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A study on the sum of the squares of generalized balancing numbers: the sum formula $\sum_{k=0}^{n} x^k W_{mk+j}^2$

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Abstract. In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} x^k W_{mk+j}^2$ for generalized balancing numbers are presented. As special cases, we give sum formulas of balancing, modified Lucas-balancing and Lucas-balancing numbers.

Keywords: Balancing numbers, modified Lucas-balancing numbers, Lucas-balancing numbers, sum formulas.

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1 Introduction

A generalized balancing sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$W_n = 6W_{n-1} - W_{n-2} \tag{1.1}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 6W_{-(n-1)} - W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

The Binet formula of generalized balancing numbers can be written as

$$W_n = \frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n,$$

where α and β are the roots of the quadratic equation $x^2 - 6x + 1 = 0$. Moreover

$$\begin{aligned} \alpha &= 3 + 2\sqrt{2}, \\ \beta &= 3 - 2\sqrt{2}. \end{aligned}$$

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Note that

$$\begin{aligned} \alpha+\beta &= 6, \\ \alpha\beta &= 1, \\ \alpha-\beta &= 4\sqrt{2}. \end{aligned}$$

Now, one defines three special cases of the sequence $\{W_n\}$. Balancing sequence $\{B_n\}_{n\geq 0}$, modified Lucas-balancing sequence $\{H_n\}_{n\geq 0}$ and Lucas-balancing sequence $\{C_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$B_n = 6B_{n-1} - B_{n-2}, \quad B_0 = 0, B_1 = 1, \tag{1.2}$$
$$H_n = 6H_{n-1} - H_{n-2}, \quad H_0 = 2, H_1 = 6, \tag{1.3}$$
$$C_n = 6C_{n-1} - C_{n-2}, \quad C_0 = 1, C_1 = 3. \tag{1.4}$$

The sequences $\{B_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} B_{-n} &= 6B_{-(n-1)} - B_{-(n-2)}, \\ H_{-n} &= 6H_{-(n-1)} - H_{-(n-2)}, \\ C_{-n} &= 6C_{-(n-1)} - C_{-(n-2)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.2)-(1.4) hold for all integer n. For more information on generalized balancing numbers, see Soykan [29].

In [1], Behera and Panda defined balancing numbers n as solutions of the diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$$

for some natural number r, called the balancer corresponding to n. The nth balancing number is denoted by B_n. Moreover, $C_n = \sqrt{8B_n^2 + 1}$ is called the nth Lucas-balancing number (see [16]). In fact, B_n and C_n satisfy the second order linear recurrence relations (1.2) and (1.4) respectively. $(B_n)_{n\geq 0}$ is the sequence $A001109$ in the OEIS [27], whereas $(B_n)_{n\geq 0}$ is the id-number $A001541$ in OEIS. Balancing and Lucas-balancing sequences have been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [1–4, 9–26].

2 The Sum Formula $\sum_{k=0}^{n} x^k W_{mk+j}^2$

The following theorem presents sum formulas of generalized balancing numbers.

Theorem 2.1. *Let x be a real (or complex) number. For all integers m and j, for generalized balancing numbers (the case $r = 6,\ s = -1$), the following sum formulas hold:*

(a) *If $(1 + (-s)^{2m}x^2 - xH_{2m})((-s)^m x - 1) \neq 0$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_1}{32(1 + x^2 - xH_{2m})(x-1)}, \tag{2.1}$$

where

$\Omega_1 = 32(x-1)(x-H_{2m})x^{n+1}W_{mn+j}^2 + 32(x-1)x^{n+1}W_{mn-m+j}^2 + 32(x-1)W_j^2 - 32(x-1)xW_{j-m}^2 + 2(W_1^2 + W_0^2 - 6W_1W_0)(x^n - 1)(H_{2m} - 2)x.$

(b) *If* $(1+(-s)^{2m}x^2-xH_{2m})((-s)^m x-1)=u(x-a)(x-b)(x-c)=0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$*, i.e.,* $x=a$ *or* $x=b$ *or* $x=c$*, then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^2=\frac{\Omega_2}{32(3x^2-2(H_{2m}+1)x+H_{2m}+1)},$$

where

$\Omega_2=32((x-H_{2m})x^{n+1}+(x-1)((n+2)x-(n+1)H_{2m})x^nW_{mn+j}^2+32((n+2)x-(n+1))x^n)W_{mn-m+j}^2+32W_j^2-32(2x-1)W_{j-m}^2+2(W_1^2+W_0^2-6W_1W_0)(x^n(n+1)-1)(H_{2m}-2).$

