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An approximate solution for the time-fractional diffusion equation

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Abstract. In this paper, a numerical method based on a finite difference scheme is proposed for solving the time-fractional diffusion equation (TFDE). The TFDE is obtained from the standard diffusion equation by replacing the first-order time derivative with Caputo fractional derivative. At first, we introduce a time discrete scheme. Then, we prove the proposed method is unconditionally stable and the approximate solution converges to the exact solution with order $O(\Delta t^{2-\alpha})$, where Δt is the time step size and α is the order of Caputo derivative. Finally, some examples are presented to verify the order of convergence and show the application of the present method.

Keywords: Time-fractional diffusion equation, Caputo derivative, Convergence rates, Stability.

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1 Introduction

In recent years, the use of fractional ordinary differential equations (FODEs) and fractional partial differential equations (FPDEs) in finance problems [10, 21, 22], hydrology problems [1,3–5,26], physics problems [2,6–8,12,18–20,23,24,29], and mathematical models have become increasingly popular. The numerical and the analytical solutions of the time-fractional partial differential equations are studied using Fourier-Laplace transforms or Green's functions (see e.g. [9, 16, 25, 27, 28]). However, published papers on the numerical solution of the time-fractional diffusion equation (TFDE) are limited. The authors of [11] have proposed finite element methods for time-fractional partial differential equations; the authors of [17] have used a meshless method for the(TFDE); Liu et al. [15] used an explicit finite-difference scheme for TFDE (this method is a lowe-order method); Lin and Xu et al. [14] have proposed finite difference/spectral methods for TFDE, they used Legendre spectral methods in space and a finite difference scheme in time and show that the methods for α order TFDE have convergence rate $O(\Delta t^{2-\alpha} + N^{-m}/(\Delta t)^{\alpha})$, where Δt, N and m are the time step size, polynomial degree and the regularity of the exact solution respectively. The convergence rate in their paper is not optimal.

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In this paper, we propose a numerical scheme based on the finite difference method for solving time-fractional diffusion equation and prove an optimal convergence rate. We consider the time-fractional diffusion equation of the form:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \qquad (x,t) \in [0,1] \times [0,T], \tag{1.1}$$

subject to the boundary and initial conditions:

$$u(0,t) = u(1,t) = 0, \qquad t \in (0,T], \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad x \in [0,1], \tag{1.3}$$

where $0 < \alpha < 1$, u_0 and f are given smooth functions, the time-fractional derivative $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ is the Caputo derivative defined by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha}. \tag{1.4}$$

2 The numerical method for the TFDE

In this section, we will estimate the time-fractional derivative $\frac{\partial^\alpha u}{\partial t^\alpha}$ at t_{m+1} by forward finite difference approximation to discretize the time-fractional derivative. Let $t_m := m\Delta t$, $m = 0,1,...,M$, where $\Delta t := T/M$ is the time step and M is a positive integer.

$$\begin{aligned} \frac{\partial^\alpha (x,t_{m+1})}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t_{m+1}} \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t_{m+1}-s)^\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)} = \sum_{k=0}^{m} \int_{t_k}^{t_{k+1}} \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t_{m+1}-s)^\alpha}. \end{aligned} \tag{2.1}$$

For the forward finite difference, we have

$$\frac{\partial u(x,s)}{\partial t} \mid_{(x,t_k)} = \frac{u(x,t_{k+1}) - u(x,t_k)}{\Delta t} + O(\Delta t), \tag{2.2}$$

Substituting (2.2) into (2.1), we obtain

$$\begin{aligned} \frac{\partial^\alpha (x,t_{m+1})}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{m} \int_{t_k}^{t_{k+1}} \frac{\partial u(x,t_{k+1}) - u(x,t_k)}{\Delta t} \frac{ds}{(t_{m+1}-s)^\alpha} + R_{\Delta t}^{k+1} \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{m} \frac{\partial u(x,t_{k+1}) - u(x,t_k)}{\Delta t} \int_{t_k}^{t_{k+1}} \frac{ds}{(t_{m+1}-s)^\alpha} + R_{\Delta t}^{k+1}, \end{aligned} \tag{2.3}$$

where $R_{\Delta t}^{k+1}$ is the truncation error, which we will get it later in proposition 2.1, for the integral at the RHS of (2.3), we have

$$\begin{aligned} \int_{t_k}^{t_{k+1}} \frac{ds}{(t_{m+1}-s)^\alpha} &= -\int_{t_{m+1}-t_k}^{t_{m+1}-t_k} p^{-\alpha} dp \\ &= \frac{(t_{m+1}-t_k)^\alpha - (t_{m+1}-t_{k+1})^{1-\alpha}}{1-\alpha}. \end{aligned} \tag{2.4}$$

