm \Gamma_{n-1}\Phi_{n-2}\Psi_{n-3}\right)},$$
$$\Psi_{n+1} = \frac{\Psi_{n-2}\Gamma_{n-3}}{\delta + \Gamma_n\left(\pm 1 \pm \Phi_{n-1}\Psi_{n-2}\Gamma_{n-3}\right)},$$
$$\Gamma_{n+1} = \frac{\Gamma_{n-2}\Phi_{n-3}}{\gamma + \Phi_n(\pm 1 \pm \Psi_{n-1}\Gamma_{n-2}\Phi_{n-3})},$$

where $\alpha, \delta$ and $\gamma$ are arbitrary real numbers. Furthermore, the solution's qualitative behavior is explored, such as local stability and boundedness. In some cases, system has periodic solutions. Finally, we provide numerical examples to support our conclusions.

**Keywords:** Rational difference equations, solution of system of difference equations, periodic solution, recursive sequences, local stability.
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## 1 Introduction

Difference equation studies, also referred to as discrete dynamical systems, is one of the most important fields of science, that is naturally developing as discrete analogs and numerical solutions to differential and delay differential equations, with applications in a variety of fields, including the natural depiction of a discrete process. The majority of studies on nonlinear rational difference equations concentrate on examining the behavior of solutions by offering a general form of solution. Since it might be challenging to get the solution expressions, researchers occasionally turn to analyze the stability characteristics of the equilibrium

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point. Analysis of equilibrium solutions to various systems of nonlinear difference equations has become one of the main problems in the theory of dynamical systems in recent years. Numerous studies have been written about the systems and behavior of rational difference equations (which can be obtained in the references).

Akrour et al. [1] have solved the following rational difference system:

$$\varkappa_{n+1} = \frac{\alpha\psi_n\varkappa_{n-1} + \beta\varkappa_{n-1} + \eta}{\psi_n\varkappa_{n-1}},$$
$$\psi_{n+1} = \frac{\alpha\varkappa_n\psi_{n-1} + \beta\psi_{n-1} + \eta}{\varkappa_n\psi_{n-1}}.$$

Alayachi et al. [3] have got the form of the solutions of the rational difference system:

$$\chi_{n+1} = \frac{\Upsilon_n\omega_{n-1}}{\Upsilon_n \pm \chi_{n-2}}, \Upsilon_{n+1} = \frac{\omega_n\chi_{n-1}}{\omega_n \pm \Upsilon_{n-2}}, \omega_{n+1} = \frac{\chi_n\Upsilon_{n-1}}{\chi_{n-1} \pm \omega_{n-2}}.$$

Alotaibi et al. [4] have got the forms of the solution of the difference equation systems:

$$\chi_{n+1} = \frac{\Upsilon_n\Upsilon_{n-2}}{\chi_{n-1} \pm \Upsilon_{n-2}}, \quad \Upsilon_{n+1} = \frac{\chi_n\chi_{n-2}}{\Upsilon_{n-1} \pm \chi_{n-2}}.$$

The following difference equation was obtained the general form and studied qualitative behavior of the solutions by Elsayed [10]

$$\varkappa_{n+1} = \frac{\varkappa_{n-2}\varkappa_{n-3}}{\varkappa_n(\pm 1 \pm \varkappa_{n-1}\varkappa_{n-2}\varkappa_{n-3})}.$$

Elsayed et al. [15] obtained the solutions and investigated the qualitative behavior of the system of rational difference equations:

$$\phi_{n+1} = \frac{\alpha_1\omega_{n-1}\psi_{n-1}}{\phi_{n-1} + \psi_{n-1} + \omega_{n-1}},$$
$$\psi_{n+1} = \frac{\alpha_2\omega_{n-1}\phi_{n-1}}{\phi_{n-1} + \psi_{n-1} + \omega_{n-1}},$$
$$\omega_{n+1} = \frac{\alpha_3\phi_{n-1}\psi_{n-1}}{\phi_{n-1} + \psi_{n-1} + \omega_{n-1}}.$$

Gümüş and Abo-Zeid [17] investigated the behavior of positive solutions of the system of rational difference equations:

$$\varkappa_{n+1} = \frac{\alpha\varkappa_{n-1}^2}{\beta + \gamma\psi_{n-2}}, \qquad \psi_{n+1} = \frac{\alpha_1\psi_{n-1}^2}{\beta_1 + \gamma_1\varkappa_{n-2}}.$$

Okumuş and Soykan [23] studied the boundedness, persistence and periodicity of the positive solutions and investigated the dynamics of the positive equilibrium points of the system of difference equations:

$$\chi_{n+1} = A + \frac{\chi_{n-1}}{\omega_n}, \quad \Upsilon_{n+1} = A + \frac{\Upsilon_{n-1}}{\omega_n}, \quad \omega_{n+1} = A + \frac{\omega_{n-1}}{\Upsilon_n}.$$

The purpose of this article is to determine the solution expressions and investigate the behavior of solutions for nonlinear rational difference systems of order four

$$\Phi_{n+1} = \frac{\Phi_{n-2}\Psi_{n-3}}{\alpha+\Psi_n\left(\pm 1 \pm \Gamma_{n-1}\Phi_{n-2}\Psi_{n-3}\right)},$$
$$\Psi_{n+1} = \frac{\Psi_{n-2}\Gamma_{n-3}}{\delta+\Gamma_n\left(\pm 1 \pm \Phi_{n-1}\Psi_{n-2}\Gamma_{n-3}\right)}, \tag{1}$$
$$\Gamma_{n+1} = \frac{\Gamma_{n-2}\Phi_{n-3}}{\gamma+\Phi_n(\pm 1 \pm \Psi_{n-1}\Gamma_{n-2}\Phi_{n-3})}, \qquad n = 0,1,2,...,$$

where the initial conditions $\Phi_{-3}$, $\Phi_{-2}, \Phi_{-1}, \Phi_0, \Psi_{-3}, \Psi_{-2}, \Psi_{-1}, \Psi_0, \Gamma_{-3}, \Gamma_{-2}, \Gamma_{-1}$ and $\Gamma_0$ are nonzero real numbers.

## 2 Main results

Assume $I_\Phi, I_\Psi$ and $I_\Gamma$ are any intervals of real numbers and $f : I^3_\Phi \times I^3_\Psi \times I^3_\Gamma \to I_\Phi$, $g : I^3_\Phi \times I^3_\Psi \times I^3_\Gamma \to I_\Psi, h : I^3_\Phi \times I^3_\Psi \times I^3_\Gamma \to I_\Gamma$ are continuously differentiable functions. Then for each initial condition $(\Phi_i, \Psi_i, \Gamma_i) \in I_\Phi \times I_\Psi \times I_\Gamma$ for $i \in \{-3,-2,-1,0\}$, the system of difference equations:

$$\Phi_{n+1} = f(\Phi_n, \Phi_{n-1}, \Phi_{n-2}, \Phi_{n-3}, \Psi_n, \Psi_{n-1}, \Psi_{n-2}, \Psi_{n-3}, \Gamma_n, \Gamma_{n-1}, \Gamma_{n-2}, \Gamma_{n-3}),$$
$$\Psi_{n+1} = g(\Phi_n, \Phi_{n-1}, \Phi_{n-2}, \Phi_{n-3}, \Psi_n, \Psi_{n-1}, \Psi_{n-2}, \Psi_{n-3}, \Gamma_n, \Gamma_{n-1}, \Gamma_{n-2}, \Gamma_{n-3}),$$
$$\Gamma_{n+1} = h(\Phi_n, \Phi_{n-1}, \Phi_{n-2}, \Phi_{n-3}, \Psi_n, \Psi_{n-1}, \Psi_{n-2}, \Psi_{n-3}, \Gamma_n, \Gamma_{n-1}, \Gamma_{n-2}, \Gamma_{n-3}),$$

has a unique solution $\{\Phi_n, \Psi_n, \Gamma_n\}_{n=-3}^{\infty}$.

**Definition 2.1.** A point $(\bar{\Phi}, \bar{\Psi},, \bar{\Gamma})$ is said to be an equilibrium point of (1) if

$$\bar{\Phi} = f(\bar{\Phi}, \bar{\Phi}, \bar{\Phi}, \bar{\Phi}, \bar{\Psi}, \bar{\Psi}, \bar{\Psi}, \bar{\Psi}, \bar{\Gamma}, \bar{\Gamma}, \bar{\Gamma}, \bar{\Gamma}),$$
$$\bar{\Psi} = g(\bar{\Phi}, \bar{\Phi}, \bar{\Phi}, \bar{\Phi}, \bar{\Psi}, \bar{\Psi}, \bar{\Psi}, \bar{\Psi}, \bar{\Gamma}, \bar{\Gamma}, \bar{\Gamma}, \bar{\Gamma}),$$
$$\bar{\Gamma} = h(\bar{\Phi}, \bar{\Phi}, \bar{\Phi}, \bar{\Phi}, \bar{\Psi}, \bar{\Psi}, \bar{\Psi}, \bar{\Psi}, \bar{\Gamma}, \bar{\Gamma}, \bar{\Gamma}, \bar{\Gamma}),$$

are satisfied.

**Definition 2.2.** Assume that $(\bar{\Phi}, \bar{\Psi}, \bar{\Gamma})$ is a fixed point of (1) .

**(i)** $(\bar{\Phi}, \bar{\Psi}, \bar{\Gamma})$ is said to be stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every initial condition $(\Phi_i, \Psi_i, \Gamma_i) \in I_\Phi \times I_\Psi \times I_\Gamma$ for $i \in \{-3,-2,-1,0\}$ if

$$\left\| \sum_{i=-3}^{0} (\Phi_i, \Psi_i, \Gamma_i) - (\bar{\Phi}, \bar{\Psi}, \bar{\Gamma}) \right\| < \delta \Rightarrow \left\| (\Phi_n, \Psi_n, \Gamma_n) - (\bar{\Phi}, \bar{\Psi}, \bar{\Gamma}) \right\| < \varepsilon, \text{ for all } n > 0.$$

**(ii)** $(\bar{\Phi}, \bar{\Psi}, \bar{\Gamma})$ is said to be unstable if it is not stable.

**(iii)** $(\bar{\Phi}, \bar{\Psi}, \bar{\Gamma})$ is called asymptotically stable if there exists $\gamma > 0$ such that

$$\left\| \sum_{i=-3}^{0} (\Phi_i, \Psi_i, \Gamma_i) - (\bar{\Phi}, \bar{\Psi}, \bar{\Gamma}) \right\| < \gamma, \ (\Phi_n, \Psi_n, \Gamma_n) \to (\bar{\Phi}, \bar{\Psi}, \bar{\Gamma}) \text{ as } n \to \infty.$$

**(iv)** $(\bar{\Phi}, \bar{\Psi}, \bar{\Gamma})$ is called global attractor if $(\Phi_n, \Psi_n, \Gamma_n) \to (\bar{\Phi}, \bar{\Psi}, \bar{\Gamma})$ as $n \to \infty$.

**(v)** $(\bar{\Phi}, \bar{\Psi}, \bar{\Gamma})$ is called globally asymptotically stable if it is a global attractor and stable.

**Theorem 2.3.** *Assume that* $(\Phi_{n+1}, \Psi_{n+1}, \Gamma_{n+1}) = F(\Phi_n, \Psi_n, \Gamma_n), n = 0, 1, ...,$ *is a system of difference equations where F is continuously differentiable on open neighborhood* $H \subseteq \mathbb{R}^{n+1}$ *and* $(\bar{\Phi}, \bar{\Psi}, \bar{\Gamma})$ *is a fixed point of F then*

(1) *If all eigenvalues of the Jacobian matrix* $J_F$ *at equilibrium point* $(\bar{\Phi}, \bar{\Psi}, \bar{\Gamma})$ *lie inside the unit disk i.e.* $|\lambda_i| < 1$ *then* $(\bar{\Phi}, \bar{\Psi}, \bar{\Gamma})$ *is locally asymptotically stable.*

(2) *If at least one eigenvalue at equilibrium point* $(\bar{\Phi}, \bar{\Psi}, \bar{\Gamma})$ *outside the unit disk, then* $(\bar{\Phi}, \bar{\Psi}, \bar{\Gamma})$ *is unstable.*

# 3 First case

In this section, we investigate the behavior of the solutions of the following system of difference equations

$$\begin{aligned} \Phi_{n+1} &= \frac{\Phi_{n-2}\Psi_{n-3}}{\alpha + \Psi_n \left(1 + \Gamma_{n-1}\Phi_{n-2}\Psi_{n-3}\right)}, \\ \Psi_{n+1} &= \frac{\Psi_{n-2}\Gamma_{n-3}}{\delta + \Gamma_n \left(1 + \Phi_{n-1}\Psi_{n-2}\Gamma_{n-3}\right)}, \\ \Gamma_{n+1} &= \frac{\Gamma_{n-2}\Phi_{n-3}}{\gamma + \Phi_n (1 + \Psi_{n-1}\Gamma_{n-2}\Phi_{n-3})}, \qquad n = 0, 1, 2, ..., \end{aligned} \tag{2}$$

and we take a special case $\alpha = \delta = \gamma = 0$, in (2) to get explicit formulations for the solutions of the following system of difference equations

$$\begin{aligned} \Phi_{n+1} &= \frac{\Phi_{n-2}\Psi_{n-3}}{\Psi_n \left(1 + \Gamma_{n-1}\Phi_{n-2}\Psi_{n-3}\right)}, \\ \Psi_{n+1} &= \frac{\Psi_{n-2}\Gamma_{n-3}}{\Gamma_n \left(1 + \Phi_{n-1}\Psi_{n-2}\Gamma_{n-3}\right)}, \\ \Gamma_{n+1} &= \frac{\Gamma_{n-2}\Phi_{n-3}}{\Phi_n (1 + \Psi_{n-1}\Gamma_{n-2}\Phi_{n-3})}, \qquad n = 0, 1, 2, ..., \end{aligned} \tag{3}$$

where the initial conditions $\Phi_{-3}$, $\Phi_{-2}, \Phi_{-1}, \Phi_0, \Psi_{-3}, \Psi_{-2}, \Psi_{-1}, \Psi_0, \Gamma_{-3}, \Gamma_{-2}, \Gamma_{-1}$ and $\Gamma_0$ are nonzero real numbers.

## 3.1 Stability of equilibrium point

In this subsection, we study the stability of critical point $O = (0, 0, 0)$ of the system (2).

**Theorem 3.1.** *The equilibrium point O is locally asymptotically stable.*

*Proof.* To investigate the stability of the equilibrium point $O$, we assume

$$\begin{gathered} X_n = \Phi_{n-3}, Y_n = \Phi_{n-2}, Z_n = \Phi_{n-1}, \\ U_n = \Psi_{n-3}, V_n = \Psi_{n-2}, W_n = \Psi_{n-1}, \\ P_n = \Gamma_{n-3}, Q_n = \Gamma_{n-2}, R_n = \Gamma_{n-1}, \end{gathered}$$

then the system (2) can be written as

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \\ Z_{n+1} \\ \Phi_{n+1} \\ U_{n+1} \\ V_{n+1} \\ W_{n+1} \\ \Psi_{n+1} \\ P_{n+1} \\ Q_{n+1} \\ R_{n+1} \\ \Gamma_{n+1} \end{pmatrix} = \begin{pmatrix} Y_n \\ Z_n \\ \Phi_n \\ \frac{Y_n U_n}{(\alpha+\Psi_n(1+R_n Y_n U_n))} \\ V_n \\ W_n \\ \Psi_n \\ \frac{V_n P_n}{(\delta+\Gamma_n(1+Z_n V_n P_n))} \\ Q_n \\ R_n \\ \Gamma_n \\ \frac{Q_n X_n}{(\gamma+\Phi_n(1+W_n Q_n X_n))} \end{pmatrix}. \tag{4}$$

The Jacobian matrix of (4) is given by

$$J_F = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A_1 & 0 & 0 & A_2 & 0 & 0 & A_3 & 0 & 0 & A_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & B_1 & 0 & 0 & B_2 & 0 & 0 & B_3 & 0 & 0 & B_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
C_1 & 0 & 0 & C_2 & 0 & 0 & C_3 & 0 & 0 & C_4 & 0 & 0
\end{pmatrix},$$

where

$$A_1 = \frac{\alpha U_n + \Psi_n U_n}{(\alpha + \Psi_n(1 + R_n Y_n U_n))^2}, \quad A_2 = \frac{\alpha Y_n + \Psi_n Y_n}{(\alpha + \Psi_n(1 + R_n Y_n U_n))^2},$$

$$A_3 = \frac{-Y_n U_n - R_n Y_n^2 U_n^2}{(\alpha + \Psi_n(1 + R_n Y_n U_n))^2}, \quad A_4 = \frac{-\Psi_n Y_n^2 U_n^2}{(\alpha + \Psi_n(1 + R_n Y_n U_n))^2},$$

$$B_1 = \frac{-\Gamma_n V_n^2 P_n^2}{(\delta + \Gamma_n(1 + Z_n V_n P_n))^2}, \quad B_2 = \frac{\delta P_n + \Gamma_n P_n}{(\delta + \Gamma_n(1 + Z_n V_n P_n))^2},$$

$$B_3 = \frac{\delta V_n + \Gamma_n V_n}{(\delta + \Gamma_n(1 + Z_n V_n P_n))^2}, \quad B_4 = \frac{-V_n P_n - Z_n V_n^2 P_n^2}{(\delta + \Gamma_n(1 + Z_n V_n P_n))^2},$$

$$C_1 = \frac{\gamma Q_n + \Phi_n Q_n}{(\gamma + \Phi_n(1 + W_n Q_n X_n))^2} \quad C_2 = \frac{-Q_n X_n - W_n Q_n^2 X_n^2}{(\gamma + \Phi_n(1 + W_n Q_n X_n))^2},$$

$$C_3 = \frac{-\Phi_n Q_n^2 X_n^2}{(\gamma + \Phi_n(1 + W_n Q_n X_n))^2}, \quad C_4 = \frac{\gamma X_n + \Phi_n X_n}{(\gamma + \Phi_n(1 + W_n Q_n X_n))^2}.$$

If we evaluate the Jacobian matrix about the equilibrium point $O$, we get all eigenvalues $|\lambda_i| = 0, i = 1, 2, ..., 12$. So, all eigenvalues are inside the unit disk. Therefor Theorem 1 ensures that the origin is locally asymptotically stable. □

### 3.2 On solution of system (3)

In this subsection, we obtain the solution of system (3).

