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# Uniqueness and stability of parameter identification in elliptic boundary value problem

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**Abstract.** This paper concerns the uniqueness and stability of an inverse problem in PDE. Our problem consists of identifying two parameters $b(x)$ and $c(x)$ in the following boundary-value problem

$$\begin{cases} Lu := -b(x)u''(x) + c(x)u'(x) = f(x), \\ u(0) = u(1) = 0, \end{cases}$$

from distributed observations $u_1$ (resp. $u_2$) associated with the source $f_1$ (resp. $f_2$). For one observation, the solution is not unique. However, we prove, under some conditions, the uniqueness of the solution $p = (b, c)$ in the case of two observations. Furthermore, we derive a Hölder-type stability result. The algorithm of reconstruction uses the least squares method. Finally, we present some numerical examples with exact and noisy data to illustrate our method.

**Keywords:** inverse problem, least squares method, Levenberg-Marquardt algorithm.

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## 1 Introduction

The direct problem is to find the weak solution of the problem (1.1)

$$\begin{cases} Lu := -b(x)u''(x) + c(x)u'(x) = f(x), \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

given $b(x)$, $c(x)$, and $f(x)$, find $u \in H_0^1(0,1)$, such that

$$\int_0^1 [u'(x)v(x)(b'(x) + c(x)) + u'(x)v'(x)b(x)]dx = \int_0^1 f(x)v(x)dx, \quad \forall v \in H_0^1(0,1). \tag{1.2}$$

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The direct problem is well-posed under standard conditions $b \in C^1[0,1]$; $b(x) \geq b_0 > 0$, $c \in C[0,1]$ and $f \in H = L^2(0,1)$.

Our inverse problem is the parameter identification. Given $(u, f)$ (for one observation) or $(u_1, f_1)$ and $(u_2, f_2)$ (for two observations), we reconstruct the pair of coefficients $p = (b, c)$.

It is well-known that such problem is typically ill-posed problem [2], that is the solution can be non-unique and unstable.

The problem of identifying parameters have many engineering applications like hydrology, geology and ecology [6].

Many articles have studied the identification of one parameter. In [2], the authors developed an abstract framework for nonlinear ill-posed problems. In [3], the author gives a condition that ensures the uniqueness in the problem of transmissivity parameter identification. The most common approach is to reformulate the inverse problem as a least squares problem which is solved by optimization methods using the gradient of the objective function [4]. We remark that a few articles concern the identification of many parameters with numerical validation.

The paper is organized as follows. In section 2, we study the stability of the inverse problem. In section 3, we propose a reconstruction algorithm of the parameters. In section 4, we show some numerical examples.

# 2 Stability

In this section, we consider the case of two observations. We give a condition for which the solution of the inverse problem is unique. The equation $\Phi(p) = (u_1, u_2)$ has a unique solution $p = (b, c)$ if and only if the linear system $\{Lu_1 = f_1;\ Lu_2 = f_2\}$ has a unique solution with respect to $(b, c)$. In addition, we prove the stability estimates for the inverse problem's solution.

## 2.1 Notations

First, we introduce some notations that will be used throughout this paper.

- In order to provide an abstract formulation of the inverse problem, we introduce the parameter space $M_{ad} = \{(b, c);\ b \in C^1[0,1],\ b(x) \geq b_0 > 0,\ \ c \in C[0,1]\}$.

- The parameter is the pair $p_1 = (b_1, c_1),\ p_2 = (b_2, c_2) \in M_{ad}$.

- Consider $f_1, f_2 \in L^\infty(0,1)$.

- The mapping $\Phi$ that relates the parameter to the observation is defined by $\Phi(p_1) = (u_1, u_2),\ \Phi(p_2) = (v_1, v_2) \in Y \times Y,\ \ Y = H^2(0,1) \cap H_0^1(0,1)$, with $u_j$ (resp. $v_j$) solution of $-b_1 u_j'' + c_1 u_j' = f_j$ (resp. $-b_1 v_j'' + c_1 v_j' = f_j$), $j = 1, 2$.

- We set $\Delta_1(x) = f_1 u_2' - f_2 u_1'$ and $\Delta_2(x) = f_1 v_2' - f_2 v_1'$.