(c) *If* $(1+(-s)^{2m}x^2-xH_{2m})((-s)^m x-1)=u(x-a)^2(x-c)=0$ *for some* $u,a,c\in\mathbb{C}$ *with* $u\neq 0$*,* $a\neq c$*, when* $x=c$*, then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^2=\frac{\Omega_3}{32(3x^2-2(H_{2m}+1)x+H_{2m}+1)},$$

where

$\Omega_3=32((x-H_{2m})x^{n+1}+(x-1)((n+2)x-(n+1)H_{2m})x^nW_{mn+j}^2+32((n+2)x-(n+1))x^n)W_{mn-m+j}^2+32W_j^2-32(2x-1)W_{j-m}^2+2(W_1^2+W_0^2-6W_1W_0)(x^n(n+1)-1)(H_{2m}-2),$

and when $x=a$*, then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^2=\frac{\Omega_4}{64(3x-1-H_{2m})},$$

where

$\Omega_4=32((n+3)(n+2)x^2-x(n+2)(n+1)(H_{2m}+1)+n(n+1)H_{2m})x^{n-1}W_{mn+j}^2+32(n+1)((2+n)x^n-nx^{n-1})W_{mn-m+j}^2-64W_{j-m}^2+2n(n+1)(W_1^2+W_0^2-6W_1W_0)(H_{2m}-2)x^{n-1}.$

(d) *If* $(1+(-s)^{2m}x^2-xH_{2m})((-s)^m x-1)=u(x-a)^3=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$*, i.e.,* $x=a$*, then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^2=\frac{\Omega_5}{192},$$

where

$\Omega_5=32(n+1)((n+3)(n+2)x^2-n(n+2)(H_{2m}+1)x+n(n-1)H_{2m})x^{n-2}W_{mn+j}^2+32n(n+1)((n+2)x+1-n)x^{n-2}W_{mn-m+j}^2+2(n-1)n(n+1)(H_{2m}-2)(W_1^2+W_0^2-6W_1W_0)x^{n-2}.$

Proof. Take $r=6,\ s=-1$ in Soykan [28], Theorem 2.1.. □

Note that (2.1) can be written in the following form

$$\sum_{k=1}^{n} x^k W_{mk+j}^2=\frac{\Omega_6}{32(1+x^2-xH_{2m})(x-1)},$$

where

$\Omega_6=32(x-1)(x-H_{2m})x^{n+1}W_{mn+j}^2+32((-s)^m x-1)x^{n+1}W_{mn-m+j}^2-32(x-1)(x-H_{2m})x\,W_j^2-32(x-1)W_{j-m}^2x+2(W_1^2+W_0^2-6W_1W_0)(x^n-1)(H_{2m}-2)x.$

As special cases of m and j in the last theorem, one obtains the following proposition.

Proposition 2.2. *For generalized balancing numbers (the case $r=6, s=-1$) one has the following sum formulas for $n \geq 0$:*

(a) $(m=1, j=0)$

If $(x-1)\left(x^2-34x+1\right) \neq 0$, *i.e.,* $x \neq 1$, $x \neq 17-12\sqrt{2}$, $x \neq 17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Delta}{(x-1)\left(x^2-34x+1\right)},$$

where

$\Delta = (x-1)(x-34)x^{n+1}W_n^2 + (x-1)x^{n+1}W_{n-1}^2 + (x-1)W_0^2 - (x-1)x(W_1-6W_0)^2 + 2(W_1^2+W_0^2-6W_1W_0)(x^n-1)x,$

and

if $(x-1)\left(x^2-34x+1\right) = 0$, *i.e.,* $x=1$ *or* $x = 17-12\sqrt{2}$ *or* $x = 17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Psi}{(3x^2-70x+35)},$$

where

$\Psi = ((x-34)x^{n+1} + (x-1)((n+2)x - 34(n+1))x^n W_n^2 + ((n+2)x-(n+1))x^n W_{n-1}^2 + W_0^2 - (2x-1)(W_1-6W_0)^2 + 2(W_1^2+W_0^2-6W_1W_0)(x^n(n+1)-1)$