**Theorem 3.2.** *Assume $\{\Phi_n, \Psi_n, \Gamma_n\}_{n=-3}^{\infty}$ is a solution of system (3). Then for n=0,1,2,...,*

$$\Phi_{6n-3} = \frac{\eta_3 \prod_{i=0}^{n-1} (1 + (3i+1)\,\eta_0\beta_1\zeta_2)\,(1 + (3i)\,\beta_1\zeta_2\eta_3)}{\prod_{i=0}^{n-1} (1 + (3i)\,\eta_0\beta_1\zeta_2)\,(1 + (3i+2)\,\beta_1\zeta_2\eta_3)},$$

$$\Phi_{6n-2} = \frac{\eta_2 \prod_{i=0}^{n-1} (1 + (3i)\,\beta_0\zeta_1\eta_2)\,(1 + (3i+2)\,\zeta_1\eta_2\beta_3)}{\prod_{i=0}^{n-1} (1 + (3i+2)\,\beta_0\zeta_1\eta_2)\,(1 + (3i+1)\,\zeta_1\eta_2\beta_3)},$$

$$\Phi_{6n-1} = \frac{\eta_1 \prod_{i=0}^{n-1} (1 + (3i+2)\,\zeta_0\eta_1\beta_2)\,(1 + (3i+1)\,\eta_1\beta_2\zeta_3)}{\prod_{i=0}^{n-1} (1 + (3i+1)\,\zeta_0\eta_1\beta_2)\,(1 + (3i+3)\,\eta_1\beta_2\zeta_3)},$$

$$\Phi_{6n} = \frac{\eta_0 \prod_{i=0}^{n-1} (1 + (3i+1)\,\eta_0\beta_1\zeta_2)\,(1 + (3i+3)\,\beta_1\zeta_2\eta_3)}{\prod_{i=0}^{n-1} (1 + (3i+3)\,\eta_0\beta_1\zeta_2)\,(1 + (3i+2)\,\beta_1\zeta_2\eta_3)},$$

$$\Phi_{6n+1} = \frac{\eta_2\beta_3 \prod_{i=0}^{n-1} (1 + (3i+3)\,\beta_0\zeta_1\eta_2)\,(1 + (3i+2)\,\zeta_1\eta_2\beta_3)}{\beta_0\,(1 + \zeta_1\eta_2\beta_3) \prod_{i=0}^{n-1} (1 + (3i+2)\,\beta_0\zeta_1\eta_2)\,(1 + (3i+4)\,\zeta_1\eta_2\beta_3)},$$

$$\Phi_{6n+2} = \frac{\zeta_0\eta_1\,(1 + \eta_1\beta_2\zeta_3) \prod_{i=0}^{n-1} (1 + (3i+2)\,\zeta_0\eta_1\beta_2)\,(1 + (3i+4)\,\eta_1\beta_2\zeta_3)}{\zeta_3\,(1 + \zeta_0\eta_1\beta_2) \prod_{i=0}^{n-1} (1 + (3i+4)\,\zeta_0\eta_1\beta_2)\,(1 + (3i+3)\,\eta_1\beta_2\zeta_3)},$$

$$\Psi_{6n-3} = \frac{\beta_3 \prod_{i=0}^{n-1} (1 + (3i+1)\,\beta_0\zeta_1\eta_2)\,(1 + (3i)\,\zeta_1\eta_2\beta_3)}{\prod_{i=0}^{n-1} (1 + (3i)\,\beta_0\zeta_1\eta_2)\,(1 + (3i+2)\,\zeta_1\eta_2\beta_3)},$$

$$\Psi_{6n-2} = \frac{\beta_2 \prod_{i=0}^{n-1} (1 + (3i)\,\zeta_0\eta_1\beta_2)\,(1 + (3i+2)\,\eta_1\beta_2\zeta_3)}{\prod_{i=0}^{n-1} (1 + (3i+2)\,\zeta_0\eta_1\beta_2)\,(1 + (3i+1)\,\eta_1\beta_2\zeta_3)},$$

$$\Psi_{6n-1} = \frac{\beta_1 \prod_{i=0}^{n-1} (1 + (3i+2)\,\eta_0\beta_1\zeta_2)\,(1 + (3i+1)\,\beta_1\zeta_2\eta_3)}{\prod_{i=0}^{n-1} (1 + (3i+1)\,\eta_0\beta_1\zeta_2)\,(1 + (3i+3)\,\beta_1\zeta_2\eta_3)},$$

$$\Psi_{6n} = \frac{\beta_0 \prod_{i=0}^{n-1} (1 + (3i+1)\,\beta_0\zeta_1\eta_2)\,(1 + (3i+3)\,\zeta_1\eta_2\beta_3)}{\prod_{i=0}^{n-1} (1 + (3i+3)\,\beta_0\zeta_1\eta_2)\,(1 + (3i+2)\,\zeta_1\eta_2\beta_3)},$$

$$\Psi_{6n+1} = \frac{\beta_2\zeta_3 \prod_{i=0}^{n-1} (1 + (3i+3)\,\zeta_0\eta_1\beta_2)\,(1 + (3i+2)\,\eta_1\beta_2\zeta_3)}{\zeta_0\,(1 + \eta_1\beta_2\zeta_3) \prod_{i=0}^{n-1} (1 + (3i+2)\,\zeta_0\eta_1\beta_2)\,(1 + (3i+4)\,\eta_1\beta_2\zeta_3)},$$

$$\Psi_{6n+2} = \frac{\eta_0\beta_1\,(1 + \beta_1\zeta_2\eta_3) \prod_{i=0}^{n-1} (1 + (3i+2)\,\eta_0\beta_1\zeta_2)\,(1 + (3i+4)\,\beta_1\zeta_2\eta_3)}{\eta_3\,(1 + \eta_0\beta_1\zeta_2) \prod_{i=0}^{n-1} (1 + (3i+4)\,\eta_0\beta_1\zeta_2)\,(1 + (3i+3)\,\beta_1\zeta_2\eta_3)},$$

$$\Gamma_{6n-3} = \frac{\zeta_3 \prod_{i=0}^{n-1} (1 + (3i+1)\,\zeta_0\eta_1\beta_2)\,(1 + (3i)\,\eta_1\beta_2\zeta_3)}{\prod_{i=0}^{n-1} (1 + (3i)\,\zeta_0\eta_1\beta_2)\,(1 + (3i+2)\,\eta_1\beta_2\zeta_3)},$$

$$\Gamma_{6n-2} = \frac{\zeta_2 \prod_{i=0}^{n-1} (1 + (3i)\,\eta_0\beta_1\zeta_2)\,(1 + (3i+2)\,\beta_1\zeta_2\eta_3)}{\prod_{i=0}^{n-1} (1 + (3i+2)\,\eta_0\beta_1\zeta_2)\,(1 + (3i+1)\,\beta_1\zeta_2\eta_3)},$$

$$\Gamma_{6n-1} = \frac{\zeta_1 \prod_{i=0}^{n-1} (1 + (3i+2)\,\beta_0\zeta_1\eta_2)\,(1 + (3i+1)\,\zeta_1\eta_2\beta_3)}{\prod_{i=0}^{n-1} (1 + (3i+1)\,\beta_0\zeta_1\eta_2)\,(1 + (3i+3)\,\zeta_1\eta_2\beta_3)},$$

$$\Gamma_{6n} = \frac{\zeta_0 \prod_{i=0}^{n-1} (1 + (3i+1)\,\zeta_0\eta_1\beta_2)\,(1 + (3i+3)\,\eta_1\beta_2\zeta_3)}{\prod_{i=0}^{n-1} (1 + (3i+3)\,\zeta_0\eta_1\beta_2)\,(1 + (3i+2)\,\eta_1\beta_2\zeta_3)},$$

$$\Gamma_{6n+1} = \frac{\zeta_2\eta_3 \prod_{i=0}^{n-1} (1 + (3i+3)\,\eta_0\beta_1\zeta_2)\,(1 + (3i+2)\,\beta_1\zeta_2\eta_3)}{\eta_0\,(1 + \beta_1\zeta_2\eta_3) \prod_{i=0}^{n-1} (1 + (3i+2)\,\eta_0\beta_1\zeta_2)\,(1 + (3i+4)\,\beta_1\zeta_2\eta_3)},$$

$$\Gamma_{6n+2} = \frac{\beta_0\zeta_1\,(1 + \zeta_1\eta_2\beta_3) \prod_{i=0}^{n-1} (1 + (3i+2)\,\beta_0\zeta_1\eta_2)\,(1 + (3i+4)\,\zeta_1\eta_2\beta_3)}{\beta_3\,(1 + \beta_0\zeta_1\eta_2) \prod_{i=0}^{n-1} (1 + (3i+4)\,\beta_0\zeta_1\eta_2)\,(1 + (3i+3)\,\zeta_1\eta_2\beta_3)},$$

*where* $\Phi_{-3} = \eta_3, \Phi_{-2} = \eta_2, \Phi_{-1} = \eta_1, \Phi_0 = \eta_0, \Psi_{-3} = \beta_3, \Psi_{-2} = \beta_2, \Psi_{-1} = \beta_1, \Psi_0 = \beta_0, \Gamma_{-3} = \zeta_3, \Gamma_{-2} = \zeta_2, \Gamma_{-1} = \zeta_1$ *and* $\Gamma_0 = \zeta_0$.

*Proof.* By using mathematical induction we will prove that the solution is true. First, for n=0 the result holds. Second, we suppose that $n > 0$ and our assumption holds for $n-1$, that is;

$$\Phi_{6n-9} = \frac{\eta_3 \prod_{i=0}^{n-2} \left(1 + (3i+1)\,\eta_0\beta_1\zeta_2\right)\left(1 + (3i)\,\beta_1\zeta_2\eta_3\right)}{\prod_{i=0}^{n-2} \left(1 + (3i)\,\eta_0\beta_1\zeta_2\right)\left(1 + (3i+2)\,\beta_1\zeta_2\eta_3\right)},$$

$$\Phi_{6n-8} = \frac{\eta_2 \prod_{i=0}^{n-2} \left(1 + (3i)\,\beta_0\zeta_1\eta_2\right)\left(1 + (3i+2)\,\zeta_1\eta_2\beta_3\right)}{\prod_{i=0}^{n-2} \left(1 + (3i+2)\,\beta_0\zeta_1\eta_2\right)\left(1 + (3i+1)\,\zeta_1\eta_2\beta_3\right)},$$

$$\Phi_{6n-7} = \frac{\eta_1 \prod_{i=0}^{n-2} \left(1 + (3i+2)\,\zeta_0\eta_1\beta_2\right)\left(1 + (3i+1)\,\eta_1\beta_2\zeta_3\right)}{\prod_{i=0}^{n-2} \left(1 + (3i+1)\,\zeta_0\eta_1\beta_2\right)\left(1 + (3i+3)\,\eta_1\beta_2\zeta_3\right)},$$

$$\Phi_{6n-6} = \frac{\eta_0 \prod_{i=0}^{n-2} \left(1 + (3i+1)\,\eta_0\beta_1\zeta_2\right)\left(1 + (3i+3)\,\beta_1\zeta_2\eta_3\right)}{\prod_{i=0}^{n-2} \left(1 + (3i+3)\,\eta_0\beta_1\zeta_2\right)\left(1 + (3i+2)\,\beta_1\zeta_2\eta_3\right)},$$

$$\Phi_{6n-5} = \frac{\eta_2\beta_3 \prod_{i=0}^{n-2} \left(1 + (3i+3)\,\beta_0\zeta_1\eta_2\right)\left(1 + (3i+2)\,\zeta_1\eta_2\beta_3\right)}{\beta_0\left(1 + \zeta_1\eta_2\beta_3\right) \prod_{i=0}^{n-2} \left(1 + (3i+2)\,\beta_0\zeta_1\eta_2\right)\left(1 + (3i+4)\,\zeta_1\eta_2\beta_3\right)},$$

$$\Phi_{6n-4} = \frac{\zeta_0\eta_1\left(1 + \eta_1\beta_2\zeta_3\right) \prod_{i=0}^{n-2} \left(1 + (3i+2)\,\zeta_0\eta_1\beta_2\right)\left(1 + (3i+4)\,\eta_1\beta_2\zeta_3\right)}{\zeta_3\left(1 + \zeta_0\eta_1\beta_2\right) \prod_{i=0}^{n-2} \left(1 + (3i+4)\,\zeta_0\eta_1\beta_2\right)\left(1 + (3i+3)\,\eta_1\beta_2\zeta_3\right)},$$

$$\Psi_{6n-9} = \frac{\beta_3 \prod_{i=0}^{n-2} \left(1 + (3i+1)\,\beta_0\zeta_1\eta_2\right)\left(1 + (3i)\,\zeta_1\eta_2\beta_3\right)}{\prod_{i=0}^{n-2} \left(1 + (3i)\,\beta_0\zeta_1\eta_2\right)\left(1 + (3i+2)\,\zeta_1\eta_2\beta_3\right)},$$

$$\Psi_{6n-8} = \frac{\beta_2 \prod_{i=0}^{n-2} \left(1 + (3i)\,\zeta_0\eta_1\beta_2\right)\left(1 + (3i+2)\,\eta_1\beta_2\zeta_3\right)}{\prod_{i=0}^{n-2} \left(1 + (3i+2)\,\zeta_0\eta_1\beta_2\right)\left(1 + (3i+1)\,\eta_1\beta_2\zeta_3\right)},$$

$$\Psi_{6n-7} = \frac{\beta_1 \prod_{i=0}^{n-2} \left(1 + (3i+2)\,\eta_0\beta_1\zeta_2\right)\left(1 + (3i+1)\,\beta_1\zeta_2\eta_3\right)}{\prod_{i=0}^{n-2} \left(1 + (3i+1)\,\eta_0\beta_1\zeta_2\right)\left(1 + (3i+3)\,\beta_1\zeta_2\eta_3\right)},$$

$$\Psi_{6n-6}=\frac{\beta_0\prod_{i=0}^{n-2}\left(1+(3i+1)\,\beta_0\zeta_1\eta_2\right)\left(1+(3i+3)\,\zeta_1\eta_2\beta_3\right)}{\prod_{i=0}^{n-2}\left(1+(3i+3)\,\beta_0\zeta_1\eta_2\right)\left(1+(3i+2)\,\zeta_1\eta_2\beta_3\right)},$$

$$\Psi_{6n-5}=\frac{\beta_2\zeta_3\prod_{i=0}^{n-2}\left(1+(3i+3)\,\zeta_0\eta_1\beta_2\right)\left(1+(3i+2)\,\eta_1\beta_2\zeta_3\right)}{\zeta_0\left(1+\eta_1\beta_2\zeta_3\right)\prod_{i=0}^{n-2}\left(1+(3i+2)\,\zeta_0\eta_1\beta_2\right)\left(1+(3i+4)\,\eta_1\beta_2\zeta_3\right)},$$

$$\Psi_{6n-4}=\frac{\eta_0\beta_1\left(1+\beta_1\zeta_2\eta_3\right)\prod_{i=0}^{n-2}\left(1+(3i+2)\,\eta_0\beta_1\zeta_2\right)\left(1+(3i+4)\,\beta_1\zeta_2\eta_3\right)}{\eta_3\left(1+\eta_0\beta_1\zeta_2\right)\prod_{i=0}^{n-2}\left(1+(3i+4)\,\eta_0\beta_1\zeta_2\right)\left(1+(3i+3)\,\beta_1\zeta_2\eta_3\right)},$$

$$\Gamma_{6n-9}=\frac{\zeta_3\prod_{i=0}^{n-2}\left(1+(3i+1)\,\zeta_0\eta_1\beta_2\right)\left(1+(3i)\,\eta_1\beta_2\zeta_3\right)}{\prod_{i=0}^{n-2}\left(1+(3i)\,\zeta_0\eta_1\beta_2\right)\left(1+(3i+2)\,\eta_1\beta_2\zeta_3\right)},$$

$$\Gamma_{6n-8}=\frac{\zeta_2\prod_{i=0}^{n-2}\left(1+(3i)\,\eta_0\beta_1\zeta_2\right)\left(1+(3i+2)\,\beta_1\zeta_2\eta_3\right)}{\prod_{i=0}^{n-2}\left(1+(3i+2)\,\eta_0\beta_1\zeta_2\right)\left(1+(3i+1)\,\beta_1\zeta_2\eta_3\right)},$$

$$\Gamma_{6n-7}=\frac{\zeta_1\prod_{i=0}^{n-2}\left(1+(3i+2)\,\beta_0\zeta_1\eta_2\right)\left(1+(3i+1)\,\zeta_1\eta_2\beta_3\right)}{\prod_{i=0}^{n-2}\left(1+(3i+1)\,\beta_0\zeta_1\eta_2\right)\left(1+(3i+3)\,\zeta_1\eta_2\beta_3\right)},$$

$$\Gamma_{6n-6}=\frac{\zeta_0\prod_{i=0}^{n-2}\left(1+(3i+1)\,\zeta_0\eta_1\beta_2\right)\left(1+(3i+3)\,\eta_1\beta_2\zeta_3\right)}{\prod_{i=0}^{n-2}\left(1+(3i+3)\,\zeta_0\eta_1\beta_2\right)\left(1+(3i+2)\,\eta_1\beta_2\zeta_3\right)},$$

$$\Gamma_{6n-5}=\frac{\zeta_2\eta_3\prod_{i=0}^{n-2}\left(1+(3i+3)\,\eta_0\beta_1\zeta_2\right)\left(1+(3i+2)\,\beta_1\zeta_2\eta_3\right)}{\eta_0\left(1+\beta_1\zeta_2\eta_3\right)\prod_{i=0}^{n-2}\left(1+(3i+2)\,\eta_0\beta_1\zeta_2\right)\left(1+(3i+4)\,\beta_1\zeta_2\eta_3\right)},$$

$$\Gamma_{6n-4}=\frac{\beta_0\zeta_1\left(1+\zeta_1\eta_2\beta_3\right)\prod_{i=0}^{n-2}\left(1+(3i+2)\,\beta_0\zeta_1\eta_2\right)\left(1+(3i+4)\,\zeta_1\eta_2\beta_3\right)}{\beta_3\left(1+\beta_0\zeta_1\eta_2\right)\prod_{i=0}^{n-2}\left(1+(3i+4)\,\beta_0\zeta_1\eta_2\right)\left(1+(3i+3)\,\zeta_1\eta_2\beta_3\right)},$$

Now, from system (3) we have

$$\Phi_{6n-3}=\frac{\Phi_{6n-6}\Psi_{6n-7}}{\Psi_{6n-4}\left(1+\Gamma_{6n-5}\Phi_{6n-6}\Psi_{6n-7}\right)}$$

$$= \frac{\left[\frac{\eta_0 \prod_{i=0}^{n-2}(1+(3i+1)\eta_0\beta_1\zeta_2)(1+(3i+3)\beta_1\zeta_2\eta_3)}{\prod_{i=0}^{n-2}(1+(3i+3)\eta_0\beta_1\zeta_2)(1+(3i+2)\beta_1\zeta_2\eta_3)}\right]\left[\frac{\beta_1 \prod_{i=0}^{n-2}(1+(3i+2)\eta_0\beta_1\zeta_2)(1+(3i+1)\beta_1\zeta_2\eta_3)}{\prod_{i=0}^{n-2}(1+(3i+1)\eta_0\beta_1\zeta_2)(1+(3i+3)\beta_1\zeta_2\eta_3)}\right]}{\left[\frac{\eta_0\beta_1(1+\beta_1\zeta_2\eta_3) \prod_{i=0}^{n-2}(1+(3i+2)\eta_0\beta_1\zeta_2)(1+(3i+4)\beta_1\zeta_2\eta_3)}{\eta_3(1+\eta_0\beta_1\zeta_2) \prod_{i=0}^{n-2}(1+(3i+4)\eta_0\beta_1\zeta_2)(1+(3i+3)\beta_1\zeta_2\eta_3)}\right]\left[\begin{array}{c} 1+\frac{\zeta_2\eta_3 \prod_{i=0}^{n-2}(1+(3i+3)\eta_0\beta_1\zeta_2)(1+(3i+2)\beta_1\zeta_2\eta_3)}{\eta_0(1+\beta_1\zeta_2\eta_3) \prod_{i=0}^{n-2}(1+(3i+2)\eta_0\beta_1\zeta_2)(1+(3i+4)\beta_1\zeta_2\eta_3)} \\ \frac{\eta_0 \prod_{i=0}^{n-2}(1+(3i+1)\eta_0\beta_1\zeta_2)(1+(3i+3)\beta_1\zeta_2\eta_3)}{\prod_{i=0}^{n-2}(1+(3i+3)\eta_0\beta_1\zeta_2)(1+(3i+2)\beta_1\zeta_2\eta_3)} \\ \frac{\beta_1 \prod_{i=0}^{n-2}(1+(3i+2)\eta_0\beta_1\zeta_2)(1+(3i+1)\beta_1\zeta_2\eta_3)}{\prod_{i=0}^{n-2}(1+(3i+1)\eta_0\beta_1\zeta_2)(1+(3i+3)\beta_1\zeta_2\eta_3)} \end{array}\right]}$$

$$= \frac{\left[\frac{\prod_{i=0}^{n-2}(1+(3i+1)\beta_1\zeta_2\eta_3)}{\prod_{i=0}^{n-2}(1+(3i+3)\eta_0\beta_1\zeta_2)(1+(3i+2)\beta_1\zeta_2\eta_3)}\right]}{\left[\frac{(1+\beta_1\zeta_2\eta_3) \prod_{i=0}^{n-2}(1+(3i+4)\beta_1\zeta_2\eta_3)}{\eta_3(1+\eta_0\beta_1\zeta_2) \prod_{i=0}^{n-2}(1+(3i+4)\eta_0\beta_1\zeta_2)(1+(3i+3)\beta_1\zeta_2\eta_3)}\right]\left[1+\frac{\beta_1\zeta_2\eta_3 \prod_{i=0}^{n-2}(1+(3i+1)\beta_1\zeta_2\eta_3)}{(1+\beta_1\zeta_2\eta_3) \prod_{i=0}^{n-2}(1+(3i+4)\beta_1\zeta_2\eta_3)}\right]}$$

$$= \frac{\left[\frac{\eta_3(1+\eta_0\beta_1\zeta_2) \prod_{i=0}^{n-2}(1+(3i+4)\eta_0\beta_1\zeta_2)(1+(3i+3)\beta_1\zeta_2\eta_3)}{\prod_{i=0}^{n-2}(1+(3i+3)\eta_0\beta_1\zeta_2)(1+(3i+2)\beta_1\zeta_2\eta_3)}\right]}{\left[\frac{(1+\beta_1\zeta_2\eta_3) \prod_{i=0}^{n-2}(1+(3i+4)\beta_1\zeta_2\eta_3)}{\prod_{i=0}^{n-2}(1+(3i+1)\beta_1\zeta_2\eta_3)}\right]\left[1+\frac{\beta_1\zeta_2\eta_3 \prod_{i=0}^{n-2}(1+(3i+1)\beta_1\zeta_2\eta_3)}{(1+\beta_1\zeta_2\eta_3) \prod_{i=0}^{n-2}(1+(3i+4)\beta_1\zeta_2\eta_3)}\right]}$$

$$= \frac{\left[\frac{\eta_3(1+\eta_0\beta_1\zeta_2) \prod_{i=0}^{n-2}(1+(3i+4)\eta_0\beta_1\zeta_2)(1+(3i+3)\beta_1\zeta_2\eta_3)}{\prod_{i=0}^{n-2}(1+(3i+3)\eta_0\beta_1\zeta_2)(1+(3i+2)\beta_1\zeta_2\eta_3)}\right]}{(1+(3n-2)\,\beta_1\zeta_2\eta_3)\left[1+\frac{\beta_1\zeta_2\eta_3}{(1+(3n-2)\beta_1\zeta_2\eta_3)}\right]}$$

$$= \frac{\left[\dfrac{\eta_3(1+\eta_0\beta_1\zeta_2)\prod_{i=0}^{n-2}(1+(3i+4)\eta_0\beta_1\zeta_2)(1+(3i+3)\beta_1\zeta_2\eta_3)}{\prod_{i=0}^{n-2}(1+(3i+3)\eta_0\beta_1\zeta_2)(1+(3i+2)\beta_1\zeta_2\eta_3)}\right]}{(1+(3n-2)\,\beta_1\zeta_2\eta_3+\beta_1\zeta_2\eta_3)}$$

$$= \frac{\eta_3\,(1+\eta_0\beta_1\zeta_2)\prod_{i=0}^{n-2}(1+(3i+4)\,\eta_0\beta_1\zeta_2)\,(1+(3i+3)\,\beta_1\zeta_2\eta_3)}{(1+(3n-1)\,\beta_1\zeta_2\eta_3)\prod_{i=0}^{n-2}(1+(3i+3)\,\eta_0\beta_1\zeta_2)\,(1+(3i+2)\,\beta_1\zeta_2\eta_3)}$$

