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Periodic Solutions of Nonsmooth Third-Order Differential Equations

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Abstract. In this paper, we study the existence of periodic solutions for the following piecewise third order differential equation

$$\dddot{x} + \dot{x} - \varepsilon \sum_{i=1}^{2} c_i |x|^i = 0,$$

with ε a real parameter sufficiently small, c_1 and c_2 real numbers. By applying new results from the averaging theory for continuous differential systems, we prove the existence of at most one periodic solution for the differential equation. An example is given to illustrate the established result.

Keywords: Averaging theory, Periodic Solutions, Differential equations.

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1 Introduction and main results

As far as we know, the study of piecewise vector fields goes back to Andronov, Vitt, and Khaikin [1] and still continues to receive strong attention from researchers. Since piecewise vector fields are widely used to model processes appearing in electronics, mechanics, economics, etc., there has been a strong interest in the mathematical community working in differential equations to understand the dynamical richness of these vector fields. For examples, see the book [8], the survey of [7], and the hundreds of references cited in these last three works.

For studying some electrical circuits, Sprott [9,10], and Sun and Sprott [11] considered the third order differential equation

$$\dddot{x} + b\ddot{x} + \dot{x} - g(x) = 0 \tag{1.1}$$

with b a real number and $g(x)$ is an elementary piecewise function. The authors in [9–11] showed that some of these equations exhibit chaos.

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Llibre et al in [5] by using the averaging method of first order investigate the periodic solutions of the differential equation (1.1) with $b = -\varepsilon$ and $g(x) = -\varepsilon a x^m$ where a, m are parameters and ε is a nonzero real parameter sufficiently small. In other words, they study the existence of periodic solutions of the following differential equation

$$\dddot{x} - \varepsilon|\ddot{x}| + \dot{x} + \varepsilon a x^m = 0, \tag{1.2}$$

and obtain that for m is even and $a > 0$ then system (1.2) has a periodic solution. However in all other cases of the parameter a and the exponent m the averaging theory of first order does not provide any information on the periodic solutions of system (1.2).

In this paper, we consider the differential equation (1.1) with $b = 0$ and $g(x) = \varepsilon \sum_{i=1}^{2} c_i |x|^i$. In other words we consider the following differential equation

$$\dddot{x} + \dot{x} - \varepsilon \sum_{i=1}^{2} c_i |x|^i = 0, \quad (x(0), \dot{x}(0), \ddot{x}(0)) = (x_0, \dot{x}_0, \ddot{x}_0), \tag{1.3}$$

where ε is a nonzero real parameter sufficiently small and c_1 and c_2 are nonzero real numbers. Here we are interested in knowing how the parameters c_1 and c_2 affect the periodic solutions of this problem.

Using the change of variable $\dot{x} = y, \dot{y} = z$, the differential equation (1.3) can be written as

$$\begin{aligned} \dot{x} &= \ y, \\ \dot{y} &= \ z, \\ \dot{z} &= \ -y + \varepsilon \sum_{i=1}^{2} c_i |x|^i, \end{aligned} \tag{1.4}$$

with $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$ and the dot denotes the derivative with respect to the variable t (i.e. $\dot{} := \frac{d}{dt}$).

We note that the existence of the term $|x|$ makes the differential system (1.4) only continuous, consequently, we are unable to apply the classical averaging theory to it in order to study its periodic solutions since that theory needs the differential system to be of class C^2. The averaging theory has recently been extended to encompass the only continuous differential systems, for more details, see Section 2.

Using the first order averaging method, we obtain the following main result.

Theorem 1.1. *For $|\ \varepsilon\ | \neq 0$ a real parameter sufficiently small, x_0 and r_0 are the initial conditions of the problem (1.4). The following statements hold.*

(a) If c_1 and c_2 have different signs, then the differential equation (1.4) has a periodic solution.

(b) If c_1 and c_2 have the same signs and $\left|\frac{x_0 + r_0}{r_0}\right| < 1$ with $r_0^2 = y_0^2 + z_0^2$, then the averaging method of first order does not provide any information on the periodic solutions of (1.4).

Remark 1.2. If either $c_1 = 0$ or $c_2 = 0$, then the averaging method of first order does not provide periodic solutions.

This paper is organized as follows: Theorem 1.1 is proved in section 3. Their proof is based on the first order averaging theory, see Theorem 2.1 in section 2. An example is given in section 4 to illustrate the established results.