(b) $(m=2, j=0)$

If $(x-1)\left(x^2-1154x+1\right) \neq 0$, *i.e.,* $x \neq 1$, $x \neq 577-408\sqrt{2}$, $x \neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{2k}^2 = \frac{\Delta}{(x-1)\left(x^2-1154x+1\right)},$$

where

$\Delta = (x-1)(x-1154)x^{n+1}W_{2n}^2 + (x-1)x^{n+1}W_{2n-2}^2 + (x-1)W_0^2 - (x-1)x(6W_1-35W_0)^2 + 72(W_1^2+W_0^2-6W_1W_0)(x^n-1)x,$

and

if $(x-1)\left(x^2-1154x+1\right) = 0$, *i.e.,* $x=1$ *or* $x = 577-408\sqrt{2}$ *or* $x = 577+408\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k W_{2k}^2 = \frac{\Psi}{3\left(x^2-770x+385\right)},$$

where

$\Psi = ((x-1154)x^{n+1} + (x-1)((n+2)x - 1154(n+1))x^n)W_{2n}^2 + ((n+2)x-(n+1))x^n W_{2n-2}^2 + W_0^2 - (2x-1)(6W_1-35W_0)^2 + 72(W_1^2+W_0^2-6W_1W_0)(x^n(n+1)-1)$

(c) $(m=2, j=1)$

If $(x-1)\left(x^2-1154x+1\right) \neq 0$, *i.e.,* $x \neq 1$, $x \neq 577-408\sqrt{2}$, $x \neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{2k+1}^2 = \frac{\Delta}{(x-1)\left(x^2-1154x+1\right)},$$

where

$\Delta = (x-1)(x-1154)x^{n+1}W_{2n+1}^2 + (x-1)x^{n+1}W_{2n-1}^2 + (x-1)W_1^2 - (x-1)x\,(W_1-6W_0)^2 + 72(W_1^2+W_0^2-6W_1W_0)(x^n-1)x,$

and

if $(x-1)\left(x^2-1154x+1\right)=0$, *i.e.,* $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k W_{2k+1}^2 = \frac{\Psi}{3\,(x^2-770x+385)},$$

where

$\Psi = ((x-1154)x^{n+1} + (x-1)((n+2)\,x - 1154(n+1))x^n)W_{2n+1}^2 + ((n+2)x-(n+1))x^nW_{2n-1}^2 + W_1^2 - (2x-1)(W_1-6W_0)^2 + 72(W_1^2+W_0^2-6W_1W_0)(x^n(n+1)-1)$

(d) $(m=-1, j=0)$

If $(x-1)\left(x^2-34x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 17-12\sqrt{2}$, $x\neq 17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-k}^2 = \frac{\Delta}{(x-1)\,(x^2-34x+1)},$$

where

$\Delta = (x-1)x^{n+1}W_{-n+1}^2 + (x-1)(x-34)x^{n+1}W_{-n}^2 + (x-1)W_0^2 - (x-1)xW_1^2 + 2(W_1^2 + W_0^2 - 6W_1W_0)(x^n-1)x,$

and

if $(x-1)\left(x^2-34x+1\right)=0$, *i.e.,* $x=1$ *or* $x=17-12\sqrt{2}$ *or* $x=17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-k}^2 = \frac{\Psi}{(3x^2-70x+35)},$$

where

$\Psi = ((n+2)x-(n+1))x^nW_{-n+1}^2 + ((x-34)x^{n+1} + (x-1)((n+2)\,x - 34(n+1))x^n)W_{-n}^2 + W_0^2 - (2x-1)W_1^2 + 2(W_1^2+W_0^2-6W_1W_0)(x^n(n+1)-1)$

(e) $(m=-2, j=0)$

If $(x-1)\left(x^2-1154x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 577-408\sqrt{2}$, $x\neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k}^2 = \frac{\Delta}{(x-1)\,(x^2-1154x+1)},$$

where

$\Delta = (x-1)x^{n+1}W_{-2n+2}^2 + (x-1)(x-1154)x^{n+1}W_{-2n}^2 + (x-1)W_0^2 - (x-1)x(W_0 - 6W_1)^2 + 72(W_1^2+W_0^2-6W_1W_0)(x^n-1)x,$

and

if $(x-1)\left(x^2-1154x+1\right)=0$, *i.e.,* $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k}^2 = \frac{\Psi}{3\,(x^2-770x+385)},$$

where

$\Psi = ((n+2)x-(n+1))x^nW_{-2n+2}^2 + ((x-1154)x^{n+1} + (x-1)((n+2)\,x - 1154(n+1))x^n)W_{-2n}^2 + W_0^2 - (2x-1)(W_0-6W_1)^2 + 72(W_1^2+W_0^2-6W_1W_0)(x^n(n+1)-1)$