By using $t_m = m\Delta t$, we have

$$t_{m+1} - t_{k+1} = (m-k)\Delta t, \qquad t_{m+1} - t_k = (m+1-k)\Delta t, \tag{2.5}$$

Substituting (2.5) into (2.4), we obtain

$$\int_{t_k}^{t_{k+1}} \frac{ds}{(t_{m+1}-s)^{\alpha}} = \frac{((m+1-k)\Delta t^{1-\alpha}) - ((m-k)\Delta t^{1-\alpha})}{1-\alpha}$$
$$= \frac{\Delta t^{1-\alpha}}{1-\alpha}((m+1-k)^{1-\alpha} - (m-k)^{1-\alpha}, \tag{2.6}$$

substituting (2.6) into (2.3), we obtain

$$\frac{\partial^{\alpha} u(x,t_{m+1})}{\partial t^{\alpha}} = \frac{\Delta^{-\alpha}(t)}{(1-\alpha)(\Gamma(1-\alpha)} \sum_{k=0}^{m} a_{m-k}(u(x,t_{k+1}) - u(x,t_k)) + R_{\Delta t}^{k+1}$$
$$= \frac{\Delta^{-\alpha}(t)}{\Gamma(2-\alpha)} \sum_{k=0}^{m} a_{m-k}(u(x,t_{k+1}) - u(x,t_k)) + R_{\Delta t}^{k+1}. \tag{2.7}$$

Here $a_k := (k+1)^{1-\alpha} - k^{1-\alpha}, k = 0,1,...,M$. Let $\gamma = \Gamma(2-\alpha)\Delta t^{\alpha}$. Substituting (2.7) into (1.1), the following form is obtained

$$u(x,t_{m+1}) - \gamma\left(\frac{\partial^2 u(x,t_{m+1})}{\partial x^2}\right)$$
$$= u(x,t_m) - \sum_{k=1}^{m} a_k(u(x,t_{m-k+1} - u(x,t_{m-k})) + \gamma f(x,t_{m+1}) + R_{\Delta t}^{(1)}, \tag{2.8}$$

where

$$R_{\Delta t}^{(1)} \leq C_0 \Delta t^2,$$

and C_0 is a constant.

Let u^m be the numerical solution to $u(x,t_m)$ and $f^{m+1} = f(x,t_{m+1})$, by removing the small term $R_{\Delta t}^{(1)}$ from (2.8), we can create the following discrete scheme for solving 1.1.

$$u^{m+1} - \gamma\left(\frac{\partial^2 u^{m+1}}{\partial x^2}\right) = u^m - \sum_{k=1}^{m} a_k(u^{m-k+1} - u^{m-k}) + \gamma f^{m+1}, m = 0,1,...,M. \tag{2.9}$$

Proposition 2.1. *The truncation error $R_{\Delta t}^{k+1}$ has the following form*

$$R_{\Delta t}^{k+1} \leq C_1 \left| \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{m} \int_{t_k}^{t_{k+1}} \frac{2s - t_{k+1} - t_k}{(t_{m+1}-s)^{\alpha}} ds + O(\Delta t^2) \right| \leq C_2 \Delta t^{2-\alpha}, \tag{2.10}$$

Where C_1 and C_2 are constant.

proof: First we show that

$$R_{\Delta t}^{k+1} \leq C_1 \left| \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{m} \int_{t_k}^{t_{k+1}} \frac{2s - t_{k+1} - t_k}{(t_{m+1}-s)^{\alpha}} ds + O(\Delta t^2) \right|.$$

By using the Taylor series, we have

$$\frac{u(x,t_{k+1}) - u(x,t_k)}{\Delta t} = \frac{\partial u(x,t_k)}{\partial t} + \frac{\Delta t}{2} \frac{\partial^2 u(x,t_k)}{\partial t^2} + O(\Delta t^2),$$

In addition, from (2.3), the truncation error has the following form

$$R_{\Delta t}^{k+1} = \frac{1}{\Gamma(1-\alpha)} \int_{t_k}^{t_{k+1}} \left(\frac{\partial u(x,s)}{s} - \frac{\partial u(x,t_k)}{\partial t} - \frac{\Delta t}{2} \frac{\partial^2 u(x,t_k)}{\partial t^2} + O(\Delta t)\right) \left(\frac{ds}{(t_{m+1}-s)^{\alpha}}\right), \tag{2.11}$$