Hence, we can get

$$\Phi_{6n-3} = \frac{\eta_3\prod_{i=0}^{n-1}(1+(3i+1)\,\eta_0\beta_1\zeta_2)\,(1+(3i)\,\beta_1\zeta_2\eta_3)}{\prod_{i=0}^{n-1}(1+(3i)\,\eta_0\beta_1\zeta_2)\,(1+(3i+2)\,\beta_1\zeta_2\eta_3)}.$$

Also, we see that from system (3)

$$\Psi_{6n-3} = \frac{\Psi_{6n-6}\Gamma_{6n-7}}{\Gamma_{6n-4}\,(1+\Phi_{6n-5}\Psi_{6n-6}\Gamma_{6n-7})}$$

$$= \frac{\left[\dfrac{\beta_0\prod_{i=0}^{n-2}(1+(3i+1)\beta_0\zeta_1\eta_2)(1+(3i+3)\zeta_1\eta_2\beta_3)}{\prod_{i=0}^{n-2}(1+(3i+3)\beta_0\zeta_1\eta_2)(1+(3i+2)\zeta_1\eta_2\beta_3)}\right]\left[\dfrac{\zeta_1\prod_{i=0}^{n-2}(1+(3i+2)\beta_0\zeta_1\eta_2)(1+(3i+1)\zeta_1\eta_2\beta_3)}{\prod_{i=0}^{n-2}(1+(3i+1)\beta_0\zeta_1\eta_2)(1+(3i+3)\zeta_1\eta_2\beta_3)}\right]}{\left[\dfrac{\beta_0\zeta_1(1+\zeta_1\eta_2\beta_3)\prod_{i=0}^{n-2}(1+(3i+2)\beta_0\zeta_1\eta_2)(1+(3i+4)\zeta_1\eta_2\beta_3)}{\beta_3(1+\beta_0\zeta_1\eta_2)\prod_{i=0}^{n-2}(1+(3i+4)\beta_0\zeta_1\eta_2)(1+(3i+3)\zeta_1\eta_2\beta_3)}\right]\left[1+\dfrac{\eta_2\beta_3\prod_{i=0}^{n-2}(1+(3i+3)\beta_0\zeta_1\eta_2)(1+(3i+2)\zeta_1\eta_2\beta_3)}{\beta_0(1+\zeta_1\eta_2\beta_3)\prod_{i=0}^{n-2}(1+(3i+2)\beta_0\zeta_1\eta_2)(1+(3i+4)\zeta_1\eta_2\beta_3)}\;\dfrac{\beta_0\prod_{i=0}^{n-2}(1+(3i+1)\beta_0\zeta_1\eta_2)(1+(3i+3)\zeta_1\eta_2\beta_3)}{\prod_{i=0}^{n-2}(1+(3i+3)\beta_0\zeta_1\eta_2)(1+(3i+2)\zeta_1\eta_2\beta_3)}\;\dfrac{\zeta_1\prod_{i=0}^{n-2}(1+(3i+2)\beta_0\zeta_1\eta_2)(1+(3i+1)\zeta_1\eta_2\beta_3)}{\prod_{i=0}^{n-2}(1+(3i+1)\beta_0\zeta_1\eta_2)(1+(3i+3)\zeta_1\eta_2\beta_3)}\right]}$$

$$= \frac{\left[\frac{\prod_{i=0}^{n-2}(1+(3i+1)\zeta_1\eta_2\beta_3)}{\prod_{i=0}^{n-2}(1+(3i+3)\beta_0\zeta_1\eta_2)(1+(3i+2)\zeta_1\eta_2\beta_3)}\right]}{\left[\frac{(1+\zeta_1\eta_2\beta_3)\prod_{i=0}^{n-2}(1+(3i+4)\zeta_1\eta_2\beta_3)}{\beta_3(1+\beta_0\zeta_1\eta_2)\prod_{i=0}^{n-2}(1+(3i+4)\beta_0\zeta_1\eta_2)(1+(3i+3)\zeta_1\eta_2\beta_3)}\right]\left[1+\frac{\zeta_1\eta_2\beta_3\prod_{i=0}^{n-2}(1+(3i+1)\zeta_1\eta_2\beta_3)}{(1+\zeta_1\eta_2\beta_3)\prod_{i=0}^{n-2}(1+(3i+4)\zeta_1\eta_2\beta_3)}\right]}$$

$$= \frac{\left[\frac{\beta_3(1+\beta_0\zeta_1\eta_2)\prod_{i=0}^{n-2}(1+(3i+4)\beta_0\zeta_1\eta_2)(1+(3i+3)\zeta_1\eta_2\beta_3)}{\prod_{i=0}^{n-2}(1+(3i+3)\beta_0\zeta_1\eta_2)(1+(3i+2)\zeta_1\eta_2\beta_3)}\right]}{\left[\frac{(1+\zeta_1\eta_2\beta_3)\prod_{i=0}^{n-2}(1+(3i+4)\zeta_1\eta_2\beta_3)}{\prod_{i=0}^{n-2}(1+(3i+1)\zeta_1\eta_2\beta_3)}\right]\left[1+\frac{\zeta_1\eta_2\beta_3\prod_{i=0}^{n-2}(1+(3i+1)\zeta_1\eta_2\beta_3)}{(1+\zeta_1\eta_2\beta_3)\prod_{i=0}^{n-2}(1+(3i+4)\zeta_1\eta_2\beta_3)}\right]}$$

$$= \frac{\left[\frac{\beta_3(1+\beta_0\zeta_1\eta_2)\prod_{i=0}^{n-2}(1+(3i+4)\beta_0\zeta_1\eta_2)(1+(3i+3)\zeta_1\eta_2\beta_3)}{\prod_{i=0}^{n-2}(1+(3i+3)\beta_0\zeta_1\eta_2)(1+(3i+2)\zeta_1\eta_2\beta_3)}\right]}{(1+(3n-2)\,\zeta_1\eta_2\beta_3)\left[1+\frac{\zeta_1\eta_2\beta_3}{(1+(3n-2)\zeta_1\eta_2\beta_3)}\right]}$$

$$= \frac{\left[\frac{\beta_3(1+\beta_0\zeta_1\eta_2)\prod_{i=0}^{n-2}(1+(3i+4)\beta_0\zeta_1\eta_2)(1+(3i+3)\zeta_1\eta_2\beta_3)}{\prod_{i=0}^{n-2}(1+(3i+3)\beta_0\zeta_1\eta_2)(1+(3i+2)\zeta_1\eta_2\beta_3)}\right]}{(1+(3n-2)\,\zeta_1\eta_2\beta_3+\zeta_1\eta_2\beta_3)}$$

$$= \frac{\beta_3\,(1+\beta_0\zeta_1\eta_2)\prod_{i=0}^{n-2}(1+(3i+4)\,\beta_0\zeta_1\eta_2)\,(1+(3i+3)\,\zeta_1\eta_2\beta_3)}{(1+(3n-1)\,\zeta_1\eta_2\beta_3)\prod_{i=0}^{n-2}(1+(3i+3)\,\beta_0\zeta_1\eta_2)\,(1+(3i+2)\,\zeta_1\eta_2\beta_3)}$$

Thus, we obtain

$$\Psi_{6n-3} = \frac{\beta_3\prod_{i=0}^{n-1}(1+(3i+1)\,\beta_0\zeta_1\eta_2)\,(1+(3i)\,\zeta_1\eta_2\beta_3)}{\prod_{i=0}^{n-1}(1+(3i)\,\beta_0\zeta_1\eta_2)\,(1+(3i+2)\,\zeta_1\eta_2\beta_3)}.$$

Furthermore, we have from system (3)

$$\Gamma_{6n-3} = \frac{\Gamma_{6n-6}\Phi_{6n-7}}{\Phi_{6n-4}(1+\Psi_{6n-5}\Gamma_{6n-6}\Phi_{6n-7})}$$

$$= \frac{\left[\dfrac{\zeta_0 \prod_{i=0}^{n-2}(1+(3i+1)\zeta_0\eta_1\beta_2)(1+(3i+3)\eta_1\beta_2\zeta_3)}{\prod_{i=0}^{n-2}(1+(3i+3)\zeta_0\eta_1\beta_2)(1+(3i+2)\eta_1\beta_2\zeta_3)}\right]\left[\dfrac{\eta_1 \prod_{i=0}^{n-2}(1+(3i+2)\zeta_0\eta_1\beta_2)(1+(3i+1)\eta_1\beta_2\zeta_3)}{\prod_{i=0}^{n-2}(1+(3i+1)\zeta_0\eta_1\beta_2)(1+(3i+3)\eta_1\beta_2\zeta_3)}\right]}{\left[\dfrac{\zeta_0\eta_1(1+\eta_1\beta_2\zeta_3)\prod_{i=0}^{n-2}(1+(3i+2)\zeta_0\eta_1\beta_2)(1+(3i+4)\eta_1\beta_2\zeta_3)}{\zeta_3(1+\zeta_0\eta_1\beta_2)\prod_{i=0}^{n-2}(1+(3i+4)\zeta_0\eta_1\beta_2)(1+(3i+3)\eta_1\beta_2\zeta_3)}\right]\left[1+\dfrac{\dfrac{\beta_2\zeta_3\prod_{i=0}^{n-2}(1+(3i+3)\zeta_0\eta_1\beta_2)(1+(3i+2)\eta_1\beta_2\zeta_3)}{\zeta_0(1+\eta_1\beta_2\zeta_3)\prod_{i=0}^{n-2}(1+(3i+2)\zeta_0\eta_1\beta_2)(1+(3i+4)\eta_1\beta_2\zeta_3)}}{\dfrac{\zeta_0\prod_{i=0}^{n-2}(1+(3i+1)\zeta_0\eta_1\beta_2)(1+(3i+3)\eta_1\beta_2\zeta_3)}{\prod_{i=0}^{n-2}(1+(3i+3)\zeta_0\eta_1\beta_2)(1+(3i+2)\eta_1\beta_2\zeta_3)}\;\dfrac{\eta_1\prod_{i=0}^{n-2}(1+(3i+2)\zeta_0\eta_1\beta_2)(1+(3i+1)\eta_1\beta_2\zeta_3)}{\prod_{i=0}^{n-2}(1+(3i+1)\zeta_0\eta_1\beta_2)(1+(3i+3)\eta_1\beta_2\zeta_3)}}\right]}$$

$$= \frac{\left[\dfrac{\prod_{i=0}^{n-2}(1+(3i+1)\eta_1\beta_2\zeta_3)}{\prod_{i=0}^{n-2}(1+(3i+3)\zeta_0\eta_1\beta_2)(1+(3i+2)\eta_1\beta_2\zeta_3)}\right]}{\left[\dfrac{(1+\eta_1\beta_2\zeta_3)\prod_{i=0}^{n-2}(1+(3i+4)\eta_1\beta_2\zeta_3)}{\zeta_3(1+\zeta_0\eta_1\beta_2)\prod_{i=0}^{n-2}(1+(3i+4)\zeta_0\eta_1\beta_2)(1+(3i+3)\eta_1\beta_2\zeta_3)}\right]\left[1+\dfrac{\eta_1\beta_2\zeta_3\prod_{i=0}^{n-2}(1+(3i+1)\eta_1\beta_2\zeta_3)}{\zeta_0(1+\eta_1\beta_2\zeta_3)\prod_{i=0}^{n-2}(1+(3i+4)\eta_1\beta_2\zeta_3)}\right]}$$

$$= \frac{\left[\dfrac{\zeta_3(1+\zeta_0\eta_1\beta_2)\prod_{i=0}^{n-2}(1+(3i+4)\zeta_0\eta_1\beta_2)(1+(3i+3)\eta_1\beta_2\zeta_3)}{\prod_{i=0}^{n-2}(1+(3i+3)\zeta_0\eta_1\beta_2)(1+(3i+2)\eta_1\beta_2\zeta_3)}\right]}{\left[\dfrac{(1+\eta_1\beta_2\zeta_3)\prod_{i=0}^{n-2}(1+(3i+4)\eta_1\beta_2\zeta_3)}{\prod_{i=0}^{n-2}(1+(3i+1)\eta_1\beta_2\zeta_3)}\right]\left[1+\dfrac{\eta_1\beta_2\zeta_3\prod_{i=0}^{n-2}(1+(3i+1)\eta_1\beta_2\zeta_3)}{(1+\eta_1\beta_2\zeta_3)\prod_{i=0}^{n-2}(1+(3i+4)\eta_1\beta_2\zeta_3)}\right]}$$

$$= \frac{\left[\dfrac{\zeta_3(1+\zeta_0\eta_1\beta_2)\prod_{i=0}^{n-2}(1+(3i+4)\zeta_0\eta_1\beta_2)(1+(3i+3)\eta_1\beta_2\zeta_3)}{\prod_{i=0}^{n-2}(1+(3i+3)\zeta_0\eta_1\beta_2)(1+(3i+2)\eta_1\beta_2\zeta_3)}\right]}{(1+(3n-2)\,\eta_1\beta_2\zeta_3)\left[1+\frac{\eta_1\beta_2\zeta_3}{(1+(3n-2)\eta_1\beta_2\zeta_3)}\right]}$$

$$= \frac{\left[\frac{\zeta_3(1+\zeta_0\eta_1\beta_2)\prod_{i=0}^{n-2}(1+(3i+4)\zeta_0\eta_1\beta_2)(1+(3i+3)\eta_1\beta_2\zeta_3)}{\prod_{i=0}^{n-2}(1+(3i+3)\zeta_0\eta_1\beta_2)(1+(3i+2)\eta_1\beta_2\zeta_3)}\right]}{(1+(3n-2)\,\eta_1\beta_2\zeta_3+\eta_1\beta_2\zeta_3)}$$

$$= \frac{\zeta_3\,(1+\zeta_0\eta_1\beta_2)\prod_{i=0}^{n-2}(1+(3i+4)\,\zeta_0\eta_1\beta_2)\,(1+(3i+3)\,\eta_1\beta_2\zeta_3)}{(1+(3n-1)\,\eta_1\beta_2\zeta_3)\prod_{i=0}^{n-2}(1+(3i+3)\,\zeta_0\eta_1\beta_2)\,(1+(3i+2)\,\eta_1\beta_2\zeta_3)}$$

Then, we have

$$\Gamma_{6n-3} = \frac{\zeta_3\prod_{i=0}^{n-1}(1+(3i+1)\,\zeta_0\eta_1\beta_2)\,(1+(3i)\,\eta_1\beta_2\zeta_3)}{\prod_{i=0}^{n-1}(1+(3i)\,\zeta_0\eta_1\beta_2)\,(1+(3i+2)\,\eta_1\beta_2\zeta_3)}.$$

In the same way, other expressions can be investigated. □

### 3.3 Boundedness of the Solution

In this subsection, we demonstrate that the positive solutions of system (3) are bounded.

**Lemma 3.3.** *Every positive solution of system (3) is bounded and converges to zero.*

*Proof.* System (3) shows that

$$\Phi_{n+1} = \frac{\Phi_{n-2}\Psi_{n-3}}{\Psi_n\,(1+\Gamma_{n-1}\Phi_{n-2}\Psi_{n-3})} \le \frac{\Phi_{n-2}\Psi_{n-3}}{\Gamma_{n-1}\Phi_{n-2}\Psi_{n-3}} \le \frac{1}{\Gamma_{n-1}},$$
$$\Psi_{n+1} = \frac{\Psi_{n-2}\Gamma_{n-3}}{\Gamma_n\,(1+\Phi_{n-1}\Psi_{n-2}\Gamma_{n-3})} \le \frac{\Psi_{n-2}\Gamma_{n-3}}{\Phi_{n-1}\Psi_{n-2}\Gamma_{n-3}} \le \frac{1}{\Phi_{n-1}},$$
$$\Gamma_{n+1} = \frac{\Gamma_{n-2}\Phi_{n-3}}{\Phi_n(1+\Psi_{n-1}\Gamma_{n-2}\Phi_{n-3})} \le \frac{\Gamma_{n-2}\Phi_{n-3}}{\Psi_{n-1}\Gamma_{n-2}\Phi_{n-3}} \le \frac{1}{\Psi_{n-1}}.$$

We conclude that

$$\Phi_{n+1} \le \frac{1}{\Gamma_{n-1}}, \quad \Psi_{n+1} \le \frac{1}{\Phi_{n-1}} \quad \text{and} \quad \Gamma_{n+1} \le \frac{1}{\Psi_{n-1}}.$$

So, for $k = 1, 2, ...$, we can get

$$\Phi_{n+1} \le \frac{1}{\Gamma_{n-1}} \le \Psi_{n-3} \le \frac{1}{\Phi_{n-5}} \le \Gamma_{n-7} \le \frac{1}{\Psi_{n-9}} \le \Phi_{n-11} \le ... \le \frac{1}{(\Phi_{n-6k+1})^{(-1)^{(k+1)}}}.$$

If we set $n = n+1, n+2, ...$, we get

$$\Phi_{n+2} \le \frac{1}{\Gamma_n} \le \Psi_{n-2} \le \frac{1}{\Phi_{n-4}} \le \Gamma_{n-6} \le \frac{1}{\Psi_{n-8}} \le \Phi_{n-10} \le ... \le \frac{1}{(\Phi_{n-6k+2})^{(-1)^{(k+1)}}},$$

$$\Phi_{n+3} \le \frac{1}{\Gamma_{n+1}} \le \Psi_{n-1} \le \frac{1}{\Phi_{n-3}} \le \Gamma_{n-5} \le \frac{1}{\Psi_{n-7}} \le \Phi_{n-9} \le ... \le \frac{1}{(\Phi_{n-6k+3})^{(-1)^{(k+1)}}},$$

$$\Phi_{n+4} \leq \frac{1}{\Gamma_{n+2}} \leq \Psi_n \leq \frac{1}{\Phi_{n-2}} \leq \Gamma_{n-4} \leq \frac{1}{\Psi_{n-6}} \leq \Phi_{n-8} \leq ... \leq \frac{1}{(\Phi_{n-6k+4})^{(-1)^{(k+1)}}},$$

$$\Phi_{n+5} \leq \frac{1}{\Gamma_{n+3}} \leq \Psi_{n+1} \leq \frac{1}{\Phi_{n-1}} \leq \Gamma_{n-3} \leq \frac{1}{\Psi_{n-5}} \leq \Phi_{n-7} \leq ... \leq \frac{1}{(\Phi_{n-6k+5})^{(-1)^{(k+1)}}},$$

$$\Phi_{n+6} \leq \frac{1}{\Gamma_{n+4}} \leq \Psi_{n+2} \leq \frac{1}{\Phi_{n}} \leq \Gamma_{n-2} \leq \frac{1}{\Psi_{n-4}} \leq \Phi_{n-6} \leq ... \leq \frac{1}{(\Phi_{n-6k+6})^{(-1)^{(k+1)}}}.$$

And so on. Similarly, for the sequences $\Psi_{n+1}$ and $\Gamma_{n+1}$, this implies that the subsequences $\{\Phi_{6n-3}\}_{n=0}^{\infty}, \{\Phi_{6n-2}\}_{n=0}^{\infty}, \{\Phi_{6n-1}\}_{n=0}^{\infty}, \{\Phi_{6n}\}_{n=0}^{\infty}, \{\Phi_{6n+1}\}_{n=0}^{\infty}, \{\Phi_{6n+2}\}_{n=0}^{\infty}, \{\Psi_{6n-3}\}_{n=0}^{\infty}, \{\Psi_{6n-2}\}_{n=0}^{\infty}, \{\Psi_{6n-1}\}_{n=0}^{\infty}, \{\Psi_{6n}\}_{n=0}^{\infty}, \{\Psi_{6n+1}\}_{n=0}^{\infty}, \{\Psi_{6n+2}\}_{n=0}^{\infty}$ and $\{\Gamma_{6n-3}\}_{n=0}^{\infty}, \{\Gamma_{6n-2}\}_{n=0}^{\infty}, \{\Gamma_{6n-1}\}_{n=0}^{\infty}, \{\Gamma_{6n}\}_{n=0}^{\infty}, \{\Gamma_{6n+1}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by

$$\Phi_{\max} = \max\{\Phi_{-3}, \Phi_{-2}, \Phi_{-1}, \Phi_0\},$$

$$\Psi_{\max} = \max\{\Psi_{-3}, \Psi_{-2}, \Psi_{-1}, \Psi_0\},$$

and

$$\Gamma_{\max} = \max\{\Gamma_{-3}, \Gamma_{-2}, \Gamma_{-1}, \Gamma_0\}.$$

□

In the following cases, we will obtain the solution expressions when we take $\alpha = \delta = \gamma = 0$, in (1).