2 Averaging method for continuous differential systems

For proving Theorem 1.1 we apply the recent result from the averaging theory for the continuous piecewise linear differential systems. In this section we present this result and some necessary remarks for their applications. From Theorem B of [6] we get the following result adapted to the next system (2.1). See also Theorems 11.5 and 11.6 of Verhulst [12]. The following theorem can be applied to general nonsmooth differential systems.

Theorem 2.1. *consider the following differential system*

$$\dot{x} = F_0(t,x) + \varepsilon F_1(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{2.1}$$

where $\varepsilon \neq 0$ is a small parameter, $D \subset \mathbb{R}^n$ is an open subset, the functions $F_i(t,x) : \mathbb{R} \times D \to \mathbb{R}^n$ for $i = 1,2$ and $R(t,x,\varepsilon) : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $T-$periodic functions in the variable t, and for each $t \in \mathbb{R}$, the functions $F_0(t,.) \in C^1$, $F_1(t,.) \in C^0$, $D_x F_0$ and $R \in C^0$ are locally Lipschitz in the second variable x. We denote by $x(t,z,\varepsilon)$ the solution of system (2.1), such that $x(0,z,\varepsilon) = z$. Assume that there exists an open and bounded subset V of D with its closure $\bar{V} \subset D$, such that for each $z \in \bar{V}$, the solution $x(t,z,0)$ is T periodic. We denote by $M_z(t)$ the fundamental matrix solution of the variational equation:

$$\dot{x}(t) = D_x F_0(t, x(t,z,0)),$$

associated with the periodic solution $x(t,z,0)$, such that $M_z(0)$ is the identity.
If $a \in V$ is a zero of the map $\mathcal{F} : \bar{V} \to \mathbb{R}^n$ defined by

$$\mathcal{F}(z) = \int_0^T M_0^{-1}(t) F_1(t, x(t,z))dt,$$

and $\det(D_z\mathcal{F}(a)) \neq 0$, then for $\varepsilon > 0$ sufficiently small, system (2.1) has a T periodic solution $x(t,a_\varepsilon,\varepsilon)$, such that $a_\varepsilon \to a$ as $\varepsilon \to 0$. Moreover, the linear stability type of the periodic solution $x(t,a_\varepsilon,\varepsilon)$ is given by the eigenvalues of the matrix $D_z\mathcal{F}(a)$.

The averaging theory is one of the best methods for obtaining analytically periodic solutions of the differential equations, see for instance [3–5] and the references therein.

3 Proof of Theorem 1.1

In this section, we will prove Theorem 1.1 by using the first order averaging method for continuous differential systems which is presented in Theorem 2.1.
Proof of statement (a) of Theorem 1.1

In order to apply the averaging method of first order, we should transform system (1.4) into the standard form (2.1). In cylindrical coordinates $x = x$, $y = r\sin\theta$ and $z = r\cos\theta$, the system (2.1) becomes

$$\begin{aligned} \dot{x} &= r\sin\theta, \\ \dot{r} &= \varepsilon\cos\theta \sum_{i=1}^{2} c_i |x|^i, \\ \dot{\theta} &= 1 - \frac{\varepsilon}{r}\sin\theta \sum_{i=1}^{2} c_i |x|^i. \end{aligned} \tag{3.1}$$

Taking θ as the new independent variable, we can write the previous differential system as

$$\begin{aligned} x' &= r\sin\theta + \varepsilon\sin^2\theta\sum_{i=1}^{2} c_i|x|^i + o(\varepsilon^2), \\ r' &= \varepsilon\cos\theta\sum_{i=1}^{2} c_i|x|^i + o(\varepsilon^2), \end{aligned} \tag{3.2}$$

with $' := \frac{d}{d\theta}$. The unperturbed system is

$$\begin{aligned} x' &= r\sin\theta, \\ r' &= 0. \end{aligned} \tag{3.3}$$

The solution of system (3.3) is given by $(x(\theta, x_0, r_0), r(\theta, x_0, r_0)) = (x_0 + r_0(1 - \cos\theta), r_0)$, with $(x(0, x_0, r_0), r(0, x_0, r_0)) = (x_0, r_0)$. It is clear that the solution of system (3.3) is 2π periodic for all $(x_0, r_0) \neq (0, 0)$. If $r_0 = 0$ we have a straight line of equilibrium points.