Now we write the Taylor expansion of $\frac{\partial u(x,s)}{\partial s}$ at t_k

$$\frac{\partial u(x,s)}{\partial s} = \frac{\partial u(x,t_k)}{\partial t} + (s-t_k)\frac{\partial^2 u(x,t_k)}{\partial t^2} + O((s-t_k)^2), \tag{2.12}$$

Substituting (2.12) into (2.11), we obtain

$$\begin{aligned} R_{\Delta t}^{k+1} &= \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{m}\int_{t_k}^{t_{k+1}}\left((s-t_k-\frac{\Delta t}{2})\frac{\partial^2 u(x,t_k)}{\partial t^2} + O(\Delta t^2)\right)\left(\frac{ds}{(t_{m+1}-s)^{\alpha}}\right) \\ &= \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{m}\int_{t_k}^{t_{k+1}}\left(\frac{2s-t_{k+1}-t_k}{2}\frac{\partial^2 u(x,t_k)}{\partial t^2} + O(\Delta t^2)\right)\left(\frac{ds}{(t_{m+1}-s)^{\alpha}}\right), \end{aligned}$$

the absolute value of the truncation error is as follows

$$R_{\Delta t}^{k+1} \le C_1\left|\frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{m}\int_{t_k}^{t_{k+1}}\frac{2s-t_{k+1}-t_k}{2}\frac{\partial^2 u(x,t_k)}{\partial t^2}ds + O(\Delta t^2)\right|,$$

where $C_1 \le \frac{1}{2}\left|\frac{\partial^2 u(x,t_k)}{\partial t^2}\right|$.

Now, we show that

$$\left|\frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{m}\int_{t_k}^{t_{k+1}}\frac{2s-t_{k+1}-t_k}{2}\frac{\partial^2 u(x,t_k)}{\partial t^2}ds + O(\Delta t^2)\right| \le C_2(\Delta t^{2-\alpha}).$$

We have

$$\begin{aligned} &\frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{m}\int_{t_k}^{t_{k+1}}\frac{2s-t_{k+1}-t_k}{2}\frac{\partial^2 u(x,t_k)}{\partial t^2}ds + O(\Delta t^2) \\ &= -\frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{m}\frac{1}{\Gamma(1-\alpha)}(2k+1)(\Delta t)^{2-\alpha}\left[(m-k)^{1-\alpha}-(m+1-k)^{1-\alpha}\right] \\ &\quad + \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{m}\frac{2}{(1-\alpha)}(\Delta t)^{2-\alpha}\left[(k+1)(m-k)^{1-\alpha}-k(m+1-k)^{1-\alpha}\right] \\ &\quad + \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{m}\frac{2}{(1-\alpha)(2-\alpha)}(\Delta t)^{2-\alpha}\left[(m-k)^{2-\alpha}-(m+1-k)^{2-\alpha}\right] \\ &= \frac{(\Delta t)^{2-\alpha}}{\Gamma(2-\alpha)}\left[(m+1)^{1-\alpha}+2(m^{1-\alpha}+(m-1)^{1-\alpha}+(m-2)^{1-\alpha}+...+1^{1-\alpha})\right] \\ &\qquad - \frac{2(\Delta t)^{2-\alpha}}{\Gamma(3-\alpha)}(m+1)^{2-\alpha} \\ &= \frac{(\Delta t)^{2-\alpha}}{\Gamma(2-\alpha)}\left[(m+1)^{1-\alpha}+2(m^{1-\alpha}+(m-1)^{1-\alpha}+(m-2)^{1-\alpha}+...+1^{1-\alpha})-\frac{2}{2-\alpha}(m+1)^{2-\alpha}\right]. \end{aligned}$$

Let

$$p(m) = (m+1)^{1-\alpha}+2(m^{1-\alpha}+(m-1)^{1-\alpha}+(m-2)^{1-\alpha}+...+1^{1-\alpha})-\frac{2}{2-\alpha}(m+1)^{2-\alpha}.$$

We will show that the $|p(m)|$ is bounded for all $\alpha \in [0,1]$ and all $m \ge 1$, as proven in the following lemma.

Lemma 2.2. *for all $\alpha \in [0,1]$ and all $m \ge 1$, we have*

$$|p(m)| \le C_3,$$

where C_3 is a constant independent of α, m.

Proof. First, for $\alpha = 0$ and $m \geq 1$, we will show that $p(m) = 0$.