## 4 Second case

In this section, we get the solution expressions of the following system of difference equations

$$\begin{aligned}
\Phi_{n+1} &= \frac{\Phi_{n-2}\Psi_{n-3}}{\Psi_n\left(1 - \Gamma_{n-1}\Phi_{n-2}\Psi_{n-3}\right)}, \\
\Psi_{n+1} &= \frac{\Psi_{n-2}\Gamma_{n-3}}{\Gamma_n\left(1 - \Phi_{n-1}\Psi_{n-2}\Gamma_{n-3}\right)}, \\
\Gamma_{n+1} &= \frac{\Gamma_{n-2}\Phi_{n-3}}{\Phi_n(1 - \Psi_{n-1}\Gamma_{n-2}\Phi_{n-3})}, \qquad n = 0, 1, 2, ...,
\end{aligned} \tag{5}$$

where the initial conditions $\Phi_{-3}$, $\Phi_{-2}, \Phi_{-1}, \Phi_0, \Psi_{-3}, \Psi_{-2}, \Psi_{-1}, \Psi_0, \Gamma_{-3}, \Gamma_{-2}, \Gamma_{-1}$ and $\Gamma_0$ are nonzero real numbers.

**Theorem 4.1.** *Assume $\{\Phi_n, \Psi_n, \Gamma_n\}_{n=-3}^{\infty}$ is a solution of system (5). Then for n=0,1,2,...,*

$$\Phi_{6n-3} = \frac{\eta_3 \prod_{i=0}^{n-1} \left(1 - (3i+1)\,\eta_0\beta_1\zeta_2\right)\left(1 - (3i)\,\beta_1\zeta_2\eta_3\right)}{\prod_{i=0}^{n-1} \left(1 - (3i)\,\eta_0\beta_1\zeta_2\right)\left(1 - (3i+2)\,\beta_1\zeta_2\eta_3\right)},$$

$$\Phi_{6n-2} = \frac{\eta_2 \prod_{i=0}^{n-1} \left(1 - (3i)\,\beta_0\zeta_1\eta_2\right)\left(1 - (3i+2)\,\zeta_1\eta_2\beta_3\right)}{\prod_{i=0}^{n-1} \left(1 - (3i+2)\,\beta_0\zeta_1\eta_2\right)\left(1 - (3i+1)\,\zeta_1\eta_2\beta_3\right)},$$

$$\Phi_{6n-1}=\frac{\eta_1\prod_{i=0}^{n-1}\left(1-(3i+2)\,\zeta_0\eta_1\beta_2\right)\left(1-(3i+1)\,\eta_1\beta_2\zeta_3\right)}{\prod_{i=0}^{n-1}\left(1-(3i+1)\,\zeta_0\eta_1\beta_2\right)\left(1-(3i+3)\,\eta_1\beta_2\zeta_3\right)},$$

$$\Phi_{6n}=\frac{\eta_0\prod_{i=0}^{n-1}\left(1-(3i+1)\,\eta_0\beta_1\zeta_2\right)\left(1-(3i+3)\,\beta_1\zeta_2\eta_3\right)}{\prod_{i=0}^{n-1}\left(1-(3i+3)\,\eta_0\beta_1\zeta_2\right)\left(1-(3i+2)\,\beta_1\zeta_2\eta_3\right)},$$

$$\Phi_{6n+1}=\frac{\eta_2\beta_3\prod_{i=0}^{n-1}\left(1-(3i+3)\,\beta_0\zeta_1\eta_2\right)\left(1-(3i+2)\,\zeta_1\eta_2\beta_3\right)}{\beta_0\left(1-\zeta_1\eta_2\beta_3\right)\prod_{i=0}^{n-1}\left(1-(3i+2)\,\beta_0\zeta_1\eta_2\right)\left(1-(3i+4)\,\zeta_1\eta_2\beta_3\right)},$$

$$\Phi_{6n+2}=\frac{\zeta_0\eta_1\left(1-\eta_1\beta_2\zeta_3\right)\prod_{i=0}^{n-1}\left(1-(3i+2)\,\zeta_0\eta_1\beta_2\right)\left(1-(3i+4)\,\eta_1\beta_2\zeta_3\right)}{\zeta_3\left(1-\zeta_0\eta_1\beta_2\right)\prod_{i=0}^{n-1}\left(1-(3i+4)\,\zeta_0\eta_1\beta_2\right)\left(1-(3i+3)\,\eta_1\beta_2\zeta_3\right)},$$

$$\Psi_{6n-3}=\frac{\beta_3\prod_{i=0}^{n-1}\left(1-(3i+1)\,\beta_0\zeta_1\eta_2\right)\left(1-(3i)\,\zeta_1\eta_2\beta_3\right)}{\prod_{i=0}^{n-1}\left(1-(3i)\,\beta_0\zeta_1\eta_2\right)\left(1-(3i+2)\,\zeta_1\eta_2\beta_3\right)},$$

$$\Psi_{6n-2}=\frac{\beta_2\prod_{i=0}^{n-1}\left(1-(3i)\,\zeta_0\eta_1\beta_2\right)\left(1-(3i+2)\,\eta_1\beta_2\zeta_3\right)}{\prod_{i=0}^{n-1}\left(1-(3i+2)\,\zeta_0\eta_1\beta_2\right)\left(1-(3i+1)\,\eta_1\beta_2\zeta_3\right)},$$

$$\Psi_{6n-1}=\frac{\beta_1\prod_{i=0}^{n-1}\left(1-(3i+2)\,\eta_0\beta_1\zeta_2\right)\left(1-(3i+1)\,\beta_1\zeta_2\eta_3\right)}{\prod_{i=0}^{n-1}\left(1-(3i+1)\,\eta_0\beta_1\zeta_2\right)\left(1-(3i+3)\,\beta_1\zeta_2\eta_3\right)},$$

$$\Psi_{6n}=\frac{\beta_0\prod_{i=0}^{n-1}\left(1-(3i+1)\,\beta_0\zeta_1\eta_2\right)\left(1-(3i+3)\,\zeta_1\eta_2\beta_3\right)}{\prod_{i=0}^{n-1}\left(1-(3i+3)\,\beta_0\zeta_1\eta_2\right)\left(1-(3i+2)\,\zeta_1\eta_2\beta_3\right)},$$

$$\Psi_{6n+1}=\frac{\beta_2\zeta_3\prod_{i=0}^{n-1}\left(1-(3i+3)\,\zeta_0\eta_1\beta_2\right)\left(1-(3i+2)\,\eta_1\beta_2\zeta_3\right)}{\zeta_0\left(1-\eta_1\beta_2\zeta_3\right)\prod_{i=0}^{n-1}\left(1-(3i+2)\,\zeta_0\eta_1\beta_2\right)\left(1-(3i+4)\,\eta_1\beta_2\zeta_3\right)},$$

$$\Psi_{6n+2}=\frac{\eta_0\beta_1\left(1-\beta_1\zeta_2\eta_3\right)\prod_{i=0}^{n-1}\left(1-(3i+2)\,\eta_0\beta_1\zeta_2\right)\left(1-(3i+4)\,\beta_1\zeta_2\eta_3\right)}{\eta_3\left(1-\eta_0\beta_1\zeta_2\right)\prod_{i=0}^{n-1}\left(1-(3i+4)\,\eta_0\beta_1\zeta_2\right)\left(1-(3i+3)\,\beta_1\zeta_2\eta_3\right)},$$

$$\Gamma_{6n-3}=\frac{\zeta_3\prod\limits_{i=0}^{n-1}\left(1-\left(3i+1\right)\zeta_0\eta_1\beta_2\right)\left(1-\left(3i\right)\eta_1\beta_2\zeta_3\right)}{\prod\limits_{i=0}^{n-1}\left(1-\left(3i\right)\zeta_0\eta_1\beta_2\right)\left(1-\left(3i+2\right)\eta_1\beta_2\zeta_3\right)},$$

$$\Gamma_{6n-2}=\frac{\zeta_2\prod\limits_{i=0}^{n-1}\left(1-\left(3i\right)\eta_0\beta_1\zeta_2\right)\left(1-\left(3i+2\right)\beta_1\zeta_2\eta_3\right)}{\prod\limits_{i=0}^{n-1}\left(1-\left(3i+2\right)\eta_0\beta_1\zeta_2\right)\left(1-\left(3i+1\right)\beta_1\zeta_2\eta_3\right)},$$

$$\Gamma_{6n-1}=\frac{\zeta_1\prod\limits_{i=0}^{n-1}\left(1-\left(3i+2\right)\beta_0\zeta_1\eta_2\right)\left(1-\left(3i+1\right)\zeta_1\eta_2\beta_3\right)}{\prod\limits_{i=0}^{n-1}\left(1-\left(3i+1\right)\beta_0\zeta_1\eta_2\right)\left(1-\left(3i+3\right)\zeta_1\eta_2\beta_3\right)},$$

$$\Gamma_{6n}=\frac{\zeta_0\prod\limits_{i=0}^{n-1}\left(1-\left(3i+1\right)\zeta_0\eta_1\beta_2\right)\left(1-\left(3i+3\right)\eta_1\beta_2\zeta_3\right)}{\prod\limits_{i=0}^{n-1}\left(1-\left(3i+3\right)\zeta_0\eta_1\beta_2\right)\left(1-\left(3i+2\right)\eta_1\beta_2\zeta_3\right)},$$

$$\Gamma_{6n+1}=\frac{\zeta_2\eta_3\prod\limits_{i=0}^{n-1}\left(1-\left(3i+3\right)\eta_0\beta_1\zeta_2\right)\left(1-\left(3i+2\right)\beta_1\zeta_2\eta_3\right)}{\eta_0\left(1-\beta_1\zeta_2\eta_3\right)\prod\limits_{i=0}^{n-1}\left(1-\left(3i+2\right)\eta_0\beta_1\zeta_2\right)\left(1-\left(3i+4\right)\beta_1\zeta_2\eta_3\right)},$$

$$\Gamma_{6n+2}=\frac{\beta_0\zeta_1\left(1-\zeta_1\eta_2\beta_3\right)\prod\limits_{i=0}^{n-1}\left(1-\left(3i+2\right)\beta_0\zeta_1\eta_2\right)\left(1-\left(3i+4\right)\zeta_1\eta_2\beta_3\right)}{\beta_3\left(1-\beta_0\zeta_1\eta_2\right)\prod\limits_{i=0}^{n-1}\left(1-\left(3i+4\right)\beta_0\zeta_1\eta_2\right)\left(1-\left(3i+3\right)\zeta_1\eta_2\beta_3\right)},$$

*where* $\Phi_{-3}=\eta_3,\Phi_{-2}=\eta_2,\Phi_{-1}=\eta_1,\Phi_0=\eta_0,\Psi_{-3}=\beta_3,\Psi_{-2}=\beta_2,\Psi_{-1}=\beta_1,\Psi_0=\beta_0,\Gamma_{-3}=\zeta_3,\Gamma_{-2}=\zeta_2,\Gamma_{-1}=\zeta_1$ *and* $\Gamma_0=\zeta_0$.

*Proof.* The same procedures used to demonstrate Theorem 3 can be utilized here. □

## 5 Third case

In this section, we obtain specific expressions of the solutions of the following system of difference equations

$$\begin{aligned}\Phi_{n+1}&=\frac{\Phi_{n-2}\Psi_{n-3}}{\Psi_n\left(-1+\Gamma_{n-1}\Phi_{n-2}\Psi_{n-3}\right)},\\ \Psi_{n+1}&=\frac{\Psi_{n-2}\Gamma_{n-3}}{\Gamma_n\left(-1+\Phi_{n-1}\Psi_{n-2}\Gamma_{n-3}\right)},\\ \Gamma_{n+1}&=\frac{\Gamma_{n-2}\Phi_{n-3}}{\Phi_n\left(-1+\Psi_{n-1}\Gamma_{n-2}\Phi_{n-3}\right)},\qquad n=0,1,2,...,\end{aligned}\tag{6}$$

where the initial conditions $\Phi_{-3}$, $\Phi_{-2},\Phi_{-1},\Phi_0,\Psi_{-3},\Psi_{-2},\Psi_{-1},\Psi_0,\Gamma_{-3},\Gamma_{-2},\Gamma_{-1}$ and $\Gamma_0$ are nonzero real numbers.

**Theorem 5.1.** *Assume* $\{\Phi_n, \Psi_n, \Gamma_n\}_{n=-3}^{\infty}$ *is a solution of system (6). Then every solution of system (6) is periodic with period 12, which takes the following form:*

$$\Phi_{12n-3} = \eta_3, \quad \Phi_{12n-2} = \eta_2, \quad \Phi_{12n-1} = \eta_1, \quad \Phi_{12n} = \eta_0,$$

$$\Phi_{12n+1} = \frac{\eta_2\beta_3}{\beta_0\left(-1+\zeta_1\eta_2\beta_3\right)}, \qquad \Phi_{12n+2} = \frac{\zeta_0\eta_1\left(-1+\eta_1\beta_2\zeta_3\right)}{\zeta_3\left(-1+\zeta_0\eta_1\beta_2\right)},$$

$$\Phi_{12n+3} = \eta_3\left(-1+\eta_0\beta_1\zeta_2\right), \qquad \Phi_{12n+4} = \frac{\eta_2}{\left(-1+\zeta_1\eta_2\beta_3\right)},$$

$$\Phi_{12n+5} = \frac{\eta_1}{\left(-1+\zeta_0\eta_1\beta_2\right)}, \qquad \Phi_{12n+6} = \eta_0\left(-1+\beta_1\zeta_2\eta_3\right),$$

$$\Phi_{12n+7} = \frac{\eta_2\beta_3\left(-1+\beta_0\zeta_1\eta_2\right)}{\beta_0\left(-1+\zeta_1\eta_2\beta_3\right)}, \qquad \Phi_{12n+8} = \frac{\zeta_0\eta_1}{\zeta_3\left(-1+\zeta_0\eta_1\beta_2\right)},$$

$$\Psi_{12n-3} = \beta_3, \quad \Psi_{12n-2} = \beta_2, \quad \Psi_{12n-1} = \beta_1, \quad \Psi_{12n} = \beta_0,$$

$$\Psi_{12n+1} = \frac{\beta_2\zeta_3}{\zeta_0\left(-1+\eta_1\beta_2\zeta_3\right)}, \qquad \Psi_{12n+2} = \frac{\eta_0\beta_1\left(-1+\beta_1\zeta_2\eta_3\right)}{\eta_3\left(-1+\eta_0\beta_1\zeta_2\right)},$$

$$\Psi_{12n+3} = \beta_3\left(-1+\beta_0\zeta_1\eta_2\right), \qquad \Psi_{12n+4} = \frac{\beta_2}{\left(-1+\eta_1\beta_2\zeta_3\right)},$$

$$\Psi_{12n+5} = \frac{\beta_1}{\left(-1+\eta_0\beta_1\zeta_2\right)}, \qquad \Psi_{12n+6} = \beta_0\left(-1+\zeta_1\eta_2\beta_3\right),$$

$$\Psi_{12n+7} = \frac{\beta_2\zeta_3\left(-1+\zeta_0\eta_1\beta_2\right)}{\zeta_0\left(-1+\eta_1\beta_2\zeta_3\right)}, \qquad \Psi_{12n+8} = \frac{\eta_0\beta_1}{\eta_3\left(-1+\eta_0\beta_1\zeta_2\right)},$$

$$\Gamma_{12n-3} = \zeta_3, \quad \Gamma_{12n-2} = \zeta_2, \quad \Gamma_{12n-1} = \zeta_1, \quad \Gamma_{12n} = \zeta_0,$$

$$\Gamma_{12n+1} = \frac{\zeta_2\eta_3}{\eta_0\left(-1+\beta_1\zeta_2\eta_3\right)}, \qquad \Gamma_{12n+2} = \frac{\beta_0\zeta_1\left(-1+\zeta_1\eta_2\beta_3\right)}{\beta_3\left(-1+\beta_0\zeta_1\eta_2\right)},$$

$$\Gamma_{12n+3} = \zeta_3\left(-1+\zeta_0\eta_1\beta_2\right), \qquad \Gamma_{12n+4} = \frac{\zeta_2}{\left(-1+\beta_1\zeta_2\eta_3\right)},$$

$$\Gamma_{12n+5} = \frac{\zeta_1}{\left(-1+\beta_0\zeta_1\eta_2\right)}, \qquad \Gamma_{12n+6} = \zeta_0\left(-1+\eta_1\beta_2\zeta_3\right),$$

$$\Gamma_{12n+7} = \frac{\zeta_2\eta_3\left(-1+\eta_0\beta_1\zeta_2\right)}{\eta_0\left(-1+\beta_1\zeta_2\eta_3\right)}, \qquad \Gamma_{12n+8} = \frac{\beta_0\zeta_1}{\beta_3\left(-1+\beta_0\zeta_1\eta_2\right)},$$

*where* $\Phi_{-3} = \eta_3, \Phi_{-2} = \eta_2, \Phi_{-1} = \eta_1, \Phi_0 = \eta_0, \Psi_{-3} = \beta_3, \Psi_{-2} = \beta_2, \Psi_{-1} = \beta_1, \Psi_0 = \beta_0, \Gamma_{-3} = \zeta_3, \Gamma_{-2} = \zeta_2, \Gamma_{-1} = \zeta_1$ *and* $\Gamma_0 = \zeta_0$.

*Proof.* For $n = 0$ the result holds. Now suppose that $n > 0$ and our assumption holds for $n-1$, that is;

$$\Phi_{12n-15} = \eta_3, \quad \Phi_{12n-14} = \eta_2, \quad \Phi_{12n-13} = \eta_1, \quad \Phi_{12n-12} = \eta_0,$$
$$\Phi_{12n-11} = \frac{\eta_2\beta_3}{\beta_0\left(-1+\zeta_1\eta_2\beta_3\right)}, \quad \Phi_{12n-10} = \frac{\zeta_0\eta_1\left(-1+\eta_1\beta_2\zeta_3\right)}{\zeta_3\left(-1+\zeta_0\eta_1\beta_2\right)},$$
$$\Phi_{12n-9} = \eta_3\left(-1+\eta_0\beta_1\zeta_2\right), \quad \Phi_{12n-8} = \frac{\eta_2}{\left(-1+\zeta_1\eta_2\beta_3\right)},$$
$$\Phi_{12n-7} = \frac{\eta_1}{\left(-1+\zeta_0\eta_1\beta_2\right)}, \quad \Phi_{12n-6} = \eta_0\left(-1+\beta_1\zeta_2\eta_3\right),$$
$$\Phi_{12n-5} = \frac{\eta_2\beta_3\left(-1+\beta_0\zeta_1\eta_2\right)}{\beta_0\left(-1+\zeta_1\eta_2\beta_3\right)}, \quad \Phi_{12n-4} = \frac{\zeta_0\eta_1}{\zeta_3\left(-1+\zeta_0\eta_1\beta_2\right)},$$

$$\Psi_{12n-15} = \beta_3, \quad \Psi_{12n-14} = \beta_2, \quad \Psi_{12n-13} = \beta_1, \quad \Psi_{12n-12} = \beta_0,$$
$$\Psi_{12n-11} = \frac{\beta_2\zeta_3}{\zeta_0\left(-1+\eta_1\beta_2\zeta_3\right)}, \quad \Psi_{12n-10} = \frac{\eta_0\beta_1\left(-1+\beta_1\zeta_2\eta_3\right)}{\eta_3\left(-1+\eta_0\beta_1\zeta_2\right)},$$
$$\Psi_{12n-9} = \beta_3\left(-1+\beta_0\zeta_1\eta_2\right), \quad \Psi_{12n-8} = \frac{\beta_2}{\left(-1+\eta_1\beta_2\zeta_3\right)},$$
$$\Psi_{12n-7} = \frac{\beta_1}{\left(-1+\eta_0\beta_1\zeta_2\right)}, \quad \Psi_{12n-6} = \beta_0\left(-1+\zeta_1\eta_2\beta_3\right),$$
$$\Psi_{12n-5} = \frac{\beta_2\zeta_3\left(-1+\zeta_0\eta_1\beta_2\right)}{\zeta_0\left(-1+\eta_1\beta_2\zeta_3\right)}, \quad \Psi_{12n-4} = \frac{\eta_0\beta_1}{\eta_3\left(-1+\eta_0\beta_1\zeta_2\right)},$$

$$\Gamma_{12n-15} = \zeta_3, \quad \Gamma_{12n-14} = \zeta_2, \quad \Gamma_{12n-13} = \zeta_1, \quad \Gamma_{12n-12} = \zeta_0,$$
$$\Gamma_{12n-11} = \frac{\zeta_2\eta_3}{\eta_0\left(-1+\beta_1\zeta_2\eta_3\right)}, \quad \Gamma_{12n-10} = \frac{\beta_0\zeta_1\left(-1+\zeta_1\eta_2\beta_3\right)}{\beta_3\left(-1+\beta_0\zeta_1\eta_2\right)},$$
$$\Gamma_{12n-9} = \zeta_3\left(-1+\zeta_0\eta_1\beta_2\right), \quad \Gamma_{12n-8} = \frac{\zeta_2}{\left(-1+\beta_1\zeta_2\eta_3\right)},$$
$$\Gamma_{12n-7} = \frac{\zeta_1}{\left(-1+\beta_0\zeta_1\eta_2\right)}, \quad \Gamma_{12n-6} = \zeta_0\left(-1+\eta_1\beta_2\zeta_3\right),$$
$$\Gamma_{12n-5} = \frac{\zeta_2\eta_3\left(-1+\eta_0\beta_1\zeta_2\right)}{\eta_0\left(-1+\beta_1\zeta_2\eta_3\right)}, \quad \Gamma_{12n-4} = \frac{\beta_0\zeta_1}{\beta_3\left(-1+\beta_0\zeta_1\eta_2\right)},$$