**Proposition 2.1.** *Suppose that $\Delta_1(x) \equiv f_1(x) u_2'(x) - f_2(x) u_1'(x) \neq 0$, a.e $x \in (0,1)$. Then, the equation $\Phi(p_1) = (u_1, u_2)$ has a unique solution $p_1 = (b_1, c_1)$.*

*Proof.* Let $x \in (0,1)$ be fixed. Assume that we have two solutions $p_1 = (b_1(x), c_1(x))$ and $p_2 = (b_2(x), c_2(x))$, then $(b(x), c(x)) = (b_1(x) - b_2(x), c_1(x) - c_2(x))$ satisfy the linear homogeneous system

$$\begin{cases} -b(x)u_1''(x) + c(x)u_1'(x) = 0, \\ -b(x)u_2''(x) + c(x)u_2'(x) = 0. \end{cases} \tag{2.1}$$

The determinant of system (2.1) is given by

$$\Delta(x) = -u_1''(x)u_2'(x) + u_2''(x)u_1'(x), \tag{2.2}$$

but from the differential equations, we have

$$u_i''(x) = \frac{1}{b_1(x)}(c_1(x)u_i'(x) - f_i(x)), \ \ i = 1, 2, \tag{2.3}$$

then

$$\Delta(x) = \frac{\Delta_1(x)}{b_1(x)}, \ a.e\ x \in (0,1).$$

Since $\Delta_1(x) \neq 0$, then $b(x) = c(x) = 0$, $\forall x \in [0,1]$.
It follows that $b_1(x) = b_2(x)$ and $c_1(x) = c_2(x)$, $\forall x \in [0,1]$.
Which completes the proof. □

**Proposition 2.2.** *Assume that:* $\exists \alpha_1 > 0$, *such that*

$$|\Delta_1(x)| \geq \alpha_1 > 0, \quad \forall x \in [0,1]. \tag{2.4}$$

*Then we have the following estimate*

$$\|p_1 - p_2\|_H \leq C(p_1)\|\Phi(p_1) - \Phi(p_2)\|_{Y \times Y}. \tag{2.5}$$

*Proof.* We set $w_1 = u_1 - v_1$ and $w_2 = u_2 - v_2$, such that $\|w_1\|_{H^2(0,1)} + \|w_2\|_{H^2(0,1)} \leq \eta$, then

$$\begin{array}{rcl} |\Delta_2(x)| & = & |f_1(x)v_2'(x) - f_2(x)v_1'(x)| = |f_2 w_1' - f_1 w_2' + \Delta_1(x)|, \\ & \geq & |\Delta_1(x)| - |f_2(x)||w_1'(x)| - |f_1(x)||w_2'(x)|, \\ & \geq & \alpha_1 - (\|f_1\|_\infty + \|f_2\|_\infty)(\|w_1'\|_\infty + \|w_2'\|_\infty), \\ & \geq & \alpha_1 - C\eta(\|f_1\|_\infty + \|f_2\|_\infty), \quad (\text{since } \|w_j'\|_\infty \leq C\|w_j\|_{H^2(0,1)}) \\ & \geq & \alpha_2 > 0 \quad (\text{for } \eta > 0 \text{ small enough}). \end{array} \tag{2.6}$$

In this case, the equation $\Phi(p_2) = (v_1, v_2)$ has a unique solution.
From the system

$$\begin{cases} \Phi(p_1) = (u_1, u_2), \\ \Phi(p_2) = (v_1, v_2), \end{cases} \tag{2.7}$$

we obtain the following systems

$$\left\{ \begin{array}{rcl} -b_1 u_1'' + c_1 u_1' & = & f_1, \\ -b_1 u_2'' + c_1 u_2' & = & f_2. \end{array} \right. ; \quad \left\{ \begin{array}{rcl} -b_2 v_1'' + c_2 v_1' & = & f_1, \\ -b_2 v_2'' + c_2 v_2' & = & f_2. \end{array} \right.$$

Combining these equations, we obtain the system

$$\left\{ \begin{array}{rcl} (b_2 - b_1)v_1'' + (c_1 - c_2)v_1' & = & b_1 w_1'' - c_1 w_1', \\ (b_2 - b_1)v_2'' + (c_1 - c_2)v_2' & = & b_1 w_2'' - c_1 w_2'. \end{array} \right. \tag{2.8}$$