We have

$$\begin{aligned} p(m) &= (m+1) + 2(m + (m-1) + (m-2) + ... + 1) - (m+1)^2 \\ &= m + 1 + 2\left[\frac{m}{2}(m+1)\right] - (m+1)^2 \\ &= (m+1)^2 - (m+1)^2 = 0. \end{aligned}$$

Now we prove for $\alpha \in (0, 1]$, we can write $p(m)$ as follows

$$p(m) = (m+1)^{1-\alpha} + 2(m^{1-\alpha} + (m-1)^{1-\alpha} + (m-2)^{1-\alpha} + \cdots + 1^{1-\alpha}) - \frac{2}{2-\alpha}(m+1)^{2-\alpha} = \sum_{i=0}^{m} b_i,$$

where

$$b_i = (i+1)^{1-\alpha} + i^{1-\alpha} - \frac{2}{2-\alpha}((i+1)^{2-\alpha} - i^{2-\alpha}).$$

It suffices to prove that the $\sum_{i=0}^{\infty} b_i$ convergent. It is well known that the series $\sum_{i=0}^{\infty} \frac{1}{i^\beta}$ is a geometric series and converges for all $\beta > 1$. Now we will show that the $|b_i| \leq \frac{1}{i^{1+\alpha}}$ for big enough i. For $i \geq 2$, we have

$$\begin{aligned} |b_i| =& i^{1-\alpha} \left| \left(1 + \frac{1}{i}\right)^{1-\alpha} + 1 - \frac{2i}{2-\alpha}\left(\left(1 + \frac{1}{i}\right) - 1\right) \right| \\ =& i^{1-\alpha} \mid 1 + 1 + (1-\alpha)\frac{1}{i} + \frac{(1-\alpha)(-\alpha)}{2!}\frac{1}{i^2} + \frac{(1-\alpha)(\alpha)(-\alpha-1)}{3!}\frac{1}{i^3} + \ldots \\ & - \frac{2i}{2-\alpha}\left(-1 + 1 + (2-\alpha)\frac{1}{i} + \frac{(2-\alpha)(1-\alpha)}{2!}\frac{1}{i^2} + \frac{(2-\alpha)(1-\alpha)(-\alpha)}{3!}\frac{1}{i^3} + \ldots\right) \mid \\ =& i^{1-\alpha} \left| \left(\frac{1}{2!} - \frac{2}{3!}\right)(1-\alpha)(-\alpha)\frac{1}{i^2} + \left(\frac{1}{3!} - \frac{2}{4!}\right)(1-\alpha)(-\alpha)(-\alpha-1)\frac{1}{i^3} + \ldots \right| \\ \leq& i^{1-\alpha}\frac{1}{3!}(1-\alpha)\alpha\frac{1}{i^2}\left(1 + \frac{2(\alpha+1)}{4}\frac{1}{i} + \frac{3(\alpha+1)(\alpha+2)}{20}\frac{1}{i^2} + \ldots\right) \\ \leq& \frac{1}{3!}(1-\alpha)\alpha\frac{1}{i^{1+\alpha}}\left(1 + \frac{1}{i} + \frac{1}{i^2} + \frac{1}{i^3} + \ldots\right) \leq \frac{2}{3!}(1-\alpha)\alpha\frac{1}{i^{1+\alpha}} \leq \frac{1}{i^{1+\alpha}}. \end{aligned}$$

The proof is completed. □

3 Stability of the method

In this section, by using the following lemma, we will prove the proposed method is unconditionally stable, in other words, we will prove the stability of Eq. (2.9).

Lemma 3.1. *[13] Let Ω be a bounded domain in R^n with piecewise smooth boundary $\partial\Omega$, if V and U are two functions defined on the closed region containing Ω and have continuous partial derivatives, then*

$$\int_\Omega V\frac{\partial U}{\partial x_i}d\Omega = \int_{\partial\Omega} VUcos(\overrightarrow{n}, x_i)dS - \int_\Omega U\frac{\partial V}{\partial x_i}d\Omega, \tag{3.1}$$

Where $\overrightarrow{n}$ is the outward vector, dS stands for the surface area element on $\partial\Omega$.