(f) $(m=-2, j=1)$

If $(x-1)\left(x^2-1154x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 577-408\sqrt{2}$, $x\neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1}^2 = \frac{\Delta}{(x-1)\,(x^2-1154x+1)},$$

where

$\Delta = (x-1)x^{n+1}W_{-2n+3}^2 + (x-1)(x-1154)x^{n+1}W_{-2n+1}^2 + (x-1)W_1^2 - (x-1)x(6W_0 - 35W_1)^2 + 72(W_1^2+W_0^2-6W_1W_0)(x^n-1)x,$

and

if $(x-1)\left(x^2-1154x+1\right)=0$, *i.e.,* $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1}^2 = \frac{\Psi}{3\,(x^2-770x+385)},$$

where

$\Psi = ((n+2)x-(n+1))x^n W_{-2n+3}^2 + ((x-1154)x^{n+1} + (x-1)((n+2)\,x - 1154(n+1))x^n)W_{-2n+1}^2 + W_1^2 - (2x-1)(6W_0-35W_1)^2 + 72(W_1^2+W_0^2-6W_1W_0)(x^n(n+1)-1)$

From the above proposition, one has the following corollary, which gives sum formulas of balancing numbers (take $W_n = B_n$ with $B_0=0$, $B_1=1$).

Corollary 2.3. *For* $n\geq 0$, *balancing numbers have the following properties:*

(a) $(m=1, j=0)$

If $(x-1)\left(x^2-34x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 17-12\sqrt{2}$, $x\neq 17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_k^2 = \frac{(x-1)(x-34)x^{n+1}B_n^2 + (x-1)x^{n+1}B_{n-1}^2 + x(2x^n-x-1)}{(x-1)\,(x^2-34x+1)},$$

and

if $(x-1)\left(x^2-34x+1\right)=0$, *i.e.,* $x=1$ *or* $x=17-12\sqrt{2}$ *or* $x=17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_k^2 = \frac{\Theta_1}{(3x^2-70x+35)},$$

where

$\Theta_1 = ((x-34)x^{n+1} + (x-1)((n+2)\,x - 34(n+1))x^n)B_n^2 + ((n+2)x-(n+1))x^n B_{n-1}^2 + 2(n+1)x^n - 2x - 1.$

(b) $(m=2, j=0)$

If $(x-1)\left(x^2-1154x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 577-408\sqrt{2}$, $x\neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{2k}^2 = \frac{(x-1)(x-1154)x^{n+1}B_{2n}^2 + (x-1)x^{n+1}B_{2n-2}^2 - 36x(-2x^n+x+1)}{(x-1)\,(x^2-1154x+1)}$$

and

if $(x-1)\left(x^2-1154x+1\right)=0$, *i.e.,* $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{2k}^2 = \frac{\Theta_2}{3\left(x^2-770x+385\right)},$$

where

$\Theta_2 = ((x-1154)x^{n+1}+(x-1)((n+2)x-1154(n+1))x^n)B_{2n}^2+((n+2)x-(n+1))x^nB_{2n-2}^2+36(2(n+1)x^n-2x-1)$.

(c) $(m=2, j=1)$

If $(x-1)\left(x^2-1154x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 577-408\sqrt{2}$, $x\neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{2k+1}^2 = \frac{(x-1)(x-1154)x^{n+1}B_{2n+1}^2+(x-1)x^{n+1}B_{2n-1}^2-(-72x^{n+1}+x^2+70x+1)}{(x-1)\left(x^2-1154x+1\right)},$$

and

if $(x-1)\left(x^2-1154x+1\right)=0$, *i.e.,* $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{2k+1}^2 = \frac{\Theta_3}{3\left(x^2-770x+385\right)},$$

where

$\Theta_3 = ((x-1154)x^{n+1}+(x-1)((n+2)x-1154(n+1))x^n)B_{2n+1}^2+((n+2)x-(n+1))x^nB_{2n-1}^2+2(36(n+1)x^n-x-35)$.

(d) $(m=-1, j=0)$

If $(x-1)\left(x^2-34x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 17-12\sqrt{2}$, $x\neq 17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{-k}^2 = \frac{(x-1)x^{n+1}B_{-n+1}^2+(x-1)(x-34)x^{n+1}B_{-n}^2+x(2x^n-x-1)}{(x-1)\left(x^2-34x+1\right)},$$

and

if $(x-1)\left(x^2-34x+1\right)=0$, *i.e.,* $x=1$ *or* $x=17-12\sqrt{2}$ *or* $x=17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{-k}^2 = \frac{\Theta_4}{(3x^2-70x+35)}$$

where

$\Theta_4 = ((n+2)x-(n+1))x^nB_{-n+1}^2+((x-34)x^{n+1}+(x-1)((n+2)x-34(n+1))x^n)B_{-n}^2+2(n+1)x^n-2x-1$.