The determinant of system (2.8) is $-\frac{\Delta_2(x)}{b_2}$ (does not vanish by (2.6)). Therefore, we obtain the inversion formulas

$$\begin{cases} \Delta b := b_2 - b_1 = -\dfrac{b_2}{\Delta_2(x)}[v_2'(b_1 w_1'' - c_1 w_1') - v_1'(b_1 w_2'' - c_1 w_2')], \\ \Delta c := c_2 - c_1 = \dfrac{b_2}{\Delta_2(x)}[-v_2''(b_1 w_1'' - c_1 w_1') + v_1''(b_1 w_2'' - c_1 w_2')]. \end{cases}$$

Using the equations $\Delta b\, w_1'' = \Delta c\, w_1'$ and $\Delta b\, w_2'' = \Delta c\, w_2'$ we deduce the system

$$\begin{cases} (\Delta b)^2 = \dfrac{-b_2}{\Delta_2}(b_1 \Delta c - c_1 \Delta b)(v_2' w_1' - w_2' v_1'), \\ (\Delta c)^2 = \dfrac{-b_2}{\Delta_2}(b_1 \Delta c - c_1 \Delta b)(v_2'' w_1'' - v_1'' w_2''), \end{cases} \tag{2.9}$$

from (2.6) and (2.9), we obtain the estimations

$$\begin{cases} |\Delta b|^2 \leq \dfrac{b_2}{\alpha_2}\sqrt{b_1^2 + c_1^2}\sqrt{\Delta c^2 + \Delta b^2}(|v_2'||w_1'| + |w_2'||v_1'|), \\ |\Delta c|^2 \leq \dfrac{b_2}{\alpha_2}\sqrt{b_1^2 + c_1^2}\sqrt{\Delta c^2 + \Delta b^2}(|v_2''||w_1''| + |w_2''||v_1''|), \end{cases} \tag{2.10}$$

therefore

$$\begin{aligned} (|\Delta b|^2 + |\Delta c|^2)^{\frac{1}{2}} \leq \; & \frac{b_2}{\alpha_2}\sqrt{b_1^2 + c_1^2}\Big(\sqrt{|v_2'|^2 + |v_1'|^2}\sqrt{|w_2'|^2 + |w_1'|^2} + \\ & \sqrt{|v_2''|^2 + |v_1''|^2}\sqrt{|w_2''|^2 + |w_1''|^2}\Big). \end{aligned} \tag{2.11}$$

From the stability of the direct problem, we have the estimation

$$\|v_j\|_{H^2(0,1)} \leq C(p_2)\|f_j\|_\infty, \quad j = 1, 2,$$

which leads to

$$\|\Delta b\|_H^2 + \|\Delta c\|_H^2 \leq M(p_1)(\|f_1\|_\infty + \|f_2\|_\infty)^2(\|w_1\|_{H^2(0,1)} + \|w_2\|_{H^2(0,1)})^2.$$

□

**Remark 2.3.** Proposition 2.2 means that the operator $\Phi : p \mapsto u = (u_1, u_2)$ from $M_{ad} \subset H \times H$ to $Y \times Y$ is invertible in a neighbourhood of $p_1$, moreover $\Phi^{-1}$ is continuous (locally Lipschitz).

# 3 Algorithm

One of the most commonly used approaches for solving the inverse problem is by setting it as a least squares problem [1]. The solution $p = (b, c)$ realizes the minimum of the functional

$$J(p) = \frac{1}{2}[\|\Phi_1(p) - d_1\|_H^2 + \|\Phi_2(p) - d_2\|_H^2], \quad \text{for } p \in M_{ad}, \tag{3.1}$$

where $\Phi_j(p) = u_j(p)$ is the operator solution and $(d_1, d_2) \in H \times H$ is the measured data. To solve the least squares problem, we apply the Levenberg-Marquardt method [5] which consists of iterating the procedure:

1. $p_0$: initial approximation,
2. $p_{n+1} = p_n + h_n$, where $h_n$ is the solution of the linearized equation

$$\Phi'^*(p_n)\Phi'(p_n)h_n + \alpha_n h_n = \Phi'^*(p_n)(d - \Phi(p_n)), \tag{3.2}$$

where $\Phi'(p)$ is the Fréchet-derivative of $\Phi$ and $\Phi'^*(p)$ is the adjoint operator of $\Phi'(p)$ [2] given by the following theorem:

**Theorem 3.1.** *The operator $\Phi$ is Fréchet-differentiable. The partial derivatives are given by:*

$$\begin{cases} \dfrac{\partial \Phi}{\partial b}(p;h) = \big(A(p)\big)^{-1}(hu''), \\ \\ \dfrac{\partial \Phi}{\partial c}(p;k) = -\big(A(p)\big)^{-1}(ku'). \end{cases} \tag{3.3}$$

*Where $A(p)$ is the differential operator defined by: $A(p) : \mathcal{D}(A(p)) \longrightarrow H$.*

$$\begin{cases} \mathcal{D}(A(p)) = H^2(0,1) \cap H_0^1(0,1), \\ A(p)\varphi = L\varphi, \quad L\varphi = -b(x)\varphi'' + c(x)\varphi'. \end{cases} \tag{3.4}$$

# 4 Numerical examples

In the following experiences, $n = 80$ designates the number of points in $(0,1)$ and $itermax = 10$ is the maximal number of iterations of the reconstruction. We choose initial guess as $p_0 = (1,1)$.
As an example, we choose the coefficients $b(x) = 1 + 0.5\sin(\pi x)$ and $c(x) = 1 + x - x^2$.

**Example 4.1.** $f_1(x) = \cos(\pi x)$ and $f_2(x) = x - x^2$. The solution $u_1$ and $u_2$ are computed using the Finite Element Method.

**Example 4.2.** $f_1(x) = \begin{cases} x & \text{if} \quad x \leq 0.5 \\ 0.5 & \text{if} \quad x > 0.5 \end{cases} \quad , \quad f_2(x) = x - x^2.$

## 4.1 Commentaries

- In example 4.1 (see figure 4.1), $\Delta_1$ does not change the sign ($\Delta_1(x) \geq 0.02$); which confirms the numerical stability (see figures 4.2, 4.3, case with $\delta = 0$).
- Figure 4.3 shows that reconstruction of $c$ is deteriorated when the noise level $\delta \geq 10^{-4}$. However, the reconstruction of $b$ is acceptable.
- Figure 4.4 shows that, if $\delta \geq 10^{-4}$, the error curve presents local minimum after few iterations, hence the need to introduce a stopping criterion based on Morozov discrepancy principle.
- In example 4.2 (see figure 4.5), $\Delta_1$ changes the sign, it vanishes at $x = 0.5$. For the parameter $c(x)$, the algorithm converges to another solution (lack of uniqueness). However, the parameter $b$ is relatively stable (see figure 4.6).

## Conclusion

We considered an inverse problem of determining two parameters in an elliptic boundary value problem from the couple $(f, u)$ where $f$ is the right-hand side and $u$ is the solution. We showed uniqueness and stability under some conditions on the data. We have proposed a reconstruction algorithm. The numerical examples valid the method when the noise level is less than $10^{-4}$. In perspective, it is important to continue this research in order to improve the reconstruction of the parameter $c$ in the case of noisy data.

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## Conflict of Interest

The authors have no conflicts of interest to declare.

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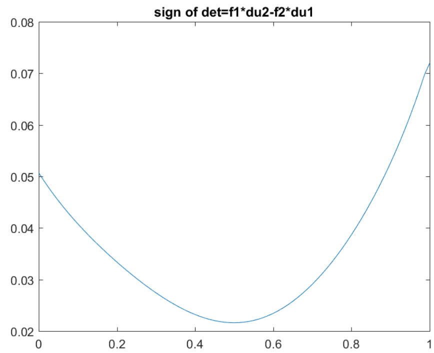


Figure 4.1: Variation of $\Delta_1(x)$.

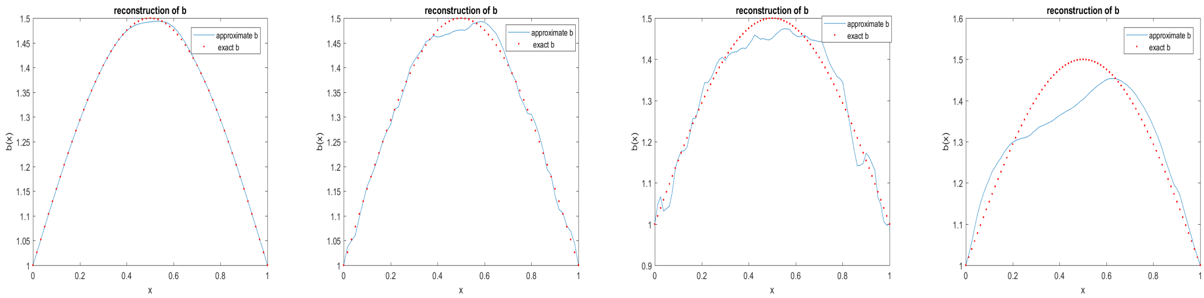


Figure 4.2: Reconstruction of $b$ with noise level $\delta = 0$, $\delta = 10^{-5}$, $\delta = 10^{-4}$ and $\delta = 10^{-3}$.