Lemma 3.2. *The coefficient, a_i, satisfy*

1. $a_i > 0, \quad i = 1, 2, ...$

2. $a_i > a_{i+1}, \quad i = 0, 1, 2, ...$

Proof. Let

$$s(i) := a_i = (i+1)^{1-\alpha} - i^{1-\alpha}, \quad i = 0, 1, 2, ...$$

We have

$$s'(i) = (1-\alpha)[(i+1)^{-\alpha} - i^{-\alpha}] < 0 \Rightarrow a_i > a_{i+1}, \quad i = 0, 1, ...$$

□

Lemma 3.3. *[30] If $u^m \in H_0^1, m = 0, 1, .., M$ is the solution of (2.9), then*

$$||u^m||_2 \leq ||u^0||_2 + \gamma a_{m-1}^{-1} \max_{0 \leq l \leq M} ||f^l||_2.$$

Proof. We will prove this Lemma by mathematical induction. When $m = 0$, by using (2.9), we will have

$$u^1 - \gamma \frac{\partial u^1}{\partial x^2} = u^0 + \gamma f^1.$$

Multiplying the above relation by u^1 and integrating on Ω, we will obtain the following relation

$$(u^1, u^1) - \gamma(\frac{\partial^2 u^1}{\partial x^2}, u^1) = (u^0, u^1) + \gamma(f^1, u^1),$$

i.e.

$$||u^1||_2^2 - \gamma \left(\frac{\partial^2 u^1}{\partial x_1^2}, u^1 \right) = (u^0, u^1) + \gamma(f^1, u^1). \tag{3.2}$$

By using Lemma 3.1, we get

$$(\frac{\partial^2 u^1}{\partial x_1{}^2}, u^1) = \int_\Omega \frac{\partial}{\partial x_1} \left(\frac{\partial u^1}{\partial x_1} \right) u^1 d\Omega = \underbrace{\int_{\partial\Omega} \frac{\partial u^1}{\partial x_1} u^1 ds}_{0} - \int_\Omega \frac{\partial u^1}{\partial x_1} \frac{\partial u^1}{\partial x_1} d\Omega$$

$$= - \int_\Omega \frac{\partial u^1}{\partial x_1} \frac{\partial u^1}{\partial x_1} d\Omega = -(\frac{\partial u^1}{\partial x_1}, \frac{\partial u^1}{\partial x_1}), \tag{3.3}$$

Substituting (3.3) into (3.2), we obtain

$$||u^1||_2^2 + \gamma \left(\frac{\partial^2 u^1}{\partial x_1^2}, \frac{\partial^2 u^1}{\partial x_1^2} \right) = (u^0, u^1) + \gamma(f^1, u^1), \tag{3.4}$$

Since $\gamma > 0$ and $\left(\frac{\partial^2 u^1}{\partial x_1^2}, \frac{\partial^2 u^1}{\partial x_1^2} \right) \geq 0$, we will rewrite the (3.4), as follows

$$||u^1||_2^2 \leq (u^0, u^1) + \lambda(f^1, u^1),$$

using Schwarz inequality, we get

$$||u^1||_2 \leq ||u^0||_2 + \gamma ||f^1||_2 \leq ||u^0||_2 + \gamma a_0^{-1} \max_{0 \leq l \leq M} ||f^l||_2.$$

Suppose now we have

$$||u^k||_2 \leq ||u^0||_2 + \gamma a_{k-1}^{-1} \max_{0 \leq l \leq M} ||f^l||_2, \quad k = 1, 2, \ldots, m. \tag{3.5}$$

Multiplying (2.9) by u^{m+1} and integrating on Ω, we will obtain

$$||u^{m+1}||_2^2 - \gamma(\frac{\partial^2 u^{m+1}}{\partial x_{m+1}^2}, u^{m+1}) = (1-a_1)(u^m, u^{m+1}) + \left(\sum_{k=1}^{m-1}(a_k - a_{k+1})u^{m-k}, u^{m+1}\right) + (a_m u^0, u^m + 1) + \gamma(f^{m+1}, u^{m+1}),$$

By using Schwarz inequality and the inequality in Lemma 3.2

$$a_k \geq a_{k+1}, \qquad k = 1, 2, ..., m.$$

We obtain

$$||u^{m+1}||_2 \leq (1 - a_1)||u^m||_2 + \sum_{k=1}^{m-1}(a_k - a_{k+1})||u^{m-k}||_2 + a_m||u^0||_2 + \gamma||f^{m+1}||_2.$$

By using (3.5), we get

$$\begin{aligned} ||u^{m+1}||_2 &\leq ||u^0||_2 + \left(\sum_{k=1}^{m-1}(a_k - a_{k+1})a_{m-k-1}^{-1} + 1\right) \max_{0 \leq l \leq M} ||\gamma f^l||_2 \\ &\leq ||u^0||_2 + \left(\sum_{k=1}^{m-1}(a_k - a_{k+1})a_m^{-1} + 1\right) \max_{0 \leq l \leq M} ||\gamma f^1||_2 \\ &\leq ||u^0|| + \gamma a_m^{-1} \max_{0 \leq l \leq M} ||f^l||_2. \end{aligned}$$

Hence, the proof is completed. □

Now, we will prove the stability theorem, to simplify the notations without loss of generality, let U^m be an exact solution of (2.9), we consider the case $f \equiv 0$ in stability analysis.