(e) $(m=-2, j=0)$

If $(x-1)\left(x^2-1154x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 577-408\sqrt{2}$, $x\neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{-2k}^2 = \frac{(x-1)x^{n+1}B_{-2n+2}^2+(x-1)(x-1154)x^{n+1}B_{-2n}^2-36x(-2x^n+x+1)}{(x-1)\left(x^2-1154x+1\right)}$$

and

if $(x-1)\left(x^2-1154x+1\right)=0$, *i.e.,* $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{-2k}^2 = \frac{\Theta_5}{3\left(x^2-770x+385\right)},$$

where

$\Theta_5 = ((n+2)x-(n+1))x^n B_{-2n+2}^2 + ((x-1154)x^{n+1} + (x-1)((n+2)x-1154(n+1))x^n)B_{-2n}^2 + 36(2(n+1)x^n - 2x - 1)$.

(f) $(m=-2, j=1)$

If $(x-1)\left(x^2-1154x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 577-408\sqrt{2}$, $x\neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{-2k+1}^2 = \frac{(x-1)x^{n+1}B_{-2n+3}^2 + (x-1)(x-1154)x^{n+1}B_{-2n+1}^2 + (72x^{n+1} - 1225x^2 + 1154x - 1)}{(x-1)\left(x^2-1154x+1\right)},$$

and

if $(x-1)\left(x^2-1154x+1\right)=0$, *i.e.,* $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{-2k+1}^2 = \frac{\Theta_6}{3\left(x^2-770x+385\right)},$$

where

$\Theta_6 = ((n+2)x-(n+1))x^n B_{-2n+3}^2 + ((x-1154)x^{n+1} + (x-1)((n+2)x-1154(n+1))x^n)B_{-2n+1}^2 + 2(36(n+1)x^n - 1225x + 577)$.

Taking $W_n = H_n$ with $H_0 = 2, H_1 = 6$ in the last proposition, one has the following corollary, which presents sum formulas of modified Lucas-balancing numbers.

Corollary 2.4. *For $n \geq 0$, modified Lucas-balancing numbers have the following properties:*

(a) $(m=1, j=0)$

If $(x-1)\left(x^2-34x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 17-12\sqrt{2}$, $x\neq 17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_k^2 = \frac{(x-1)(x-34)x^{n+1}H_n^2 + (x-1)x^{n+1}H_{n-1}^2 - 4(16x^{n+1} + 9x^2 - 26x + 1)}{(x-1)\left(x^2-34x+1\right)},$$

and

if $(x-1)\left(x^2-34x+1\right)=0$, *i.e.,* $x=1$ *or* $x=17-12\sqrt{2}$ *or* $x=17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_k^2 = \frac{\Theta_7}{(3x^2-70x+35)},$$

where

$\Theta_7 = ((x-34)x^{n+1} + (x-1)((n+2)x-34(n+1))x^n)H_n^2 + ((n+2)x-(n+1))x^n)H_{n-1}^2 - 8(8(n+1)x^n + 9x - 13)$.

(b) $(m=2, j=0)$

If $(x-1)\left(x^2-1154x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 577-408\sqrt{2}$, $x\neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{2k}^2 = \frac{(x-1)(x-1154)x^{n+1}H_{2n}^2+(x-1)x^{n+1}H_{2n-2}^2-4(576x^{n+1}+289x^2-866x+1)}{(x-1)\left(x^2-1154x+1\right)},$$

and

if $(x-1)\left(x^2-1154x+1\right)=0$, *i.e.,* $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{2k}^2 = \frac{\Theta_8}{3\left(x^2-770x+385\right)},$$

where

$\Theta_8=((x-1154)x^{n+1}+(x-1)((n+2)x-1154(n+1))x^n)H_{2n}^2+((n+2)x-(n+1))x^nH_{2n-2}^2-8\left(288(n+1)x^n+289x-433\right))$.