Now, we find from system (6) that

$$\begin{aligned}
\Phi_{12n-3} &= \frac{\Phi_{12n-6}\Psi_{12n-7}}{\Psi_{12n-4}\left(-1+\Gamma_{12n-5}\Phi_{12n-6}\Psi_{12n-7}\right)} \\
&= \frac{\left[\eta_0\left(-1+\beta_1\zeta_2\eta_3\right)\right]\left[\frac{\beta_1}{\left(-1+\eta_0\beta_1\zeta_2\right)}\right]}{\left[\frac{\eta_0\beta_1}{\eta_3\left(-1+\eta_0\beta_1\zeta_2\right)}\right]\left[-1+\frac{\zeta_2\eta_3\left(-1+\eta_0\beta_1\zeta_2\right)}{\eta_0\left(-1+\beta_1\zeta_2\eta_3\right)}\eta_0\left(-1+\beta_1\zeta_2\eta_3\right)\frac{\beta_1}{\left(-1+\eta_0\beta_1\zeta_2\right)}\right]} \\
&= \frac{\eta_3\left(-1+\beta_1\zeta_2\eta_3\right)}{\left[-1+\beta_1\zeta_2\eta_3\right]} = \eta_3.
\end{aligned}$$

Similarly, by using the same method, we can investigate the relations

$$\begin{aligned}\Psi_{12n-3} &= \frac{\Psi_{12n-6}\Gamma_{12n-7}}{\Gamma_{12n-4}\left(-1+\Phi_{12n-5}\Psi_{12n-6}\Gamma_{12n-7}\right)}\\ &= \frac{\left[\beta_0\left(-1+\zeta_1\eta_2\beta_3\right)\right]\left[\frac{\zeta_1}{(-1+\beta_0\zeta_1\eta_2)}\right]}{\left[\frac{\beta_0\zeta_1}{\beta_3(-1+\beta_0\zeta_1\eta_2)}\right]\left[-1+\frac{\eta_2\beta_3(-1+\beta_0\zeta_1\eta_2)}{\beta_0(-1+\zeta_1\eta_2\beta_3)}\beta_0\left(-1+\zeta_1\eta_2\beta_3\right)\frac{\zeta_1}{(-1+\beta_0\zeta_1\eta_2)}\right]}\\ &= \frac{\beta_3\left(-1+\zeta_1\eta_2\beta_3\right)}{\left[-1+\zeta_1\eta_2\beta_3\right]} = \beta_3.\end{aligned}$$

Also, we see that from system (6)

$$\begin{aligned}\Gamma_{12n-3} &= \frac{\Gamma_{12n-6}\Phi_{12n-7}}{\Phi_{12n-4}(-1+\Psi_{12n-5}\Gamma_{12n-6}\Phi_{12n-7})}\\ &= \frac{\left[\zeta_0\left(-1+\eta_1\beta_2\zeta_3\right)\right]\left[\frac{\eta_1}{(-1+\zeta_0\eta_1\beta_2)}\right]}{\left[\frac{\zeta_0\eta_1}{\zeta_3(-1+\zeta_0\eta_1\beta_2)}\right]\left[-1+\frac{\beta_2\zeta_3(-1+\zeta_0\eta_1\beta_2)}{\zeta_0(-1+\eta_1\beta_2\zeta_3)}\zeta_0\left(-1+\eta_1\beta_2\zeta_3\right)\frac{\eta_1}{(-1+\zeta_0\eta_1\beta_2)}\right]}\\ &= \frac{\zeta_3\left(-1+\eta_1\beta_2\zeta_3\right)}{\left[-1+\zeta_0\eta_1\beta_2\right]} = \zeta_3.\end{aligned}$$

Likewise,

$$\begin{aligned}\Phi_{12n-2} &= \frac{\Phi_{12n-5}\Psi_{12n-6}}{\Psi_{12n-3}\left(-1+\Gamma_{12n-4}\Phi_{12n-5}\Psi_{12n-6}\right)}\\ &= \frac{\left[\frac{\eta_2\beta_3(-1+\beta_0\zeta_1\eta_2)}{\beta_0(-1+\zeta_1\eta_2\beta_3)}\right]\left[\beta_0\left(-1+\zeta_1\eta_2\beta_3\right)\right]}{\beta_3\left[-1+\frac{\beta_0\zeta_1}{\beta_3(-1+\beta_0\zeta_1\eta_2)}\frac{\eta_2\beta_3(-1+\beta_0\zeta_1\eta_2)}{\beta_0(-1+\zeta_1\eta_2\beta_3)}\beta_0\left(-1+\zeta_1\eta_2\beta_3\right)\right]}\\ &= \frac{\eta_2\left(-1+\beta_0\zeta_1\eta_2\right)}{\left[-1+\beta_0\zeta_1\eta_2\right]} = \eta_2.\end{aligned}$$

$$\begin{aligned}\Psi_{12n-2} &= \frac{\Psi_{12n-5}\Gamma_{12n-6}}{\Gamma_{12n-3}\left(-1+\Phi_{12n-4}\Psi_{12n-5}\Gamma_{12n-6}\right)}\\ &= \frac{\left[\frac{\beta_2\zeta_3(-1+\zeta_0\eta_1\beta_2)}{\zeta_0(-1+\eta_1\beta_2\zeta_3)}\right]\left[\zeta_0\left(-1+\eta_1\beta_2\zeta_3\right)\right]}{\zeta_3\left[-1+\frac{\zeta_0\eta_1}{\zeta_3(-1+\zeta_0\eta_1\beta_2)}\frac{\beta_2\zeta_3(-1+\zeta_0\eta_1\beta_2)}{\zeta_0(-1+\eta_1\beta_2\zeta_3)}\zeta_0\left(-1+\eta_1\beta_2\zeta_3\right)\right]}\\ &= \frac{\beta_2\left(-1+\zeta_0\eta_1\beta_2\right)}{\left[-1+\zeta_0\eta_1\beta_2\right]} = \beta_2.\end{aligned}$$

$$\begin{aligned}\Gamma_{12n-2} &= \frac{\Gamma_{12n-5}\Phi_{12n-6}}{\Phi_{12n-3}(-1+\Psi_{12n-4}\Gamma_{12n-5}\Phi_{12n-6})}\\ &= \frac{\left[\frac{\zeta_2\eta_3(-1+\eta_0\beta_1\zeta_2)}{\eta_0(-1+\beta_1\zeta_2\eta_3)}\right]\left[\eta_0\left(-1+\beta_1\zeta_2\eta_3\right)\right]}{\eta_3\left[-1+\frac{\eta_0\beta_1}{\eta_3(-1+\eta_0\beta_1\zeta_2)}\frac{\zeta_2\eta_3(-1+\eta_0\beta_1\zeta_2)}{\eta_0(-1+\beta_1\zeta_2\eta_3)}\eta_0\left(-1+\beta_1\zeta_2\eta_3\right)\right]}\\ &= \frac{\zeta_2\left(-1+\eta_0\beta_1\zeta_2\right)}{\left[-1+\eta_0\beta_1\zeta_2\right]} = \zeta_2.\end{aligned}$$

Other relations can be proven in the same way. □

**Lemma 5.2.** *System (6) has aperiodic solution of period six iff* $\zeta_1\eta_2\beta_3 = \eta_1\beta_2\zeta_3 = \zeta_0\eta_1\beta_2 = \eta_0\beta_1\zeta_2 = \beta_1\zeta_2\eta_3 = \beta_0\zeta_1\eta_2 = 2$ *and* $\{\Phi_n, \Psi_n, \Gamma_n\}_{n=-3}^{\infty}$ *will take the form*

$$\Phi_n = \left\{\eta_3, \eta_2, \eta_1, \eta_0, \frac{\eta_2\beta_3}{\beta_0}, \frac{\zeta_0\eta_1}{\zeta_3}, \eta_3, \eta_2, \eta_1, \eta_0, \frac{\eta_2\beta_3}{\beta_0}, \frac{\zeta_0\eta_1}{\zeta_3}, ...\right\},$$
$$\Psi_n = \left\{\beta_3, \beta_2, \beta_1, \beta_0, \frac{\beta_2\zeta_3}{\zeta_0}, \frac{\eta_0\beta_1}{\eta_3}, \beta_3, \beta_2, \beta_1, \beta_0, \frac{\beta_2\zeta_3}{\zeta_0}, \frac{\eta_0\beta_1}{\eta_3}, ...\right\},$$
$$\Gamma_n = \left\{\zeta_3, \zeta_2, \zeta_1, \zeta_0, \frac{\zeta_2\eta_3}{\eta_0}, \frac{\beta_0\zeta_1}{\beta_3}, \zeta_3, \zeta_2, \zeta_1, \zeta_0, \frac{\zeta_2\eta_3}{\eta_0}, \frac{\beta_0\zeta_1}{\beta_3}, ...\right\}.$$

## 6 Fourth case

In this section, we obtain specific expressions of the solutions of the following system of difference equations

$$\begin{aligned}
\Phi_{n+1} &= \frac{\Phi_{n-2}\Psi_{n-3}}{\Psi_n\left(-1-\Gamma_{n-1}\Phi_{n-2}\Psi_{n-3}\right)}, \\
\Psi_{n+1} &= \frac{\Psi_{n-2}\Gamma_{n-3}}{\Gamma_n\left(-1-\Phi_{n-1}\Psi_{n-2}\Gamma_{n-3}\right)}, \\
\Gamma_{n+1} &= \frac{\Gamma_{n-2}\Phi_{n-3}}{\Phi_n\left(-1-\Psi_{n-1}\Gamma_{n-2}\Phi_{n-3}\right)}, \qquad n = 0, 1, 2, ...,
\end{aligned} \tag{7}$$

where the initial conditions $\Phi_{-3}$, $\Phi_{-2}, \Phi_{-1}, \Phi_0, \Psi_{-3}, \Psi_{-2}, \Psi_{-1}, \Psi_0, \Gamma_{-3}, \Gamma_{-2}, \Gamma_{-1}$ and $\Gamma_0$ are nonzero real numbers.

**Theorem 6.1.** *Assume* $\{\Phi_n, \Psi_n, \Gamma_n\}_{n=-3}^{\infty}$ *is a solution of system (7). Then every solution of system (7) is periodic with period 12, which takes the following form:*

$$\Phi_{12n-3} = \eta_3, \quad \Phi_{12n-2} = \eta_2, \quad \Phi_{12n-1} = \eta_1, \quad \Phi_{12n} = \eta_0,$$

$$\Phi_{12n+1} = \frac{\eta_2\beta_3}{\beta_0\left(-1-\zeta_1\eta_2\beta_3\right)}, \qquad \Phi_{12n+2} = \frac{\zeta_0\eta_1\left(-1-\eta_1\beta_2\zeta_3\right)}{\zeta_3\left(-1-\zeta_0\eta_1\beta_2\right)},$$

$$\Phi_{12n+3} = \eta_3\left(-1-\eta_0\beta_1\zeta_2\right), \qquad \Phi_{12n+4} = \frac{\eta_2}{\left(-1-\zeta_1\eta_2\beta_3\right)},$$

$$\Phi_{12n+5} = \frac{\eta_1}{\left(-1-\zeta_0\eta_1\beta_2\right)}, \qquad \Phi_{12n+6} = \eta_0\left(-1-\beta_1\zeta_2\eta_3\right),$$

$$\Phi_{12n+7} = \frac{\eta_2\beta_3\left(-1-\beta_0\zeta_1\eta_2\right)}{\beta_0\left(-1-\zeta_1\eta_2\beta_3\right)}, \qquad \Phi_{12n+8} = \frac{\zeta_0\eta_1}{\zeta_3\left(-1-\zeta_0\eta_1\beta_2\right)},$$

$$\Psi_{12n-3} = \beta_3, \quad \Psi_{12n-2} = \beta_2, \quad \Psi_{12n-1} = \beta_1, \quad \Psi_{12n} = \beta_0,$$

$$\Psi_{12n+1} = \frac{\beta_2\zeta_3}{\zeta_0\left(-1-\eta_1\beta_2\zeta_3\right)}, \qquad \Psi_{12n+2} = \frac{\eta_0\beta_1\left(-1-\beta_1\zeta_2\eta_3\right)}{\eta_3\left(-1-\eta_0\beta_1\zeta_2\right)},$$

$$\Psi_{12n+3} = \beta_3\left(-1-\beta_0\zeta_1\eta_2\right), \qquad \Psi_{12n+4} = \frac{\beta_2}{\left(-1-\eta_1\beta_2\zeta_3\right)},$$

$$\Psi_{12n+5} = \frac{\beta_1}{(-1-\eta_0\beta_1\zeta_2)}, \qquad \Psi_{12n+6} = \beta_0\left(-1-\zeta_1\eta_2\beta_3\right),$$

$$\Psi_{12n+7} = \frac{\beta_2\zeta_3\left(-1-\beta_0\zeta_1\eta_2\right)}{\zeta_0\left(-1-\eta_1\beta_2\zeta_3\right)}, \qquad \Psi_{12n+8} = \frac{\eta_0\beta_1}{\eta_3\left(-1-\eta_0\beta_1\zeta_2\right)},$$

$$\Gamma_{12n-3} = \zeta_3, \quad \Gamma_{12n-2} = \zeta_2, \quad \Gamma_{12n-1} = \zeta_1, \quad \Gamma_{12n} = \zeta_0,$$

$$\Gamma_{12n+1} = \frac{\zeta_2\eta_3}{\eta_0\left(-1-\beta_1\zeta_2\eta_3\right)}, \qquad \Gamma_{12n+2} = \frac{\beta_0\zeta_1\left(-1-\zeta_1\eta_2\beta_3\right)}{\beta_3\left(-1-\beta_0\zeta_1\eta_2\right)},$$

$$\Gamma_{12n+3} = \zeta_3\left(-1-\zeta_0\eta_1\beta_2\right), \qquad \Gamma_{12n+4} = \frac{\zeta_2}{(-1-\beta_1\zeta_2\eta_3)},$$

$$\Gamma_{12n+5} = \frac{\zeta_1}{(-1-\beta_0\zeta_1\eta_2)}, \qquad \Gamma_{12n+6} = \zeta_0\left(-1-\eta_1\beta_2\zeta_3\right),$$

$$\Gamma_{12n+7} = \frac{\zeta_2\eta_3\left(-1-\eta_0\beta_1\zeta_2\right)}{\eta_0\left(-1-\beta_1\zeta_2\eta_3\right)}, \qquad \Gamma_{12n+8} = \frac{\beta_0\zeta_1}{\beta_3\left(-1-\beta_0\zeta_1\eta_2\right)},$$

*where* $\Phi_{-3} = \eta_3, \Phi_{-2} = \eta_2, \Phi_{-1} = \eta_1, \Phi_0 = \eta_0, \Psi_{-3} = \beta_3, \Psi_{-2} = \beta_2, \Psi_{-1} = \beta_1, \Psi_0 = \beta_0, \Gamma_{-3} = \zeta_3, \Gamma_{-2} = \zeta_2, \Gamma_{-1} = \zeta_1$ *and* $\Gamma_0 = \zeta_0$.

*Proof.* We can use the same steps used to prove Theorem 5. □

**Lemma 6.2.** *System (7) has aperiodic solution of period six iff* $\zeta_1\eta_2\beta_3 = \eta_1\beta_2\zeta_3 = \zeta_0\eta_1\beta_2 = \eta_0\beta_1\zeta_2 = \beta_1\zeta_2\eta_3 = \beta_0\zeta_1\eta_2 = -2$ *and* $\{\Phi_n, \Psi_n, \Gamma_n\}_{n=-3}^{\infty}$ *will take the form*

$$\begin{aligned}
\Phi_n &= \left\{\eta_3, \eta_2, \eta_1, \eta_0, \frac{\eta_2\beta_3}{\beta_0}, \frac{\zeta_0\eta_1}{\zeta_3}, \eta_3, \eta_2, \eta_1, \eta_0, \frac{\eta_2\beta_3}{\beta_0}, \frac{\zeta_0\eta_1}{\zeta_3}, \ldots\right\},\\
\Psi_n &= \left\{\beta_3, \beta_2, \beta_1, \beta_0, \frac{\beta_2\zeta_3}{\zeta_0}, \frac{\eta_0\beta_1}{\eta_3}, \beta_3, \beta_2, \beta_1, \beta_0, \frac{\beta_2\zeta_3}{\zeta_0}, \frac{\eta_0\beta_1}{\eta_3}, \ldots\right\},\\
\Gamma_n &= \left\{\zeta_3, \zeta_2, \zeta_1, \zeta_0, \frac{\zeta_2\eta_3}{\eta_0}, \frac{\beta_0\zeta_1}{\beta_3}, \zeta_3, \zeta_2, \zeta_1, \zeta_0, \frac{\zeta_2\eta_3}{\eta_0}, \frac{\beta_0\zeta_1}{\beta_3}, \ldots\right\}.
\end{aligned}$$

## 7 Fifth case

In this section, we investigate the solutions of the following difference equations system

$$\begin{aligned}
\Phi_{n+1} &= \frac{\Phi_{n-2}\Psi_{n-3}}{\Psi_n\left(1+\Gamma_{n-1}\Phi_{n-2}\Psi_{n-3}\right)},\\
\Psi_{n+1} &= \frac{\Psi_{n-2}\Gamma_{n-3}}{\Gamma_n\left(1+\Phi_{n-1}\Psi_{n-2}\Gamma_{n-3}\right)},\\
\Gamma_{n+1} &= \frac{\Gamma_{n-2}\Phi_{n-3}}{\Phi_n(1-\Psi_{n-1}\Gamma_{n-2}\Phi_{n-3})}, \qquad n = 0, 1, 2, \ldots,
\end{aligned} \tag{8}$$

where the initial conditions $\Phi_{-3}$, $\Phi_{-2}, \Phi_{-1}, \Phi_0, \Psi_{-3}, \Psi_{-2}, \Psi_{-1}, \Psi_0, \Gamma_{-3}, \Gamma_{-2}, \Gamma_{-1}$ and $\Gamma_0$ are nonzero real numbers.

**Theorem 7.1.** *Assume $\{\Phi_n, \Psi_n, \Gamma_n\}_{n=-3}^{\infty}$ is a solution of system (8). Then for n=0,1,2,...,*

$$\Phi_{6n-3} = \eta_3 (1 + (n) \eta_0 \beta_1 \zeta_2), \qquad \Phi_{6n-2} = \frac{\eta_2 (1 + (n+1) \zeta_1 \eta_2 \beta_3)}{(1 + \zeta_1 \eta_2 \beta_3)},$$

$$\Phi_{6n-1} = \frac{\eta_1 (1 + (n+1) \zeta_0 \eta_1 \beta_2)}{(1 + \zeta_0 \eta_1 \beta_2)}, \qquad \Phi_{6n} = \eta_0 (1 + (n) \beta_1 \zeta_2 \eta_3),$$

$$\Phi_{6n+1} = \frac{\eta_2 \beta_3 (1 + (n) \beta_0 \zeta_1 \eta_2)}{\beta_0 (1 + \zeta_1 \eta_2 \beta_3)}, \qquad \Phi_{6n+2} = \frac{\zeta_0 \eta_1 (1 + (n+1) \eta_1 \beta_2 \zeta_3)}{\zeta_3 (1 + \zeta_0 \eta_1 \beta_2)},$$

$$\Psi_{6n-3} = \frac{\beta_3 (1 - \beta_0 \zeta_1 \eta_2)(1 + \zeta_1 \eta_2 \beta_3)}{(1 + (n) \zeta_1 \eta_2 \beta_3)(1 + (n-1) \beta_0 \zeta_1 \eta_2)(1 + (n+1) \zeta_1 \eta_2 \beta_3)},$$

$$\Psi_{6n-2} = \frac{\beta_2 (1 + \zeta_0 \eta_1 \beta_2)}{(1 + (n) \eta_1 \beta_2 \zeta_3)(1 + (n) \zeta_0 \eta_1 \beta_2)(1 + (n+1) \zeta_0 \eta_1 \beta_2)},$$

$$\Psi_{6n-1} = \frac{\beta_1 (1 - \beta_1 \zeta_2 \eta_3)}{(1 + (n-1) \beta_1 \zeta_2 \eta_3)(1 + (n) \eta_0 \beta_1 \zeta_2)(1 + (n) \beta_1 \zeta_2 \eta_3)},$$

$$\Psi_{6n} = \frac{\beta_0 (1 - \beta_0 \zeta_1 \eta_2)(1 + \zeta_1 \eta_2 \beta_3)}{(1 + (n-1) \beta_0 \zeta_1 \eta_2)(1 + (n+1) \zeta_1 \eta_2 \beta_3)(1 + (n) \beta_0 \zeta_1 \eta_2)},$$

$$\Psi_{6n+1} = \frac{\beta_2 \zeta_3 (1 + \zeta_0 \eta_1 \beta_2)}{\zeta_0 (1 + (n) \eta_1 \beta_2 \zeta_3)(1 + (n+1) \zeta_0 \eta_1 \beta_2)(1 + (n+1) \eta_1 \beta_2 \zeta_3)},$$

$$\Psi_{6n+2} = \frac{\eta_0 \beta_1 (1 - \beta_1 \zeta_2 \eta_3)}{\eta_3 (1 + (n) \eta_0 \beta_1 \zeta_2)(1 + (n) \beta_1 \zeta_2 \eta_3)(1 + (n+1) \eta_0 \beta_1 \zeta_2)},$$

$$\Gamma_{6n-3} = \zeta_3 (1 + (n) \zeta_0 \eta_1 \beta_2), \qquad \Gamma_{6n-2} = \frac{\zeta_2 (1 + (n-1) \beta_1 \zeta_2 \eta_3)}{(1 - \beta_1 \zeta_2 \eta_3)},$$

$$\Gamma_{6n-1} = \frac{\zeta_1 (1 + (n-1) \beta_0 \zeta_1 \eta_2)}{(1 - \beta_0 \zeta_1 \eta_2)}, \qquad \Gamma_{6n} = \zeta_0 (1 + (n) \eta_1 \beta_2 \zeta_3),$$

$$\Gamma_{6n+1} = \frac{\zeta_2 \eta_3 (1 + (n) \eta_0 \beta_1 \zeta_2)}{\eta_0 (1 - \beta_1 \zeta_2 \eta_3)}, \qquad \Gamma_{6n+2} = \frac{\beta_0 \zeta_1 (1 + (n+1) \zeta_1 \eta_2 \beta_3)}{\beta_3 (1 - \beta_0 \zeta_1 \eta_2)},$$

*where* $\Phi_{-3} = \eta_3, \Phi_{-2} = \eta_2, \Phi_{-1} = \eta_1, \Phi_0 = \eta_0, \Psi_{-3} = \beta_3, \Psi_{-2} = \beta_2, \Psi_{-1} = \beta_1, \Psi_0 = \beta_0, \Gamma_{-3} = \zeta_3, \Gamma_{-2} = \zeta_2, \Gamma_{-1} = \zeta_1$ *and* $\Gamma_0 = \zeta_0$.