Theorem 3.4 (Stability theorem)**.** *The numerical implicit method defined by* (2.9)*, is unconditionally stable.*

Proof. Denote the error:

$$\xi^m = U^m - u^m, \tag{3.6}$$

It satisfies

$$\xi^{m+1} - \gamma\frac{\partial^2 \xi^{m+1}}{\partial^2 x^{m+1}} = (1 - a_1)\xi^m + \sum_{k=1}^{m-1}(a_k - a_{k+1})\xi^{m-k} + a_m\xi^0, \tag{3.7}$$

and

$$\xi^{m+1}|_{\partial\Omega} = 0, \qquad t \in [0, T].$$

By using Lemma 3.1, similar to the proof of Lemma 3.3, we will obtain

$$||\xi^m||_2 \leq ||\xi^0||_2, \qquad m = 1, 2, ..., M. \tag{3.8}$$

This proves the theorem. □

4 Convergence of the method

In this section, we will show that the approximate solution converges to the exact solution with order $O(\Delta t^{2-\alpha})$ and we will obtain an error bound for the time discrete scheme.

Theorem 4.1. *Let $u^m, m = 0,1,2,...,M$ be the approximate solution of Eq. (2.9) and the $u(x,t^m), m = 0,1,...,M$ be the exact solution of Eq. (1.1) with the above initial and boundary condition, then we have the following error estimates*

$$||u(x,t^m) - u^m||_2 \le C^\star(\Delta t^{2-\alpha}), \quad m = 1,2,...,M. \tag{4.1}$$

Where $C^\star$ is, constant.

Proof. Denote

$$\epsilon^m = u(x,t^m) - u^m. \tag{4.2}$$

From (2.8) and (2.9), we get

$$\epsilon^{m+1} - \gamma\frac{\partial^2\epsilon^{m+1}}{\partial^2 x^{m+1}} = \epsilon^m - \sum_{k=1}^{m} a_k(\epsilon^{m-k+1} - \epsilon^{m-k}) + R^1(\Delta t), \tag{4.3}$$

$$\epsilon^0 = 0, \quad \epsilon^m|\partial_\Omega = 0,$$

by using Lemma 3.3, we obtain

$$||\epsilon^m||_2 \le a_{m-1}^{-1} \max_{0\le l\le M} ||R^l||_2 \le C_3(\Delta t)^2. \tag{4.4}$$

Because

$$\lim_{m\to\infty} a_{m-1}^{-1} m^{-\alpha} = \lim_{m\to\infty} \frac{m^{-\alpha}}{m^{1-\alpha} - (m-1)^{1-\alpha}} = \lim_{m\to\infty} \frac{1}{m - (m-1)(m/m-1)^\alpha} = \frac{1}{1-\alpha}, \tag{4.5}$$

thus, $a_{m-1}^{-1}(\Delta t)^2$ is bounded, from (4.4), we will obtain

$$||u(x,t^m) - u^m||_2 \le C^\star(\Delta t^{2-\alpha}), \quad m = 1,2,\ldots,M. \tag{4.6}$$

This proves the theorem. □

5 Numerical results

In this section, we present an example to verify our theoretical finding. In this example, we will check the convergence of the numerical solution with respect to Δt.

Example 5.1. We consider the same equation as that in [11]:

$$\frac{\partial^{[\alpha} u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \qquad (x,t) \in [0,1] \times [0,1], \tag{5.1}$$

with

$$f(x,t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} sin(2\pi x) + 4\pi^2 t^2 sin(2\pi x),$$

subject to the initial condition $u_0(x) = 0$ and the homogeneous boundary conditions: $u(0,t) = u(1,t) = 0$.