(c) $(m=2, j=1)$

If $(x-1)\left(x^2-1154x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 577-408\sqrt{2}$, $x\neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{2k+1}^2 = \frac{(x-1)(x-1154)x^{n+1}H_{2n+1}^2+(x-1)x^{n+1}H_{2n-1}^2-36(64x^{n+1}+x^2-66x+1)}{(x-1)\left(x^2-1154x+1\right)},$$

and

if $(x-1)\left(x^2-1154x+1\right)=0$, *i.e.,* $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{2k+1}^2 = \frac{\Theta_9}{3\left(x^2-770x+385\right)},$$

where

$\Theta_9=((x-1154)x^{n+1}+(x-1)((n+2)x-1154(n+1))x^n)H_{2n+1}^2+((n+2)x-(n+1))x^nH_{2n-1}^2-72(32(n+1)x^n+x-33)$.

(d) $(m=-1, j=0)$

If $(x-1)\left(x^2-34x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 17-12\sqrt{2}$, $x\neq 17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{-k}^2 = \frac{(x-1)x^{n+1}H_{-n+1}^2+(x-1)(x-34)x^{n+1}H_{-n}^2-4(16x^{n+1}+9x^2-26x+1)}{(x-1)\left(x^2-34x+1\right)},$$

and

if $(x-1)\left(x^2-34x+1\right)=0$, *i.e.,* $x=1$ *or* $x=17-12\sqrt{2}$ *or* $x=17+12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k H_{-k}^2 = \frac{\Theta_{10}}{(3x^2-70x+35)},$$

where

$\Theta_{10}=((n+2)x-(n+1))x^nH_{-n+1}^2+((x-34)x^{n+1}+(x-1)((n+2)x-34(n+1))x^n)H_{-n}^2-8(8(n+1)x^n+9x-13)$.

(e) $(m=-2, j=0)$

If $(x-1)\left(x^2-1154x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 577-408\sqrt{2}$, $x\neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{-2k}^2 = \frac{(x-1)x^{n+1}H_{-2n+2}^2+(x-1)(x-1154)x^{n+1}H_{-2n}^2-4(576x^{n+1}+289x^2-866x+1)}{(x-1)\left(x^2-1154x+1\right)},$$

and

if $(x-1)\left(x^2-1154x+1\right)=0$, *i.e.,* $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{-2k}^2 = \frac{\Theta_{11}}{3\left(x^2-770x+385\right)},$$

where

$\Theta_{11} = ((n+2)x-(n+1))x^nH_{-2n+2}^2+((x-1154)x^{n+1}+(x-1)((n+2)x-1154(n+1))x^n)H_{-2n}^2-8(288(n+1)x^n+289x-433)$.

(f) $(m=-2, j=1)$

If $(x-1)\left(x^2-1154x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 577-408\sqrt{2}$, $x\neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{-2k+1}^2 = \frac{(x-1)x^{n+1}H_{-2n+3}^2+(x-1)(x-1154)x^{n+1}H_{-2n+1}^2-36(64x^{n+1}+1089x^2-1154x+1)}{(x-1)\left(x^2-1154x+1\right)}$$

and

if $(x-1)\left(x^2-1154x+1\right)=0$, *i.e.,* $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{-2k+1}^2 = \frac{\Theta_{12}}{3\left(x^2-770x+385\right)},$$

where

$\Theta_{12} = ((n+2)x-(n+1))x^nH_{-2n+3}^2+((x-1154)x^{n+1}+(x-1)((n+2)x-1154(n+1))x^n)H_{-2n+1}^2-72(32(n+1)x^n+1089x-577)$.

From the above proposition, one has the following corollary, which gives sum formulas of Lucas-balancing numbers (take $W_n=C_n$ with $C_0=1, C_1=3$).

Corollary 2.5. *For* $n\geq 0$, *Lucas-balancing numbers have the following properties:*

(a) $(m=1, j=0)$

If $(x-1)\left(x^2-34x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 17-12\sqrt{2}$, $x\neq 17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_k^2 = \frac{(x-1)(x-34)x^{n+1}C_n^2+(x-1)x^{n+1}C_{n-1}^2-(16x^{n+1}+9x^2-26x+1)}{(x-1)\left(x^2-34x+1\right)},$$

and

if $(x-1)\left(x^2-34x+1\right)=0$, *i.e.,* $x=1$ *or* $x=17-12\sqrt{2}$ *or* $x=17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_k^2 = \frac{\Theta_{13}}{(3x^2-70x+35)},$$

where

$\Theta_{13} = ((x-34)x^{n+1}+(x-1)((n+2)x-34(n+1))x^n)C_n^2+((n+2)x-(n+1))x^n)C_{n-1}^2-2(8(n+1)x^n+9x-13)$.