Now, note that the function $F_0(\theta, (x, r)) = (r\sin\theta, 0)$ is C^1 and that the function $F_1(\theta, (x, r)) = \sum_{i=1}^{2} c_i|x|^i(\sin^2\theta, \cos\theta)$ is C^0, and both are Lipschitz. Therefore, system (3.1) satisfies the assumptions of Theorem 2.1. Then, by Theorem 2.1, we need to compute the averaged function

$$\mathcal{F}(z) = \int_0^T M_0^{-1}(t)F_1(t, x(t, z))dt,$$

where

$$M_0(\theta) = \begin{pmatrix} 1 & 1 - \cos\theta \\ 0 & 1 \end{pmatrix}, \tag{3.4}$$

is the fundamental matrix of the variational differential system associated with system (3.3) evaluated on the periodic solution $(x_0 + r_0(1 - \cos\theta), r_0)$, such that $M(\theta)$ is the identity matrix. Therefore, we have

$$\begin{aligned} \mathcal{F}(x_0, r_0) &= \int_0^{2\pi} \sum_{i=1}^{2} c_i|x_0 + r_0(1 - \cos\theta)|^i \begin{pmatrix} 1 & 1 - \cos\theta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin^2\theta \\ \cos\theta \end{pmatrix} d\theta \\ &= \int_0^{2\pi} \sum_{i=1}^{2} c_i|x_0 + r_0(1 - \cos\theta)|^i \begin{pmatrix} 1 - \cos\theta \\ \cos\theta \end{pmatrix} d\theta \\ &= \int_0^{2\pi} f(\theta)d\theta, \end{aligned}$$

and considering the change of variable $\theta = \phi + \pi$ in the interval $[0, 2\pi]$ and the symmetry $\cos(\theta + \pi) = -\cos\theta$, we have that

$$\mathcal{F}(x_0, r_0) = 2\int_0^{\pi} f(\theta)d\theta.$$

To calculate the explicit form of $\mathcal{F}(x_0, r_0)$, we should study the zeros of $x_0 + r_0(1 - \cos\theta)$. So $x_0 + r_0(1 - \cos\theta) = 0$ if and only if $\theta = \pm\arccos\left(\frac{x_0 + r_0}{r_0}\right)$ and the function $\arccos x$ takes real

value when $x \in [-1,1]$. Now, we must consider the following three cases.
Case 1 $\frac{x_0+r_0}{r_0} \le -1$, then $x_0 + r_0(1-\cos\theta) \le 0$ in $[0,\pi]$.
Case 2 $\left|\frac{x_0+r_0}{r_0}\right| < 1$, then $x_0 + r_0(1-\cos\theta) < 0$ if $\theta \in (0, \arccos(\frac{x_0+r_0}{r_0}))$ and $x_0 + r_0(1-\cos\theta) > 0$ if $\theta \in (\arccos(\frac{x_0+r_0}{r_0}), \pi)$.
Case 3 $\frac{x_0+r_0}{r_0} \ge 1$, then $x_0 + r_0(1-\cos\theta) \ge 0$ in $[0,\pi]$.

So, the explicit form of $\mathcal{F}(x_0, r_0)$ depends on the three previous cases.

Case 1 The averaged function is

$$\begin{aligned}\mathcal{F}(x_0,r_0) &= 2\int_0^{\pi}\sum_{i=1}^{2} c_i(-(x_0+r_0(1-\cos\theta))^i \begin{pmatrix}1-\cos\theta\\ \cos\theta\end{pmatrix} d\theta \\ &= \pi\left(\left(5\,{r_0}^2+6\,r_0x_0+2\,{x_0}^2\right)c_2-3\,c_1\left(r_0+2/3\,x_0\right), -2\,r_0\left(\left(r_0+x_0\right)c_2-1/2\,c_1\right)\right).\end{aligned}$$

In order to determine the existence of periodic solutions for system (1.4), we should solve the system

$$\left(5\,{r_0}^2+6\,r_0x_0+2\,{x_0}^2\right)c_2-3\,c_1\left(r_0+2/3\,x_0\right)=0, \tag{3.5}$$
$$-2\,r_0\left(\left(r_0+x_0\right)c_2-1/2\,c_1\right)=0. \tag{3.6}$$

From (3.6), we get $x_0 = \frac{c_1}{2c_2} - r_0$. Now by substituting in (3.5), we obtain

$$-\frac{-2\,{r_0}^2{c_2}^2+{c_1}^2}{2c_2}=0. \tag{3.7}$$

Solving this equation in the variable r_0 and since r_0 must be positive, we get

$$^{\pm}r_0^* = \pm\frac{c_1}{\sqrt{2}c_2}.$$

Then

$$^{-}x_0^* = \left(\sqrt{2}+1\right)\frac{c_1}{2c_2}, \quad ^{+}x_0^* = -\left(\sqrt{2}-1\right)\frac{c_1}{2c_2}.$$

Note that c_1 and c_2 have different signs ensure the existence of solution $^{-}r_0^* = -\frac{c_1}{\sqrt{2}c_2}$ and if c_1 and c_2 have same sign then the equation (3.7) has one real positive solution $^{+}r_0^* = \frac{c_1}{\sqrt{2}c_2}$.