Table 5.1: The error and the convergence rate for $\alpha = 0.1$

M	N	The error	Convergence rate
30	30	0.00355	–
60	60	$8.90771 * 10^{-4}$	1.99469
100	100	$3.20611 * 10^{-4}$	2.00041
150	150	$1.42463 * 10^{-4}$	2.00053
200	200	$8.01563 * 10^{-5}$	1.99910

Table 5.2: The error and the convergence rate for $\alpha = 0.5$

M	N	The error	Convergence rate
30	30	0.00358	–
60	60	$9.05205 * 10^{-4}$	1.98357
100	100	$3.28401 * 10^{-4}$	1.98497
150	150	$1.47078 * 10^{-4}$	1.98111
200	200	$8.33012 * 10^{-5}$	1.97614

The exact solution to the problem is given by $u = t^2 sin(2\pi x)$. Taking $\Delta(t) = \frac{1}{M}$ and $h = \frac{1}{N}$, where N and M are the numbers of meshes in space and time, in this example, we use $N = M$. The $\frac{\partial^2 u(x,t_{m+1})}{\partial x^2}$ is approximated as follows:

$$\frac{\partial^2 u(x, t_{m+1})}{\partial x^2} \approx \frac{u(x_{n+1}, t_{m+1}) - 2u(x_n, t_{m+1}) + u(x_{n-1}, t_{m+1})}{h^2}. \tag{5.2}$$

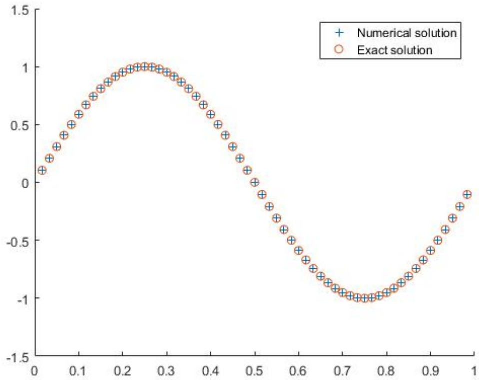
The rates of convergence are computed by

$$\text{rate} = \frac{Ln(e_{\text{new}}/e_{\text{old}})}{Ln\left((\Delta t)_{\text{new}}/(\Delta t)_{\text{old}}\right)}.$$

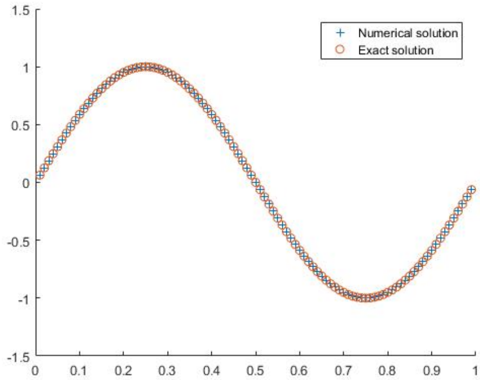
The errors in our examples are denoted by $max\{|u^m - U^m| : m = 1, 2, ..., M\}$. The convergence rate and the errors for different α and M are presented in Tables (5.1-5.4). We can see that the convergence rate for time is close to Δt^2. The numerical results are consistent with our theoretical results in theorem 3.4. The comparison of the exact and approximate solutions with $\alpha = 0.1$ at different M and the comparison of the exact and approximate solutions with $\alpha = 0.9$ at different M are shown (see Figs5.1 and 5.2). All the calculations in this example are performed using MATLAB 2016.

Table 5.3: The error and the convergence rate for $\alpha = 0.7$

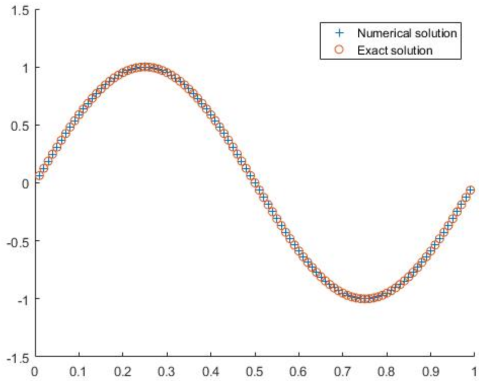
M	N	The error	Convergence rate
30	30	0.00369	–
60	60	$9.55878 * 10^{-4}$	1.94872
100	100	$3.56404 * 10^{-4}$	1.93132
150	150	$1.64371 * 10^{-4}$	1.97877
200	200	$9.55343 * 10^{-5}$	1.88625



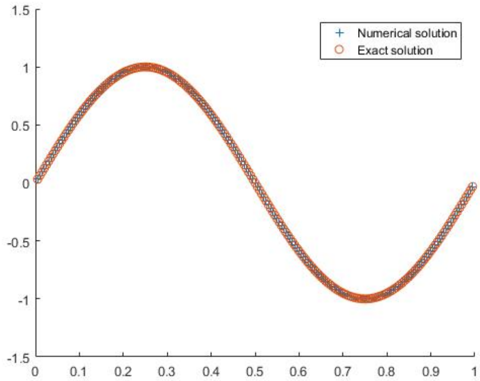
(a) $M = 60$



(b)