(b) $(m = 2, j = 0)$

If $(x-1)\left(x^2-1154x+1\right) \neq 0$, *i.e.,* $x \neq 1$, $x \neq 577-408\sqrt{2}$, $x \neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{2k}^2 = \frac{(x-1)(x-1154)x^{n+1}C_{2n}^2 + (x-1)x^{n+1}C_{2n-2}^2 - (576x^{n+1}+289x^2-866x+1)}{(x-1)\left(x^2-1154x+1\right)},$$

and

if $(x-1)\left(x^2-1154x+1\right) = 0$, *i.e.,* $x = 1$ *or* $x = 577-408\sqrt{2}$ *or* $x = 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{2k}^2 = \frac{\Theta_{14}}{3\left(x^2-770x+385\right)},$$

where

$\Theta_{14} = ((x-1154)x^{n+1} + (x-1)((n+2)x - 1154(n+1))x^n)C_{2n}^2 + ((n+2)x-(n+1))x^n C_{2n-2}^2 - 2(288(n+1)x^n + 289x - 433)$.

(c) $(m = 2, j = 1)$

If $(x-1)\left(x^2-1154x+1\right) \neq 0$, *i.e.,* $x \neq 1$, $x \neq 577-408\sqrt{2}$, $x \neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{2k+1}^2 = \frac{(x-1)(x-1154)x^{n+1}C_{2n+1}^2 + (x-1)x^{n+1}C_{2n-1}^2 - 9(64x^{n+1}+x^2-66x+1)}{(x-1)\left(x^2-1154x+1\right)},$$

and

if $(x-1)\left(x^2-1154x+1\right) = 0$, *i.e.,* $x = 1$ *or* $x = 577-408\sqrt{2}$ *or* $x = 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{2k+1}^2 = \frac{\Theta_{15}}{3\left(x^2-770x+385\right)},$$

where

$\Theta_{15} = ((x-1154)x^{n+1} + (x-1)((n+2)x - 1154(n+1))x^n)C_{2n+1}^2 + ((n+2)x - (n+1))x^n C_{2n-1}^2 - 18(32(n+1)x^n + x - 33)$.

(d) $(m = -1, j = 0)$

If $(x-1)\left(x^2-34x+1\right) \neq 0$, *i.e.,* $x \neq 1$, $x \neq 17-12\sqrt{2}$, $x \neq 17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{-k}^2 = \frac{(x-1)x^{n+1}C_{-n+1}^2 + (x-1)(x-34)x^{n+1}C_{-n}^2 - (16x^{n+1}+9x^2-26x+1)}{(x-1)\left(x^2-34x+1\right)},$$

and

if $(x-1)\left(x^2-34x+1\right) = 0$, *i.e.,* $x = 1$ *or* $x = 17-12\sqrt{2}$ *or* $x = 17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{-k}^2 = \frac{\Theta_{16}}{(3x^2-70x+35)},$$

where

$\Theta_{16} = ((n+2)x-(n+1))x^n C_{-n+1}^2 + ((x-34)x^{n+1} + (x-1)((n+2)x - 34(n+1))x^n)C_{-n}^2 - 2(8(n+1)x^n + 9x - 13)$.

(e) $(m=-2, j=0)$

If $(x-1)\left(x^2-1154x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 577-408\sqrt{2}$, $x\neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{-2k}^2=\frac{(x-1)x^{n+1}C_{-2n+2}^2+(x-1)(x-1154)x^{n+1}C_{-2n}^2-(576x^{n+1}+289x^2-866x+1)}{(x-1)\left(x^2-1154x+1\right)},$$

and

if $(x-1)\left(x^2-1154x+1\right)=0$, *i.e.,* $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{-2k}^2=\frac{\Theta_{17}}{3\left(x^2-770x+385\right)},$$

where

$\Theta_{17}=((n+2)x-(n+1))x^nC_{-2n+2}^2+((x-1154)x^{n+1}+(x-1)((n+2)x-1154(n+1))x^n)C_{-2n}^2-2(288(n+1)x^n+289x-433).$

(f) $(m=-2, j=1)$

If $(x-1)\left(x^2-1154x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 577-408\sqrt{2}$, $x\neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{-2k+1}^2=\frac{(x-1)x^{n+1}C_{-2n+3}^2+(x-1)(x-1154)x^{n+1}C_{-2n+1}^2-9(64x^{n+1}+1089x^2-1154x+1)}{(x-1)\left(x^2-1154x+1\right)},$$

and

if $(x-1)\left(x^2-1154x+1\right)=0$, *i.e.,* $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{-2k+1}^2=\frac{\Theta_{18}}{3\left(x^2-770x+385\right)},$$

where

$\Theta_{18}=((n+2)x-(n+1))x^nC_{-2n+3}^2+((x-1154)x^{n+1}+(x-1)((n+2)x-1154(n+1))x^n)C_{-2n+1}^2-18(32(n+1)x^n+1089x-577).$

Taking $x=1$ in the last two corollaries, one gets the following corollary.