*Proof.* For $n = 0$ the result holds. Now suppose that $n > 0$ and our assumption holds for $n - 1$, that is;

$$\Phi_{6n-9} = \eta_3 (1 + (n-1) \eta_0 \beta_1 \zeta_2), \qquad \Phi_{6n-8} = \frac{\eta_2 (1 + (n) \zeta_1 \eta_2 \beta_3)}{(1 + \zeta_1 \eta_2 \beta_3)},$$

$$\Phi_{6n-7} = \frac{\eta_1 (1 + (n) \zeta_0 \eta_1 \beta_2)}{(1 + \zeta_0 \eta_1 \beta_2)}, \qquad \Phi_{6n-6} = \eta_0 (1 + (n-1) \beta_1 \zeta_2 \eta_3),$$

$$\Phi_{6n-5}=\frac{\eta_2\beta_3\left(1+(n-1)\,\beta_0\zeta_1\eta_2\right)}{\beta_0\left(1+\zeta_1\eta_2\beta_3\right)},\qquad \Phi_{6n-4}=\frac{\zeta_0\eta_1\left(1+(n)\,\eta_1\beta_2\zeta_3\right)}{\zeta_3\left(1+\zeta_0\eta_1\beta_2\right)},$$

$$\Psi_{6n-9}=\frac{\beta_3\left(1-\beta_0\zeta_1\eta_2\right)\left(1+\zeta_1\eta_2\beta_3\right)}{\left(1+(n-1)\,\zeta_1\eta_2\beta_3\right)\left(1+(n-2)\,\beta_0\zeta_1\eta_2\right)\left(1+(n)\,\zeta_1\eta_2\beta_3\right)},$$

$$\Psi_{6n-8}=\frac{\beta_2\left(1+\zeta_0\eta_1\beta_2\right)}{\left(1+(n-1)\,\eta_1\beta_2\zeta_3\right)\left(1+(n-1)\,\zeta_0\eta_1\beta_2\right)\left(1+(n)\,\zeta_0\eta_1\beta_2\right)},$$

$$\Psi_{6n-7}=\frac{\beta_1\left(1-\beta_1\zeta_2\eta_3\right)}{\left(1+(n-2)\,\beta_1\zeta_2\eta_3\right)\left(1+(n-1)\,\eta_0\beta_1\zeta_2\right)\left(1+(n-1)\,\beta_1\zeta_2\eta_3\right)},$$

$$\Psi_{6n-6}=\frac{\beta_0\left(1-\beta_0\zeta_1\eta_2\right)\left(1+\zeta_1\eta_2\beta_3\right)}{\left(1+(n-2)\,\beta_0\zeta_1\eta_2\right)\left(1+(n)\,\zeta_1\eta_2\beta_3\right)\left(1+(n-1)\,\beta_0\zeta_1\eta_2\right)},$$

$$\Psi_{6n-5}=\frac{\beta_2\zeta_3\left(1+\zeta_0\eta_1\beta_2\right)}{\zeta_0\left(1+(n-1)\,\eta_1\beta_2\zeta_3\right)\left(1+(n)\,\zeta_0\eta_1\beta_2\right)\left(1+(n)\,\eta_1\beta_2\zeta_3\right)},$$

$$\Psi_{6n-4}=\frac{\eta_0\beta_1\left(1-\beta_1\zeta_2\eta_3\right)}{\eta_3\left(1+(n-1)\,\eta_0\beta_1\zeta_2\right)\left(1+(n-1)\,\beta_1\zeta_2\eta_3\right)\left(1+(n)\,\eta_0\beta_1\zeta_2\right)},$$

$$\Gamma_{6n-9}=\zeta_3\left(1+(n-1)\,\zeta_0\eta_1\beta_2\right),\qquad \Gamma_{6n-8}=\frac{\zeta_2\left(1+(n-2)\,\beta_1\zeta_2\eta_3\right)}{\left(1-\beta_1\zeta_2\eta_3\right)},$$

$$\Gamma_{6n-7}=\frac{\zeta_1\left(1+(n-2)\,\beta_0\zeta_1\eta_2\right)}{\left(1-\beta_0\zeta_1\eta_2\right)},\qquad \Gamma_{6n-6}=\zeta_0\left(1+(n-1)\,\eta_1\beta_2\zeta_3\right),$$

$$\Gamma_{6n-5}=\frac{\zeta_2\eta_3\left(1+(n-1)\,\eta_0\beta_1\zeta_2\right)}{\eta_0\left(1-\beta_1\zeta_2\eta_3\right)},\qquad \Gamma_{6n-4}=\frac{\beta_0\zeta_1\left(1+(n)\,\zeta_1\eta_2\beta_3\right)}{\beta_3\left(1-\beta_0\zeta_1\eta_2\right)}.$$

Now, from system (8) we see that

$$\begin{aligned}\Phi_{6n-3}&=\frac{\Phi_{6n-6}\Psi_{6n-7}}{\Psi_{6n-4}\left(1+\Gamma_{6n-5}\Phi_{6n-6}\Psi_{6n-7}\right)}\\&=\frac{\left[\eta_0\left(1+(n-1)\,\beta_1\zeta_2\eta_3\right)\right]\left[\frac{\beta_1(1-\beta_1\zeta_2\eta_3)}{(1+(n-2)\beta_1\zeta_2\eta_3)(1+(n-1)\eta_0\beta_1\zeta_2)(1+(n-1)\beta_1\zeta_2\eta_3)}\right]}{\left[\frac{\eta_0\beta_1(1-\beta_1\zeta_2\eta_3)}{\eta_3(1+(n-1)\eta_0\beta_1\zeta_2)(1+(n-1)\beta_1\zeta_2\eta_3)(1+(n)\eta_0\beta_1\zeta_2)}\right]\left[\begin{array}{c}1+\frac{\zeta_2\eta_3(1+(n-1)\eta_0\beta_1\zeta_2)}{\eta_0(1-\beta_1\zeta_2\eta_3)}\eta_0\left(1+(n-1)\,\beta_1\zeta_2\eta_3\right)\\ \frac{\beta_1(1-\beta_1\zeta_2\eta_3)}{(1+(n-2)\beta_1\zeta_2\eta_3)(1+(n-1)\eta_0\beta_1\zeta_2)(1+(n-1)\beta_1\zeta_2\eta_3)}\end{array}\right]}\\&=\frac{\left[\frac{1}{(1+(n-2)\beta_1\zeta_2\eta_3)}\right]}{\left[\frac{1}{\eta_3(1+(n-1)\beta_1\zeta_2\eta_3)(1+(n)\eta_0\beta_1\zeta_2)}\right]\left[1+\frac{\beta_1\zeta_2\eta_3}{(1+(n-2)\beta_1\zeta_2\eta_3)}\right]}\\&=\frac{\left[\frac{\eta_3(1+(n-1)\beta_1\zeta_2\eta_3)(1+(n)\eta_0\beta_1\zeta_2)}{(1+(n-2)\beta_1\zeta_2\eta_3)}\right]}{\left[\frac{(1+(n-1)\beta_1\zeta_2\eta_3)}{(1+(n-2)\beta_1\zeta_2\eta_3)}\right]}\end{aligned}$$

Therefore, we can get

$$\Phi_{6n-3}=\eta_3\left(1+(n)\,\eta_0\beta_1\zeta_2\right).$$

Also, it follows from system (8) that

$$\Psi_{6n-3} = \frac{\Psi_{6n-6}\Gamma_{6n-7}}{\Gamma_{6n-4}\left(1+\Phi_{6n-5}\Psi_{6n-6}\Gamma_{6n-7}\right)}$$

$$= \frac{\left[\frac{\beta_0(1-\beta_0\zeta_1\eta_2)(1+\zeta_1\eta_2\beta_3)}{(1+(n-2)\beta_0\zeta_1\eta_2)(1+(n)\zeta_1\eta_2\beta_3)(1+(n-1)\beta_0\zeta_1\eta_2)}\right]\left[\frac{\zeta_1(1+(n-2)\beta_0\zeta_1\eta_2)}{(1-\beta_0\zeta_1\eta_2)}\right]}{\left[\frac{\beta_0\zeta_1(1+(n)\zeta_1\eta_2\beta_3)}{\beta_3(1-\beta_0\zeta_1\eta_2)}\right]\left[\begin{array}{c} 1+\frac{\eta_2\beta_3(1+(n-1)\beta_0\zeta_1\eta_2)}{\beta_0(1+\zeta_1\eta_2\beta_3)}\frac{\zeta_1(1+(n-2)\beta_0\zeta_1\eta_2)}{(1-\beta_0\zeta_1\eta_2)} \\ \frac{\beta_0(1-\beta_0\zeta_1\eta_2)(1+\zeta_1\eta_2\beta_3)}{(1+(n-2)\beta_0\zeta_1\eta_2)(1+(n)\zeta_1\eta_2\beta_3)(1+(n-1)\beta_0\zeta_1\eta_2)}\end{array}\right]}$$

$$= \frac{\left[\frac{(1+\zeta_1\eta_2\beta_3)}{(1+(n)\zeta_1\eta_2\beta_3)(1+(n-1)\beta_0\zeta_1\eta_2)}\right]}{\left[\frac{(1+(n)\zeta_1\eta_2\beta_3)}{\beta_3(1-\beta_0\zeta_1\eta_2)}\right]\left[1+\frac{\zeta_1\eta_2\beta_3}{(1+(n)\zeta_1\eta_2\beta_3)}\right]}$$

$$= \frac{\left[\frac{\beta_3(1-\beta_0\zeta_1\eta_2)(1+\zeta_1\eta_2\beta_3)}{(1+(n)\zeta_1\eta_2\beta_3)(1+(n)\zeta_1\eta_2\beta_3)(1+(n-1)\beta_0\zeta_1\eta_2)}\right]}{\left[\frac{(1+(n+1)\zeta_1\eta_2\beta_3)}{(1+(n)\zeta_1\eta_2\beta_3)}\right]}$$

Hence, we get

$$\Psi_{6n-3} = \frac{\beta_3\left(1-\beta_0\zeta_1\eta_2\right)\left(1+\zeta_1\eta_2\beta_3\right)}{\left(1+(n)\,\zeta_1\eta_2\beta_3\right)\left(1+(n-1)\,\beta_0\zeta_1\eta_2\right)\left(1+(n+1)\,\zeta_1\eta_2\beta_3\right)}.$$

Additionally, we can observe from system (8)

$$\Gamma_{6n-3} = \frac{\Gamma_{6n-6}\Phi_{6n-7}}{\Phi_{6n-4}(1-\Psi_{6n-5}\Gamma_{6n-6}\Phi_{6n-7})}$$

$$= \frac{\left[\zeta_0\left(1+(n-1)\,\eta_1\beta_2\zeta_3\right)\right]\left[\frac{\eta_1(1+(n)\zeta_0\eta_1\beta_2)}{(1+\zeta_0\eta_1\beta_2)}\right]}{\left[\frac{\zeta_0\eta_1(1+(n)\eta_1\beta_2\zeta_3)}{\zeta_3(1+\zeta_0\eta_1\beta_2)}\right]\left[\begin{array}{c} 1-\frac{\beta_2\zeta_3(1+\zeta_0\eta_1\beta_2)}{\zeta_0(1+(n-1)\eta_1\beta_2\zeta_3)(1+(n)\zeta_0\eta_1\beta_2)(1+(n)\eta_1\beta_2\zeta_3)} \\ \zeta_0\left(1+(n-1)\,\eta_1\beta_2\zeta_3\right)\frac{\eta_1(1+(n)\zeta_0\eta_1\beta_2)}{(1+\zeta_0\eta_1\beta_2)}\end{array}\right]}$$

$$= \frac{\left[\left(1+(n-1)\,\eta_1\beta_2\zeta_3\right)\right]\left[\left(1+(n)\,\zeta_0\eta_1\beta_2\right)\right]}{\left[\frac{(1+(n)\eta_1\beta_2\zeta_3)}{\zeta_3}\right]\left[1-\frac{\eta_1\beta_2\zeta_3}{(1+(n)\eta_1\beta_2\zeta_3)}\right]}$$

$$= \frac{\zeta_3\left(1+(n-1)\,\eta_1\beta_2\zeta_3\right)\left(1+(n)\,\zeta_0\eta_1\beta_2\right)}{\left(1+(n)\,\eta_1\beta_2\zeta_3\right)\left[\frac{(1+(n-1)\eta_1\beta_2\zeta_3)}{(1+(n)\eta_1\beta_2\zeta_3)}\right]}$$

Consequently, we obtain

$$\Gamma_{6n-3} = \zeta_3\left(1+(n)\,\zeta_0\eta_1\beta_2\right).$$

A similar technique can be used to prove the following cases. □

## 8 Sixth case

In this section, we obtain explicit formulations for the solutions of the following difference equations system

$$\begin{aligned}\Phi_{n+1} &= \frac{\Phi_{n-2}\Psi_{n-3}}{\Psi_n\left(1+\Gamma_{n-1}\Phi_{n-2}\Psi_{n-3}\right)},\\ \Psi_{n+1} &= \frac{\Psi_{n-2}\Gamma_{n-3}}{\Gamma_n\left(1-\Phi_{n-1}\Psi_{n-2}\Gamma_{n-3}\right)},\\ \Gamma_{n+1} &= \frac{\Gamma_{n-2}\Phi_{n-3}}{\Phi_n(1+\Psi_{n-1}\Gamma_{n-2}\Phi_{n-3})}, \qquad n=0,1,2,...,\end{aligned} \tag{9}$$

where the initial conditions $\Phi_{-3}$, $\Phi_{-2}, \Phi_{-1}, \Phi_0, \Psi_{-3}, \Psi_{-2}, \Psi_{-1}, \Psi_0, \Gamma_{-3}, \Gamma_{-2}, \Gamma_{-1}$ and $\Gamma_0$ are nonzero real numbers.

**Theorem 8.1.** *Assume $\{\Phi_n, \Psi_n, \Gamma_n\}_{n=-3}^{\infty}$ is a solution of system (9). Then for n=0,1,2,...,*

$$\Phi_{6n-3} = \frac{\eta_3\left(1-\eta_0\beta_1\zeta_2\right)\left(1+\beta_1\zeta_2\eta_3\right)}{\left(1+(n)\,\beta_1\zeta_2\eta_3\right)\left(1+(n-1)\,\eta_0\beta_1\zeta_2\right)\left(1+(n+1)\,\beta_1\zeta_2\eta_3\right)},$$

$$\Phi_{6n-2} = \frac{\eta_2\left(1+\beta_0\zeta_1\eta_2\right)}{\left(1+(n)\,\beta_0\zeta_1\eta_2\right)\left(1+(n)\,\zeta_1\eta_2\beta_3\right)\left(1+(n+1)\,\beta_0\zeta_1\eta_2\right)},$$

$$\Phi_{6n-1} = \frac{\eta_1\left(1-\eta_1\beta_2\zeta_3\right)}{\left(1+(n-1)\,\eta_1\beta_2\zeta_3\right)\left(1+(n)\,\zeta_0\eta_1\beta_2\right)\left(1+(n)\,\eta_1\beta_2\zeta_3\right)},$$

$$\Phi_{6n} = \frac{\eta_0\left(1-\eta_0\beta_1\zeta_2\right)\left(1+\beta_1\zeta_2\eta_3\right)}{\left(1+(n-1)\,\eta_0\beta_1\zeta_2\right)\left(1+(n+1)\,\beta_1\zeta_2\eta_3\right)\left(1+(n)\,\eta_0\beta_1\zeta_2\right)},$$

$$\Phi_{6n+1} = \frac{\eta_2\beta_3\left(1+\beta_0\zeta_1\eta_2\right)}{\beta_0\left(1+(n)\,\zeta_1\eta_2\beta_3\right)\left(1+(n+1)\,\beta_0\zeta_1\eta_2\right)\left(1+(n+1)\,\zeta_1\eta_2\beta_3\right)},$$

$$\Phi_{6n+2} = \frac{\zeta_0\eta_1\left(1-\eta_1\beta_2\zeta_3\right)}{\zeta_3\left(1+(n)\,\zeta_0\eta_1\beta_2\right)\left(1+(n)\,\eta_1\beta_2\zeta_3\right)\left(1+(n+1)\,\zeta_0\eta_1\beta_2\right)},$$

$$\Psi_{6n-3} = \beta_3\left(1+(n)\,\beta_0\zeta_1\eta_2\right), \qquad \Psi_{6n-2} = \frac{\beta_2\left(1+(n-1)\,\eta_1\beta_2\zeta_3\right)}{\left(1-\eta_1\beta_2\zeta_3\right)},$$

$$\Psi_{6n-1} = \frac{\beta_1\left(1+(n-1)\,\eta_0\beta_1\zeta_2\right)}{\left(1-\eta_0\beta_1\zeta_2\right)}, \qquad \Psi_{6n} = \beta_0\left(1+(n)\,\zeta_1\eta_2\beta_3\right),$$

$$\Psi_{6n+1} = \frac{\beta_2\zeta_3\left(1+(n)\,\zeta_0\eta_1\beta_2\right)}{\zeta_0\left(1-\eta_1\beta_2\zeta_3\right)}, \qquad \Psi_{6n+2} = \frac{\eta_0\beta_1\left(1+(n+1)\,\beta_1\zeta_2\eta_3\right)}{\eta_3\left(1-\eta_0\beta_1\zeta_2\right)},$$

$$\Gamma_{6n-3} = \zeta_3\left(1+(n)\,\zeta_0\eta_1\beta_2\right), \qquad \Gamma_{6n-2} = \frac{\zeta_2\left(1+(n+1)\,\beta_1\zeta_2\eta_3\right)}{\left(1+\beta_1\zeta_2\eta_3\right)},$$

$$\Gamma_{6n-1} = \frac{\zeta_1\left(1+(n+1)\,\beta_0\zeta_1\eta_2\right)}{\left(1+\beta_0\zeta_1\eta_2\right)}, \qquad \Gamma_{6n} = \zeta_0\left(1+(n)\,\eta_1\beta_2\zeta_3\right),$$

$$\Gamma_{6n+1}=\frac{\zeta_2\eta_3\left(1+(n)\,\eta_0\beta_1\zeta_2\right)}{\eta_0\left(1+\beta_1\zeta_2\eta_3\right)},\qquad \Gamma_{6n+2}=\frac{\beta_0\zeta_1\left(1+(n+1)\,\zeta_1\eta_2\beta_3\right)}{\beta_3\left(1+\beta_0\zeta_1\eta_2\right)},$$

*where* $\Phi_{-3}=\eta_3,\Phi_{-2}=\eta_2,\Phi_{-1}=\eta_1,\Phi_0=\eta_0,\Psi_{-3}=\beta_3,\Psi_{-2}=\beta_2,\Psi_{-1}=\beta_1,\Psi_0=\beta_0,\Gamma_{-3}=\zeta_3,\Gamma_{-2}=\zeta_2,\Gamma_{-1}=\zeta_1$ *and* $\Gamma_0=\zeta_0$.