(c) $M = 150$

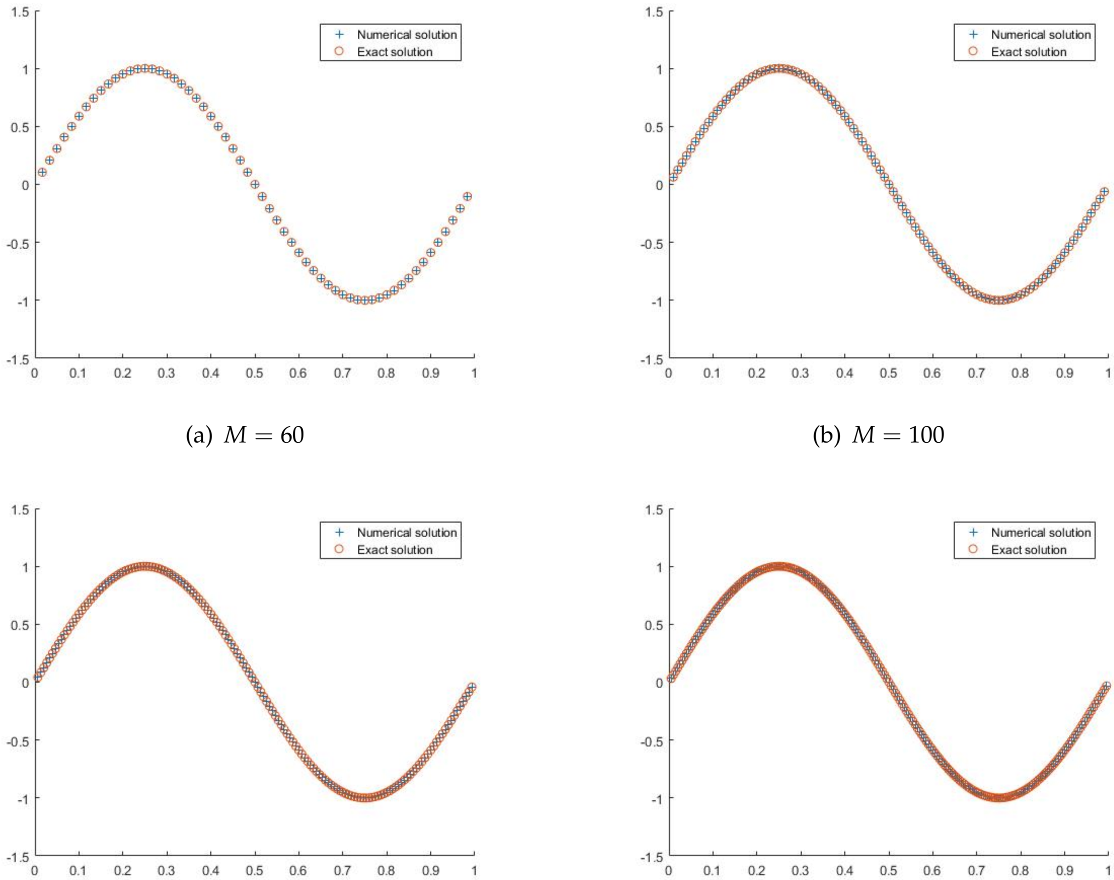


(d) $M = 200$

Figure 5.1: The comparison of the exact and approximate solutions with $\alpha = 0.1$ at different M for test problem 5.1.

Table 5.4: The error and the convergence rate for $\alpha = 0.9$

M	N	The error	Convergence rate
30	30	0.00400	–
60	60	0.00112	1.83650
100	100	$4.53413 * 10^{-4}$	1.77023
150	150	$2.28800 * 10^{-4}$	1.68684
200	200	$1.43598 * 10^{-4}$	1.61925



(a) $M = 60$ (b) $M = 100$

(c) $M = 150$ (d) $M = 200$

Figure 5.2: The comparison of the exact and approximate solutions with $\alpha = 0.9$ at different M for test problem 5.1.

6 Concluding remarks

In this paper, we studied an implicit discrete scheme to solve the time-fractional diffusion equation. The error estimates and the stability of the proposed method are discussed. The convergence rate of the proposed method was proved to be optimal. An example was provided to illustrate the capability and accuracy of the method. Constructing more efficient algorithms is also our goal in future works.

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Conflict of Interest

The authors have no conflicts of interest to declare.

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