Corollary 2.6. *For* $n\geq 0$, *balancing numbers, modified Lucas-balancing and Lucas-balancing numbers have the following properties:*

1.

(a) $\sum_{k=0}^{n} B_k^2=\frac{1}{32}(33B_n^2-B_{n-1}^2-2n+1)$.

(b) $\sum_{k=0}^{n} B_{2k}^2=\frac{1}{1152}(1153B_{2n}^2-B_{2n-2}^2-72n+36)$.

(c) $\sum_{k=0}^{n} B_{2k+1}^2=\frac{1}{1152}(1153B_{2n+1}^2-B_{2n-1}^2-72n)$.

(d) $\sum_{k=0}^{n} B_{-k}^2=\frac{1}{32}(-B_{-n+1}^2+33B_{-n}^2-2n+1)$.

(e) $\sum_{k=0}^{n} B_{-2k}^2=\frac{1}{1152}(-B_{-2n+2}^2+1153B_{-2n}^2-72n+36)$.

(f) $\sum_{k=0}^{n} B_{-2k+1}^2=\frac{1}{1152}(-B_{-2n+3}^2+1153B_{-2n+1}^2-72n+1224)$.

2.

(a) $\sum_{k=0}^{n} H_k^2 = \frac{1}{32}(33H_n^2 - H_{n-1}^2 + 64n + 32)$.

(b) $\sum_{k=0}^{n} H_{2k}^2 = \frac{1}{1152}(1153H_{2n}^2 - H_{2n-2}^2 + 2304n + 1152)$.

(c) $\sum_{k=0}^{n} H_{2k+1}^2 = \frac{1}{1152}(1153H_{2n+1}^2 - H_{2n-1}^2 + 2304n)$.

(d) $\sum_{k=0}^{n} H_{-k}^2 = \frac{1}{32}(-H_{-n+1}^2 + 33H_{-n}^2 + 64n + 32)$.

(e) $\sum_{k=0}^{n} H_{-2k}^2 = \frac{1}{1152}(-H_{-2n+2}^2 + 1153H_{-2n}^2 + 2304n + 1152)$.

(f) $\sum_{k=0}^{n} H_{-2k+1}^2 = \frac{1}{1152}(-H_{-2n+3}^2 + 1153H_{-2n+1}^2 + 2304n + 39168)$.

3.

(a) $\sum_{k=0}^{n} C_k^2 = \frac{1}{32}(33C_n^2 - C_{n-1}^2 + 16n + 8)$.

(b) $\sum_{k=0}^{n} C_{2k}^2 = \frac{1}{1152}(1153C_{2n}^2 - C_{2n-2}^2 + 576n + 288)$.

(c) $\sum_{k=0}^{n} C_{2k+1}^2 = \frac{1}{1152}(1153C_{2n+1}^2 - C_{2n-1}^2 + 576n)$.

(d) $\sum_{k=0}^{n} C_{-k}^2 = \frac{1}{32}(-C_{-n+1}^2 + 33C_{-n}^2 + 16n + 8)$.

(e) $\sum_{k=0}^{n} C_{-2k}^2 = \frac{1}{1152}(-C_{-2n+2}^2 + 1153C_{-2n}^2 + 576n + 288)$.

(f) $\sum_{k=0}^{n} C_{-2k+1}^2 = \frac{1}{1152}(-C_{-2n+3}^2 + 1153C_{-2n+1}^2 + 576n + 9792)$.

3 Conclusion

Recently, there have been so many studies of the sequences of numbers in the literature. The sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics, and engineering. In this work, sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized balancing sequence. Then, we have presented the formulas as special cases, the corresponding identity for the balancing, modified Lucas-balancing, and Lucas-balancing numbers. All the listed identities in the corollaries may be proved by induction, but that proof method gives no clue about their discovery. We have provided proofs to show how these identities were discovered in general.

Computations of the Frobenius norm, spectral norm, maximum column length norm, and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences require the sum of the numbers of the sequences. So, our results can be used to study r-circulant matrices with m-order linear recurrence sequences.

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Conflict of Interest

The authors have no conflicts of interest to declare.

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