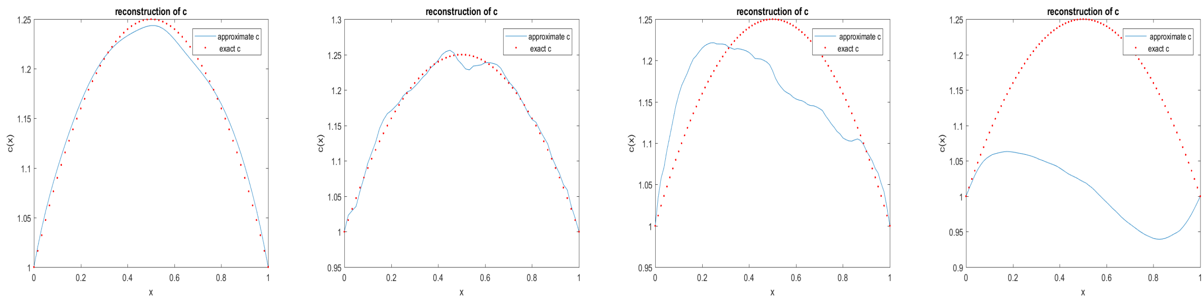


Figure 4.3: Reconstruction of $c$ with noise level $\delta = 0$, $\delta = 10^{-5}$, $\delta = 10^{-4}$ and $\delta = 10^{-3}$.

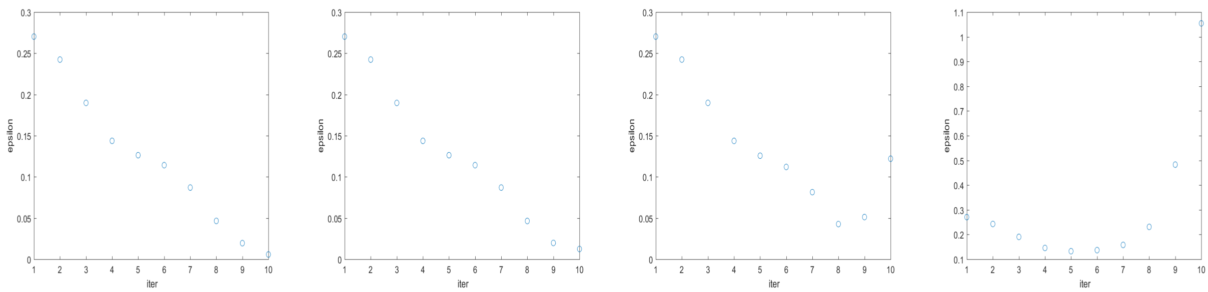


Figure 4.4: Evolution of the error $\|p - p_{ex}\|_2$ with the noise level $\delta = 0$, $\delta = 10^{-5}$, $\delta = 10^{-4}$ and $\delta = 10^{-3}$.

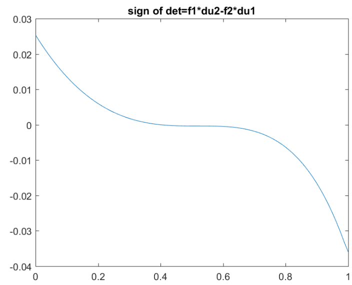


Figure 4.5: Variation of $\Delta_1(x)$.

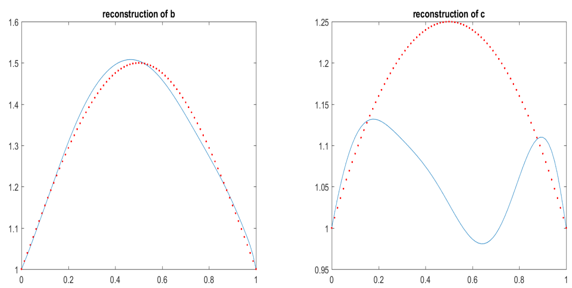


Figure 4.6: Reconstruction without noise.