*Proof.* For $n=0$ the result holds. Now suppose that $n>0$ and our assumption holds for $n-1$, that is;

$$\Phi_{6n-9}=\frac{\eta_3\left(1-\eta_0\beta_1\zeta_2\right)\left(1+\beta_1\zeta_2\eta_3\right)}{\left(1+(n-1)\,\beta_1\zeta_2\eta_3\right)\left(1+(n-2)\,\eta_0\beta_1\zeta_2\right)\left(1+(n)\,\beta_1\zeta_2\eta_3\right)},$$

$$\Phi_{6n-8}=\frac{\eta_2\left(1+\beta_0\zeta_1\eta_2\right)}{\left(1+(n-1)\,\beta_0\zeta_1\eta_2\right)\left(1+(n-1)\,\zeta_1\eta_2\beta_3\right)\left(1+(n)\,\beta_0\zeta_1\eta_2\right)},$$

$$\Phi_{6n-7}=\frac{\eta_1\left(1-\eta_1\beta_2\zeta_3\right)}{\left(1+(n-2)\,\eta_1\beta_2\zeta_3\right)\left(1+(n-1)\,\zeta_0\eta_1\beta_2\right)\left(1+(n-1)\,\eta_1\beta_2\zeta_3\right)},$$

$$\Phi_{6n-6}=\frac{\eta_0\left(1-\eta_0\beta_1\zeta_2\right)\left(1+\beta_1\zeta_2\eta_3\right)}{\left(1+(n-2)\,\eta_0\beta_1\zeta_2\right)\left(1+(n)\,\beta_1\zeta_2\eta_3\right)\left(1+(n-1)\,\eta_0\beta_1\zeta_2\right)},$$

$$\Phi_{6n-5}=\frac{\eta_2\beta_3\left(1+\beta_0\zeta_1\eta_2\right)}{\beta_0\left(1+(n-1)\,\zeta_1\eta_2\beta_3\right)\left(1+(n)\,\beta_0\zeta_1\eta_2\right)\left(1+(n)\,\zeta_1\eta_2\beta_3\right)},$$

$$\Phi_{6n-4}=\frac{\zeta_0\eta_1\left(1-\eta_1\beta_2\zeta_3\right)}{\zeta_3\left(1+(n-1)\,\zeta_0\eta_1\beta_2\right)\left(1+(n-1)\,\eta_1\beta_2\zeta_3\right)\left(1+(n)\,\zeta_0\eta_1\beta_2\right)},$$

$$\Psi_{6n-9}=\beta_3\left(1+(n-1)\,\beta_0\zeta_1\eta_2\right),\qquad \Psi_{6n-8}=\frac{\beta_2\left(1+(n-2)\,\eta_1\beta_2\zeta_3\right)}{\left(1-\eta_1\beta_2\zeta_3\right)},$$

$$\Psi_{6n-7}=\frac{\beta_1\left(1+(n-2)\,\eta_0\beta_1\zeta_2\right)}{\left(1-\eta_0\beta_1\zeta_2\right)},\qquad \Psi_{6n-6}=\beta_0\left(1+(n-1)\,\zeta_1\eta_2\beta_3\right),$$

$$\Psi_{6n-5}=\frac{\beta_2\zeta_3\left(1+(n-1)\,\zeta_0\eta_1\beta_2\right)}{\zeta_0\left(1-\eta_1\beta_2\zeta_3\right)},\qquad \Psi_{6n-4}=\frac{\eta_0\beta_1\left(1+(n)\,\beta_1\zeta_2\eta_3\right)}{\eta_3\left(1-\eta_0\beta_1\zeta_2\right)},$$

$$\Gamma_{6n-9}=\zeta_3\left(1+(n-1)\,\zeta_0\eta_1\beta_2\right),\qquad \Gamma_{6n-8}=\frac{\zeta_2\left(1+(n)\,\beta_1\zeta_2\eta_3\right)}{\left(1+\beta_1\zeta_2\eta_3\right)},$$

$$\Gamma_{6n-7}=\frac{\zeta_1\left(1+(n)\,\beta_0\zeta_1\eta_2\right)}{\left(1+\beta_0\zeta_1\eta_2\right)},\qquad \Gamma_{6n-6}=\zeta_0\left(1+(n-1)\,\eta_1\beta_2\zeta_3\right),$$

$$\Gamma_{6n-5}=\frac{\zeta_2\eta_3\left(1+(n-1)\,\eta_0\beta_1\zeta_2\right)}{\eta_0\left(1+\beta_1\zeta_2\eta_3\right)},\qquad \Gamma_{6n-4}=\frac{\beta_0\zeta_1\left(1+(n)\,\zeta_1\eta_2\beta_3\right)}{\beta_3\left(1+\beta_0\zeta_1\eta_2\right)},$$

Now, we find from system (9) that

$$\Phi_{6n-3}=\frac{\Phi_{6n-6}\Psi_{6n-7}}{\Psi_{6n-4}\left(1+\Gamma_{6n-5}\Phi_{6n-6}\Psi_{6n-7}\right)}$$

$$= \frac{\left[\frac{\eta_0(1-\eta_0\beta_1\zeta_2)(1+\beta_1\zeta_2\eta_3)}{(1+(n-2)\eta_0\beta_1\zeta_2)(1+(n)\beta_1\zeta_2\eta_3)(1+(n-1)\eta_0\beta_1\zeta_2)}\right]\left[\frac{\beta_1(1+(n-2)\eta_0\beta_1\zeta_2)}{(1-\eta_0\beta_1\zeta_2)}\right]}{\left[\frac{\eta_0\beta_1(1+(n)\beta_1\zeta_2\eta_3)}{\eta_3(1-\eta_0\beta_1\zeta_2)}\right]\left[\begin{array}{c} 1+\frac{\zeta_2\eta_3(1+(n-1)\eta_0\beta_1\zeta_2)}{\eta_0(1+\beta_1\zeta_2\eta_3)}\frac{\beta_1(1+(n-2)\eta_0\beta_1\zeta_2)}{(1-\eta_0\beta_1\zeta_2)} \\ \frac{\eta_0(1-\eta_0\beta_1\zeta_2)(1+\beta_1\zeta_2\eta_3)}{(1+(n-2)\eta_0\beta_1\zeta_2)(1+(n)\beta_1\zeta_2\eta_3)(1+(n-1)\eta_0\beta_1\zeta_2)} \end{array}\right]}$$

$$= \frac{\left[\frac{(1+\beta_1\zeta_2\eta_3)}{(1+(n)\beta_1\zeta_2\eta_3)(1+(n-1)\eta_0\beta_1\zeta_2)}\right]}{\left[\frac{(1+(n)\beta_1\zeta_2\eta_3)}{\eta_3(1-\eta_0\beta_1\zeta_2)}\right]\left[1+\frac{\beta_1\zeta_2\eta_3}{(1+(n)\beta_1\zeta_2\eta_3)}\right]}$$

$$= \frac{\left[\frac{\eta_3(1-\eta_0\beta_1\zeta_2)(1+\beta_1\zeta_2\eta_3)}{(1+(n)\beta_1\zeta_2\eta_3)(1+(n)\beta_1\zeta_2\eta_3)(1+(n-1)\eta_0\beta_1\zeta_2)}\right]}{\left[\frac{(1+(n+1)\beta_1\zeta_2\eta_3)}{(1+(n)\beta_1\zeta_2\eta_3)}\right]}$$

Then, we obtain

$$\Phi_{6n-3} = \frac{\eta_3\left(1-\eta_0\beta_1\zeta_2\right)\left(1+\beta_1\zeta_2\eta_3\right)}{\left(1+(n)\beta_1\zeta_2\eta_3\right)\left(1+(n-1)\eta_0\beta_1\zeta_2\right)\left(1+(n+1)\beta_1\zeta_2\eta_3\right)}.$$

In a similar manner, we employ the same previous procedure to investigate the solution

$$\Psi_{6n-3} = \frac{\Psi_{6n-6}\Gamma_{6n-7}}{\Gamma_{6n-4}\left(1-\Phi_{6n-5}\Psi_{6n-6}\Gamma_{6n-7}\right)}$$

$$= \frac{\left[\beta_0\left(1+(n-1)\zeta_1\eta_2\beta_3\right)\right]\left[\frac{\zeta_1(1+(n)\beta_0\zeta_1\eta_2)}{(1+\beta_0\zeta_1\eta_2)}\right]}{\left[\frac{\beta_0\zeta_1(1+(n)\zeta_1\eta_2\beta_3)}{\beta_3(1+\beta_0\zeta_1\eta_2)}\right]\left[\begin{array}{c} 1-\frac{\eta_2\beta_3(1+\beta_0\zeta_1\eta_2)}{\beta_0(1+(n-1)\zeta_1\eta_2\beta_3)(1+(n)\beta_0\zeta_1\eta_2)(1+(n)\zeta_1\eta_2\beta_3)} \\ \beta_0\left(1+(n-1)\zeta_1\eta_2\beta_3\right)\frac{\zeta_1(1+(n)\beta_0\zeta_1\eta_2)}{(1+\beta_0\zeta_1\eta_2)} \end{array}\right]}$$

$$= \frac{\left[\left(1+(n-1)\zeta_1\eta_2\beta_3\right)\right]\left[\left(1+(n)\beta_0\zeta_1\eta_2\right)\right]}{\left[\frac{(1+(n)\zeta_1\eta_2\beta_3)}{\beta_3}\right]\left[1-\frac{\zeta_1\eta_2\beta_3}{(1+(n)\zeta_1\eta_2\beta_3)}\right]}$$

$$= \frac{\beta_3\left(1+(n-1)\zeta_1\eta_2\beta_3\right)\left(1+(n)\beta_0\zeta_1\eta_2\right)}{\left(1+(n)\zeta_1\eta_2\beta_3\right)\left[\frac{(1+(n-1)\zeta_1\eta_2\beta_3)}{(1+(n)\zeta_1\eta_2\beta_3)}\right]}$$

Then, we obtain

$$\Psi_{6n-3} = \beta_3\left(1+(n)\beta_0\zeta_1\eta_2\right).$$

In a similar manner, we have from system (9) that

$$\Gamma_{6n-3} = \frac{\Gamma_{6n-6}\Phi_{6n-7}}{\Phi_{6n-4}(1+\Psi_{6n-5}\Gamma_{6n-6}\Phi_{6n-7})}$$

$$= \frac{\left[\zeta_0\left(1+(n-1)\eta_1\beta_2\zeta_3\right)\right]\left[\frac{\eta_1(1-\eta_1\beta_2\zeta_3)}{(1+(n-2)\eta_1\beta_2\zeta_3)(1+(n-1)\zeta_0\eta_1\beta_2)(1+(n-1)\eta_1\beta_2\zeta_3)}\right]}{\begin{array}{c}\left[\frac{\zeta_0\eta_1(1-\eta_1\beta_2\zeta_3)}{\zeta_3(1+(n-1)\zeta_0\eta_1\beta_2)(1+(n-1)\eta_1\beta_2\zeta_3)(1+(n)\zeta_0\eta_1\beta_2)}\right] \\ \left[\begin{array}{c} 1+\frac{\beta_2\zeta_3(1+(n-1)\zeta_0\eta_1\beta_2)}{\zeta_0(1-\eta_1\beta_2\zeta_3)}\zeta_0\left(1+(n-1)\eta_1\beta_2\zeta_3\right) \\ \frac{\eta_1(1-\eta_1\beta_2\zeta_3)}{(1+(n-2)\eta_1\beta_2\zeta_3)(1+(n-1)\zeta_0\eta_1\beta_2)(1+(n-1)\eta_1\beta_2\zeta_3)} \end{array}\right]\end{array}}$$

$$= \frac{\left[\frac{1}{(1+(n-2)\eta_1\beta_2\zeta_3)}\right]}{\left[\frac{1}{\zeta_3(1+(n-1)\eta_1\beta_2\zeta_3)(1+(n)\zeta_0\eta_1\beta_2)}\right]\left[1+\frac{\eta_1\beta_2\zeta_3}{(1+(n-2)\eta_1\beta_2\zeta_3)}\right]}$$

$$= \frac{\left[\frac{\zeta_3(1+(n-1)\eta_1\beta_2\zeta_3)(1+(n)\zeta_0\eta_1\beta_2)}{(1+(n-2)\eta_1\beta_2\zeta_3)}\right]}{\left[\frac{(1+(n-1)\eta_1\beta_2\zeta_3)}{(1+(n-2)\eta_1\beta_2\zeta_3)}\right]}$$

Consequently, we obtain

$$\Gamma_{6n-3} = \zeta_3\left(1+(n)\,\zeta_0\eta_1\beta_2\right).$$

Similarly, by using the same method, we can investigate other relations. □

## 9 Seventh case

In this section, we get explicit formulations for the solutions of the following system of difference equations

$$\begin{aligned}
\Phi_{n+1} &= \frac{\Phi_{n-2}\Psi_{n-3}}{\Psi_n\left(1-\Gamma_{n-1}\Phi_{n-2}\Psi_{n-3}\right)},\\
\Psi_{n+1} &= \frac{\Psi_{n-2}\Gamma_{n-3}}{\Gamma_n\left(1+\Phi_{n-1}\Psi_{n-2}\Gamma_{n-3}\right)},\\
\Gamma_{n+1} &= \frac{\Gamma_{n-2}\Phi_{n-3}}{\Phi_n\left(1+\Psi_{n-1}\Gamma_{n-2}\Phi_{n-3}\right)}, \qquad n=0,1,2,...,
\end{aligned} \tag{10}$$

where the initial conditions $\Phi_{-3}$, $\Phi_{-2}, \Phi_{-1}, \Phi_0, \Psi_{-3}, \Psi_{-2}, \Psi_{-1}, \Psi_0, \Gamma_{-3}, \Gamma_{-2}, \Gamma_{-1}$ and $\Gamma_0$ are nonzero real numbers.

**Theorem 9.1.** *Assume $\{\Phi_n, \Psi_n, \Gamma_n\}_{n=-3}^{\infty}$ is a solution of system (10). Then for n=0,1,2,...,*

$$\Phi_{6n-3} = \eta_3\left(1+(n)\,\eta_0\beta_1\zeta_2\right), \qquad \Phi_{6n-2} = \frac{\eta_2\left(1+(n-1)\,\zeta_1\eta_2\beta_3\right)}{\left(1-\zeta_1\eta_2\beta_3\right)},$$

$$\Phi_{6n-1} = \frac{\eta_1\left(1+(n-1)\,\zeta_0\eta_1\beta_2\right)}{\left(1-\zeta_0\eta_1\beta_2\right)}, \qquad \Phi_{6n} = \eta_0\left(1+(n)\,\beta_1\zeta_2\eta_3\right),$$

$$\Phi_{6n+1} = \frac{\eta_2\beta_3\left(1+(n)\,\beta_0\zeta_1\eta_2\right)}{\beta_0\left(1-\zeta_1\eta_2\beta_3\right)}, \qquad \Phi_{6n+2} = \frac{\zeta_0\eta_1\left(1+(n+1)\,\eta_1\beta_2\zeta_3\right)}{\zeta_3\left(1-\zeta_0\eta_1\beta_2\right)},$$

$$\Psi_{6n-3} = \beta_3\left(1+(n)\,\beta_0\zeta_1\eta_2\right), \qquad \Psi_{6n-2} = \frac{\beta_2\left(1+(n+1)\,\eta_1\beta_2\zeta_3\right)}{\left(1+\eta_1\beta_2\zeta_3\right)},$$

$$\Psi_{6n-1} = \frac{\beta_1\left(1+(n+1)\,\eta_0\beta_1\zeta_2\right)}{\left(1+\eta_0\beta_1\zeta_2\right)}, \qquad \Psi_{6n} = \beta_0\left(1+(n)\,\zeta_1\eta_2\beta_3\right),$$

$$\Psi_{6n+1} = \frac{\beta_2\zeta_3\left(1+(n)\,\zeta_0\eta_1\beta_2\right)}{\zeta_0\left(1+\eta_1\beta_2\zeta_3\right)}, \qquad \Psi_{6n+2} = \frac{\eta_0\beta_1\left(1+(n+1)\,\beta_1\zeta_2\eta_3\right)}{\eta_3\left(1+\eta_0\beta_1\zeta_2\right)},$$

$$\Gamma_{6n-3} = \frac{\zeta_3\left(1-\zeta_0\eta_1\beta_2\right)\left(1+\eta_1\beta_2\zeta_3\right)}{\left(1+(n)\,\eta_1\beta_2\zeta_3\right)\left(1+(n-1)\,\zeta_0\eta_1\beta_2\right)\left(1+(n+1)\,\eta_1\beta_2\zeta_3\right)},$$

$$\Gamma_{6n-2} = \frac{\zeta_2 (1 + \eta_0\beta_1\zeta_2)}{(1 + (n)\, \eta_0\beta_1\zeta_2)\, (1 + (n)\, \beta_1\zeta_2\eta_3)\, (1 + (n+1)\, \eta_0\beta_1\zeta_2)},$$

$$\Gamma_{6n-1} = \frac{\zeta_1 (1 - \zeta_1\eta_2\beta_3)}{(1 + (n-1)\, \zeta_1\eta_2\beta_3)\, (1 + (n)\, \beta_0\zeta_1\eta_2)\, (1 + (n)\, \zeta_1\eta_2\beta_3)},$$

$$\Gamma_{6n} = \frac{\zeta_0 (1 - \zeta_0\eta_1\beta_2)\, (1 + \eta_1\beta_2\zeta_3)}{(1 + (n-1)\, \zeta_0\eta_1\beta_2)\, (1 + (n+1)\, \eta_1\beta_2\zeta_3)\, (1 + (n)\, \zeta_0\eta_1\beta_2)},$$

$$\Gamma_{6n+1} = \frac{\zeta_2\eta_3 (1 + \eta_0\beta_1\zeta_2)}{\eta_0 (1 + (n)\, \beta_1\zeta_2\eta_3)\, (1 + (n+1)\, \eta_0\beta_1\zeta_2)\, (1 + (n+1)\, \beta_1\zeta_2\eta_3)},$$

$$\Gamma_{6n+2} = \frac{\beta_0\zeta_1 (1 - \zeta_1\eta_2\beta_3)}{\beta_3 (1 + (n)\, \beta_0\zeta_1\eta_2)\, (1 + (n)\, \zeta_1\eta_2\beta_3)\, (1 + (n+1)\, \beta_0\zeta_1\eta_2)},$$

*where* $\Phi_{-3} = \eta_3, \Phi_{-2} = \eta_2, \Phi_{-1} = \eta_1, \Phi_0 = \eta_0, \Psi_{-3} = \beta_3, \Psi_{-2} = \beta_2, \Psi_{-1} = \beta_1, \Psi_0 = \beta_0, \Gamma_{-3} = \zeta_3, \Gamma_{-2} = \zeta_2, \Gamma_{-1} = \zeta_1$ *and* $\Gamma_0 = \zeta_0$.

*Proof.* For $n = 0$ the result holds. Now suppose that $n > 0$ and our assumption holds for $n - 1$, that is;

$$\Phi_{6n-9} = \eta_3 (1 + (n-1)\, \eta_0\beta_1\zeta_2), \qquad \Phi_{6n-8} = \frac{\eta_2 (1 + (n-2)\, \zeta_1\eta_2\beta_3)}{(1 - \zeta_1\eta_2\beta_3)},$$

$$\Phi_{6n-7} = \frac{\eta_1 (1 + (n-2)\, \zeta_0\eta_1\beta_2)}{(1 - \zeta_0\eta_1\beta_2)}, \qquad \Phi_{6n-6} = \eta_0 (1 + (n-1)\, \beta_1\zeta_2\eta_3),$$

$$\Phi_{6n-5} = \frac{\eta_2\beta_3 (1 + (n-1)\, \beta_0\zeta_1\eta_2)}{\beta_0 (1 - \zeta_1\eta_2\beta_3)}, \qquad \Phi_{6n-4} = \frac{\zeta_0\eta_1 (1 + (n)\, \eta_1\beta_2\zeta_3)}{\zeta_3 (1 - \zeta_0\eta_1\beta_2)},$$

$$\Psi_{6n-9} = \beta_3 (1 + (n-1)\, \beta_0\zeta_1\eta_2), \qquad \Psi_{6n-8} = \frac{\beta_2 (1 + (n)\, \eta_1\beta_2\zeta_3)}{(1 + \eta_1\beta_2\zeta_3)},$$

$$\Psi_{6n-7} = \frac{\beta_1 (1 + (n)\, \eta_0\beta_1\zeta_2)}{(1 + \eta_0\beta_1\zeta_2)}, \qquad \Psi_{6n-6} = \beta_0 (1 + (n-1)\, \zeta_1\eta_2\beta_3),$$

$$\Psi_{6n-5} = \frac{\beta_2\zeta_3 (1 + (n-1)\, \zeta_0\eta_1\beta_2)}{\zeta_0 (1 + \eta_1\beta_2\zeta_3)}, \qquad \Psi_{6n-4} = \frac{\eta_0\beta_1 (1 + (n)\, \beta_1\zeta_2\eta_3)}{\eta_3 (1 + \eta_0\beta_1\zeta_2)},$$

$$\Gamma_{6n-9} = \frac{\zeta_3 (1 - \zeta_0\eta_1\beta_2)\, (1 + \eta_1\beta_2\zeta_3)}{(1 + (n-1)\, \eta_1\beta_2\zeta_3)\, (1 + (n-2)\, \zeta_0\eta_1\beta_2)\, (1 + (n)\, \eta_1\beta_2\zeta_3)},$$

$$\Gamma_{6n-8} = \frac{\zeta_2 (1 + \eta_0\beta_1\zeta_2)}{(1 + (n-1)\, \eta_0\beta_1\zeta_2)\, (1 + (n-1)\, \beta_1\zeta_2\eta_3)\, (1 + (n)\, \eta_0\beta_1\zeta_2)},$$

$$\Gamma_{6n-7} = \frac{\zeta_1 (1 - \zeta_1\eta_2\beta_3)}{(1 + (n-2)\, \zeta_1\eta_2\beta_3)\, (1 + (n-1)\, \beta_0\zeta_1\eta_2)\, (1 + (n-1)\, \zeta_1\eta_2\beta_3)},$$

$$\Gamma_{6n-6} = \frac{\zeta_0 (1 - \zeta_0\eta_1\beta_2)\, (1 + \eta_1\beta_2\zeta_3)}{(1 + (n-2)\, \zeta_0\eta_1\beta_2)\, (1 + (n)\, \eta_1\beta_2\zeta_3)\, (1 + (n-1)\, \zeta_0\eta_1\beta_2)},$$

$$\Gamma_{6n-5} = \frac{\zeta_2\eta_3\left(1+\eta_0\beta_1\zeta_2\right)}{\eta_0\left(1+(n-1)\beta_1\zeta_2\eta_3\right)\left(1+(n)\eta_0\beta_1\zeta_2\right)\left(1+(n)\beta_1\zeta_2\eta_3\right)},$$

$$\Gamma_{6n-4} = \frac{\beta_0\zeta_1\left(1-\zeta_1\eta_2\beta_3\right)}{\beta_3\left(1+(n-1)\beta_0\zeta_1\eta_2\right)\left(1+(n-1)\zeta_1\eta_2\beta_3\right)\left(1+(n)\beta_0\zeta_1\eta_2\right)},$$

Now, we prove that the results are holds for $n$. From system (10), it follows that

$$\begin{aligned}\Phi_{6n-3} &= \frac{\Phi_{6n-6}\Psi_{6n-7}}{\Psi_{6n-4}\left(1-\Gamma_{6n-5}\Phi_{6n-6}\Psi_{6n-7}\right)}\\ &= \frac{\left[\eta_0\left(1+(n-1)\beta_1\zeta_2\eta_3\right)\right]\left[\frac{\beta_1(1+(n)\eta_0\beta_1\zeta_2)}{(1+\eta_0\beta_1\zeta_2)}\right]}{\left[\frac{\eta_0\beta_1(1+(n)\beta_1\zeta_2\eta_3)}{\eta_3(1+\eta_0\beta_1\zeta_2)}\right]\left[\begin{array}{c}1-\frac{\zeta_2\eta_3(1+\eta_0\beta_1\zeta_2)}{\eta_0(1+(n-1)\beta_1\zeta_2\eta_3)(1+(n)\eta_0\beta_1\zeta_2)(1+(n)\beta_1\zeta_2\eta_3)}\\ \eta_0\left(1+(n-1)\beta_1\zeta_2\eta_3\right)\frac{\beta_1(1+(n)\eta_0\beta_1\zeta_2)}{(1+\eta_0\beta_1\zeta_2)}\end{array}\right]}\\ &= \frac{\left[\left(1+(n-1)\beta_1\zeta_2\eta_3\right)\right]\left[\left(1+(n)\eta_0\beta_1\zeta_2\right)\right]}{\left[\frac{(1+(n)\beta_1\zeta_2\eta_3)}{\eta_3}\right]\left[1-\frac{\beta_1\zeta_2\eta_3}{(1+(n)\beta_1\zeta_2\eta_3)}\right]}\\ &= \frac{\eta_3\left(1+(n-1)\beta_1\zeta_2\eta_3\right)\left(1+(n)\eta_0\beta_1\zeta_2\right)}{\left[\left(1+(n)\beta_1\zeta_2\eta_3\right)\right]\left[\frac{(1+(n-1)\beta_1\zeta_2\eta_3)}{(1+(n)\beta_1\zeta_2\eta_3)}\right]}\end{aligned}$$

So, we have

$$\Phi_{6n-3} = \eta_3\left(1+(n)\eta_0\beta_1\zeta_2\right).$$

Similarly, by using the same method, we can investigate the relations

$$\begin{aligned}\Psi_{6n-3} &= \frac{\Psi_{6n-6}\Gamma_{6n-7}}{\Gamma_{6n-4}\left(1+\Phi_{6n-5}\Psi_{6n-6}\Gamma_{6n-7}\right)}\\ &= \frac{\left[\beta_0\left(1+(n-1)\zeta_1\eta_2\beta_3\right)\right]\left[\frac{\zeta_1(1-\zeta_1\eta_2\beta_3)}{(1+(n-2)\zeta_1\eta_2\beta_3)(1+(n-1)\beta_0\zeta_1\eta_2)(1+(n-1)\zeta_1\eta_2\beta_3)}\right]}{\left[\frac{\beta_0\zeta_1(1-\zeta_1\eta_2\beta_3)}{\beta_3(1+(n-1)\beta_0\zeta_1\eta_2)(1+(n-1)\zeta_1\eta_2\beta_3)(1+(n)\beta_0\zeta_1\eta_2)}\right]\left[\begin{array}{c}1+\frac{\eta_2\beta_3(1+(n-1)\beta_0\zeta_1\eta_2)}{\beta_0(1-\zeta_1\eta_2\beta_3)}\beta_0\left(1+(n-1)\zeta_1\eta_2\beta_3\right)\\ \frac{\zeta_1(1-\zeta_1\eta_2\beta_3)}{(1+(n-2)\zeta_1\eta_2\beta_3)(1+(n-1)\beta_0\zeta_1\eta_2)(1+(n-1)\zeta_1\eta_2\beta_3)}\end{array}\right]}\\ &= \frac{\left[\frac{1}{(1+(n-2)\zeta_1\eta_2\beta_3)}\right]}{\left[\frac{1}{\beta_3(1+(n-1)\zeta_1\eta_2\beta_3)(1+(n)\beta_0\zeta_1\eta_2)}\right]\left[1+\frac{\zeta_1\eta_2\beta_3}{(1+(n-2)\zeta_1\eta_2\beta_3)}\right]}\\ &= \frac{\left[\frac{\beta_3(1+(n-1)\zeta_1\eta_2\beta_3)(1+(n)\beta_0\zeta_1\eta_2)}{(1+(n-2)\zeta_1\eta_2\beta_3)}\right]}{\left[\frac{(1+(n-1)\zeta_1\eta_2\beta_3)}{(1+(n-2)\zeta_1\eta_2\beta_3)}\right]}\end{aligned}$$

Hence, we obtain

$$\Psi_{6n-3} = \beta_3\left(1+(n)\beta_0\zeta_1\eta_2\right).$$

In a similar manner, we employ the same previous procedure to investigate the solution

$$\Gamma_{6n-3} = \frac{\Gamma_{6n-6}\Phi_{6n-7}}{\Phi_{6n-4}(1+\Psi_{6n-5}\Gamma_{6n-6}\Phi_{6n-7})}$$

$$= \frac{\left[\frac{\zeta_0(1-\zeta_0\eta_1\beta_2)(1+\eta_1\beta_2\zeta_3)}{(1+(n-2)\zeta_0\eta_1\beta_2)(1+(n)\eta_1\beta_2\zeta_3)(1+(n-1)\zeta_0\eta_1\beta_2)}\right]\left[\frac{\eta_1(1+(n-2)\zeta_0\eta_1\beta_2)}{(1-\zeta_0\eta_1\beta_2)}\right]}{\left[\frac{\zeta_0\eta_1(1+(n)\eta_1\beta_2\zeta_3)}{\zeta_3(1-\zeta_0\eta_1\beta_2)}\right]\left[1+\frac{\frac{\beta_2\zeta_3(1+(n-1)\zeta_0\eta_1\beta_2)}{\zeta_0(1+\eta_1\beta_2\zeta_3)}\frac{\eta_1(1+(n-2)\zeta_0\eta_1\beta_2)}{(1-\zeta_0\eta_1\beta_2)}}{\frac{\zeta_0(1-\zeta_0\eta_1\beta_2)(1+\eta_1\beta_2\zeta_3)}{(1+(n-2)\zeta_0\eta_1\beta_2)(1+(n)\eta_1\beta_2\zeta_3)(1+(n-1)\zeta_0\eta_1\beta_2)}}\right]}$$

$$= \frac{\left[\frac{(1+\eta_1\beta_2\zeta_3)}{(1+(n)\eta_1\beta_2\zeta_3)(1+(n-1)\zeta_0\eta_1\beta_2)}\right]}{\left[\frac{(1+(n)\eta_1\beta_2\zeta_3)}{\zeta_3(1-\zeta_0\eta_1\beta_2)}\right]\left[1+\frac{\eta_1\beta_2\zeta_3}{(1+(n)\eta_1\beta_2\zeta_3)}\right]}$$

$$= \frac{\left[\frac{\zeta_3(1-\zeta_0\eta_1\beta_2)(1+\eta_1\beta_2\zeta_3)}{(1+(n)\eta_1\beta_2\zeta_3)(1+(n)\eta_1\beta_2\zeta_3)(1+(n-1)\zeta_0\eta_1\beta_2)}\right]}{\left[\frac{(1+(n+1)\eta_1\beta_2\zeta_3)}{(1+(n)\eta_1\beta_2\zeta_3)}\right]}$$

So, we can get

$$\Gamma_{6n-3} = \frac{\zeta_3\left(1-\zeta_0\eta_1\beta_2\right)\left(1+\eta_1\beta_2\zeta_3\right)}{\left(1+(n)\,\eta_1\beta_2\zeta_3\right)\left(1+(n-1)\,\zeta_0\eta_1\beta_2\right)\left(1+(n+1)\,\eta_1\beta_2\zeta_3\right)}.$$

Similarly to that, we can investigate other relations by using the same methodology. □

## 10 Numerical examples

In this section, we provide several numerical examples that reflect various types of system (1) solutions in order to complement our theoretical discussion as well as to illustrate our previous results.

**Example 1.** Figure 1 depicts the numerical solution of system (3) under the initial conditions $\Phi_{-3} = 5, \Phi_{-2} = 12, \Phi_{-1} = 7, \Phi_0 = 10, \Psi_{-3} = 1.9, \Psi_{-2} = 7, \Psi_{-1} = 4, \Psi_0 = 5.6, \Gamma_{-3} = 0.9, \Gamma_{-2} = 0.2, \Gamma_{-1} = 1.8$ and $\Gamma_0 = 0.6$.

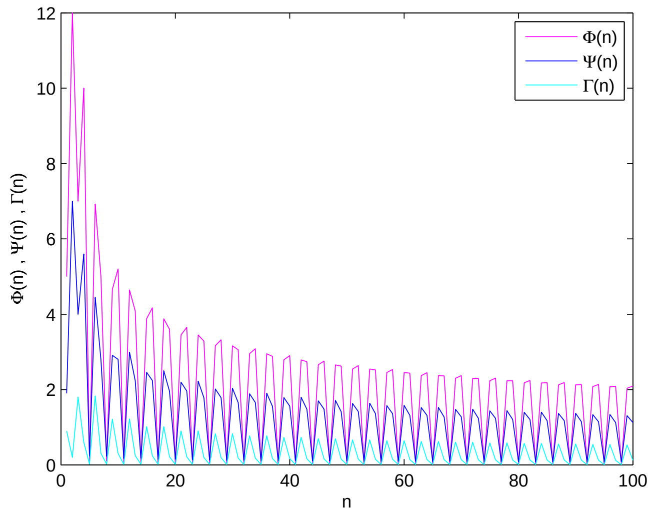


Figure 1: Plotting numerical solutions of system (3).

**Example 2.** Here, we demonstrate how system (5) can be solved numerically with the initial conditions are $\Phi_{-3} = 1.5, \Phi_{-2} = 5.12, \Phi_{-1} = 2.7, \Phi_0 = 3.1, \Psi_{-3} = 3.9, \Psi_{-2} = 7, \Psi_{-1} = 4.5, \Psi_0 = 8.6, \Gamma_{-3} = 0.9, \Gamma_{-2} = 0.2, \Gamma_{-1} = 0.8$ and $\Gamma_0 = 0.6$.

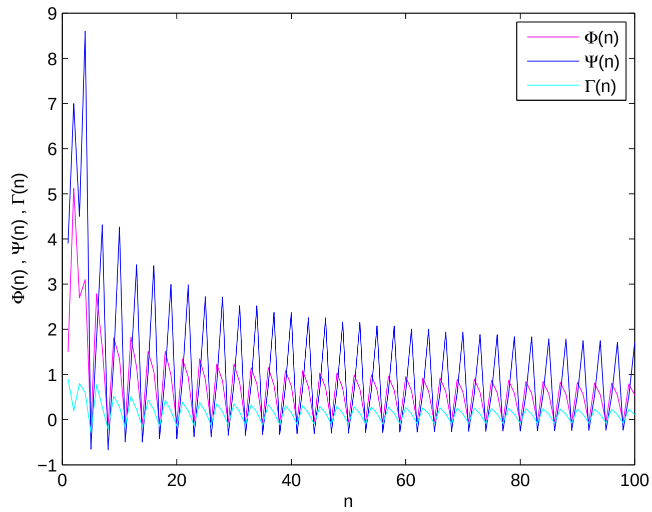


Figure 2: Plotting numerical solutions of system (5).

**Example 3.** Figure 3 indicating that the solution of system (6) is periodic when the initial values are $\Phi_{-3} = 4.5, \Phi_{-2} = 1.2, \Phi_{-1} = 3.4, \Phi_0 = 6.5, \Psi_{-3} = 2.5, \Psi_{-2} = -2.7, \Psi_{-1} = 1.5, \Psi_0 = -5.6, \Gamma_{-3} = -3.9, \Gamma_{-2} = 2.2, \Gamma_{-1} = -1.8$ and $\Gamma_0 = 5.6$.

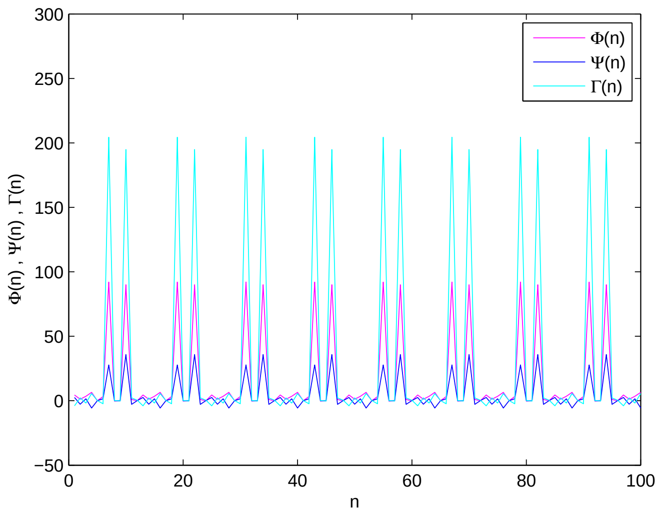


Figure 3: Plotting numerical solutions of system (6).

**Example 4.** For system (7) the initial conditions are set as follows: $\Phi_{-3} = -2.5, \Phi_{-2} = 8.2, \Phi_{-1} = -1.4, \Phi_0 = 4.5, \Psi_{-3} = -2.5, \Psi_{-2} = 4.7, \Psi_{-1} = -1.5, \Psi_0 = 1.6, \Gamma_{-3} = 5, \Gamma_{-2} = -2.2, \Gamma_{-1} = 3.8$ and $\Gamma_0 = -1.6$, we observe that the solutions are periodic with period 12, the results shown in Figure 4.

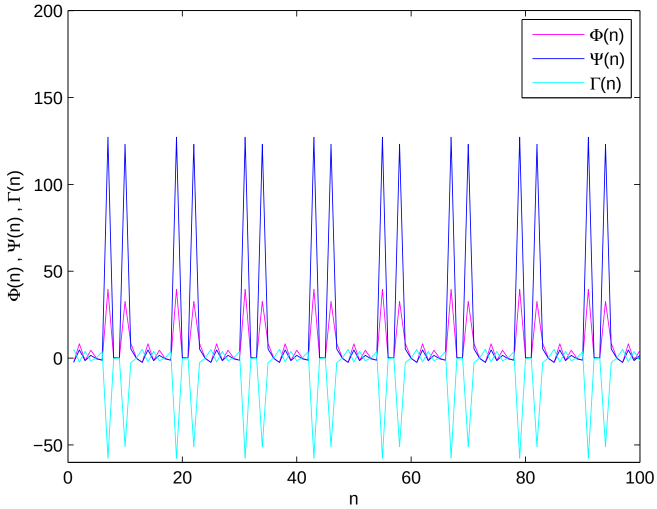


Figure 4: Plotting numerical solutions of system (7).

**Example 5.** To illustrate the numerical solution of system (8), we use the initial conditions as follows: $\Phi_{-3} = 1.5, \Phi_{-2} = 5.5, \Phi_{-1} = 2.7, \Phi_0 = -0.5, \Psi_{-3} = 4.8, \Psi_{-2} = 0.5, \Psi_{-1} = 2.4, \Psi_0 = 0.9, \Gamma_{-3} = 3.8, \Gamma_{-2} = -1.2, \Gamma_{-1} = 0.8$ and $\Gamma_0 = 1.6$, the results shown in Figure 5.

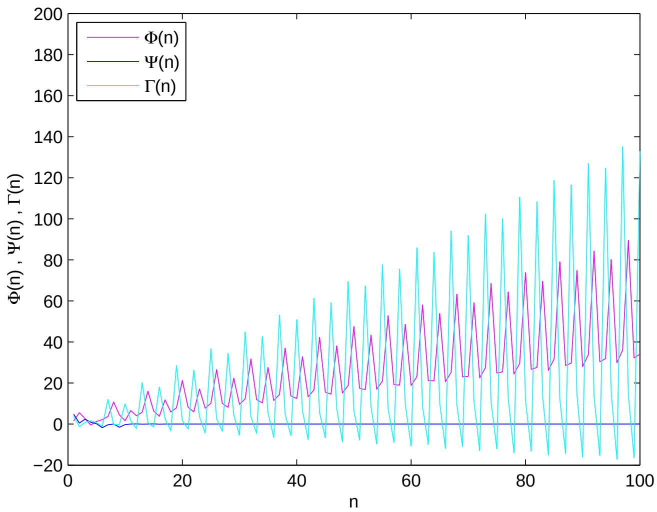


Figure 5: Plotting numerical solutions of system (8).

**Example 6.** In numerical simulation, we assumed that for system (9) the initial values are $\Phi_{-3} = -1.8, \Phi_{-2} = 4.1, \Phi_{-1} = 2.5, \Phi_0 = 7, \Psi_{-3} = -1.9, \Psi_{-2} = 1.7, \Psi_{-1} = -3.1, \Psi_0 = 2.6, \Gamma_{-3} = 3.9, \Gamma_{-2} = 2.2, \Gamma_{-1} = 0.8$ and $\Gamma_0 = 1.6$. Then the solution appear in Figure 6.

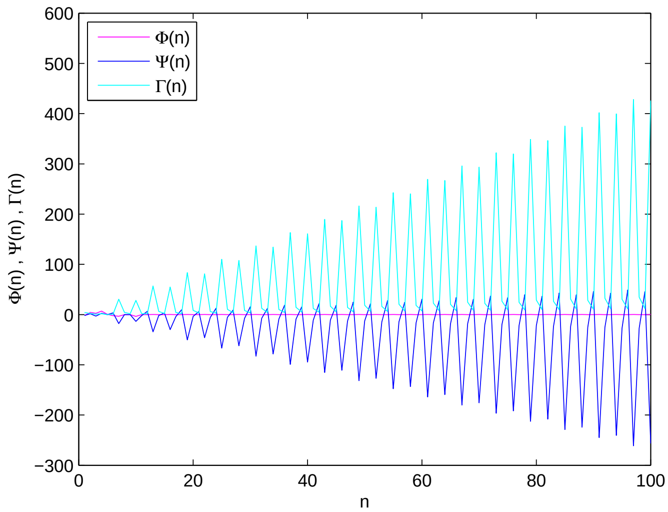


Figure 6: Plotting numerical solutions of system (9).

**Example 7.** Numerically when the initial values are $\Phi_{-3} = 5.1, \Phi_{-2} = 1.9, \Phi_{-1} = 6.8, \Phi_0 = 0.2, \Psi_{-3} = 2.1, \Psi_{-2} = 0.2, \Psi_{-1} = 3.9, \Psi_0 = 1.6, \Gamma_{-3} = 1.9, \Gamma_{-2} = 0.2, \Gamma_{-1} = 1.8$ and $\Gamma_0 = 0.6$. Figure 7 shows the results of system (10).

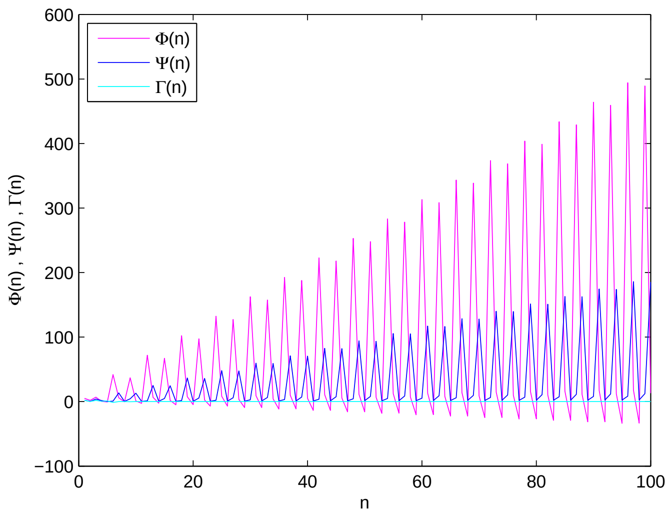


Figure 7: Plotting numerical solutions of system (10).

## 11 Conclusions

Analyzing the dynamics of systems in higher dimensions is a very fascinating mathematical issue. In this article, we have found the expressions of solutions in some special cases as applications of rational difference systems of order four. First, in case 1, we have investigated the solutions qualitative behavior is explored of system 2, such as local stability. Also, we have obtained the general form of the solution of system 3 and proven that the solution is bounded. After that, in cases 2,3,4,5.6 and 7, we have obtained expressions of solutions of six special cases of the studied system 5,6,7,8,9 and 10. In systems 6 and 7, we discovered that the system's solutions are periodic. Finally, to support our results, we provided some illustrative numerical examples.

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### Authors' contributions

The author declares that the study was realized in collaboration with the same responsibility. Both authors read and proved the final manuscript.

### Conflict of interest

The authors declare that they have no conflicts of interest.

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