In summary, for c_1 and c_2 are nonzero real numbers, the averaged function $\mathcal{F}(x_0,r_0)$ has the unique zero either $(^{-}x_0^*, ^{-}r_0^*)$ or $(^{+}x_0^*, ^{+}r_0^*)$.
In addition we have

$$\det(D\mathcal{F}(^{\pm}x_0^*, ^{\pm}r_0^*)) = 2(c_1\pi)^2.$$

Hence, the assumptions of Theorem 2.1 are verified for the continuous differential system (4.3).
From Theorem 2.1, it follows that system (4.3) has the periodic solution

$$x(\theta,\varepsilon) = {}^{\pm}x_0^* + o(\varepsilon), \quad r(\theta,\varepsilon) = {}^{\pm}r_0^* + o(\varepsilon).$$

Moreover, the eigenvalues of the Jacobian matrix of the map (f_1, f_2) evaluated at the solution $(^{\pm}x_0, ^{\pm}r_0)$ are $\pm i\pi\sqrt{2}c_1$, so the periodic solution is linearly stable.

Going back through the change of cylindrical coordinates, system (1.4) has the periodic solution

$$
\begin{aligned}
x(t,\varepsilon) &= {}^{-}x_0^* + o(\varepsilon),\\
y(t,\varepsilon) &= {}^{-}r_0^* \sin t + o(\varepsilon),\\
z(t,\varepsilon) &= {}^{-}r_0^* \cos t + o(\varepsilon).
\end{aligned}
$$

with c_1 and c_2 have a different sign, or

$$
\begin{aligned}
x(t,\varepsilon) &= {}^{+}x_0^* + o(\varepsilon),\\
y(t,\varepsilon) &= {}^{+}r_0^* \sin t + o(\varepsilon),\\
z(t,\varepsilon) &= {}^{+}r_0^* \cos t + o(\varepsilon).
\end{aligned}
$$

with c_1 and c_2 have the same sign.
Case 3 Similar to **Case 1**, we have

$$
\mathcal{F}(x_0,r_0) = \pi(\,(5\,{r_0}^2 + 6\,r_0x_0 + 2\,{x_0}^2)\,c_2 + 3\,c_1\,(r_0 + 2/3\,x_0)\,, -2\,((r_0+x_0)\,c_2 + 1/2\,c_1)\,r_0).
$$

Analogously to Case 1, we obtain that system (1.4) has the periodic solution

$$
\begin{aligned}
x(t,\varepsilon) &= {}^{+}x_0^* + o(\varepsilon),\\
y(t,\varepsilon) &= {}^{-}r_0^* \sin t + o(\varepsilon),\\
z(t,\varepsilon) &= {}^{-}r_0^* \cos t + o(\varepsilon),
\end{aligned}
$$

with c_1 and c_2 have a different sign, or

$$
\begin{aligned}
x(t,\varepsilon) &= {}^{-}x_0^* + o(\varepsilon),\\
y(t,\varepsilon) &= {}^{+}r_0^* \sin t + o(\varepsilon),\\
z(t,\varepsilon) &= {}^{+}r_0^* \cos t + o(\varepsilon),
\end{aligned}
$$

with c_1 and c_2 have the same sign.
Case 2 The averaged function is

$$
\begin{aligned}
\mathcal{F}(x_0,r_0) &= 2\int_0^{\pi} \sum_{i=1}^{2} c_i |x_0 + r_0(1-\cos\theta)|^i \begin{pmatrix} 1-\cos\theta \\ \cos\theta \end{pmatrix} d\theta\\
&= \begin{pmatrix} f_1(x_0,r_0) \\ f_2(x_0,r_0) \end{pmatrix},
\end{aligned}
$$

with

$$
\begin{aligned}
f_1(x_0,r_0) = & \int_0^{\pi} (1-\cos\theta) \sum_{i=1}^{2} c_i |x_0 + r_0(1-\cos\theta)|^i d\theta\\
= & \int_0^{\arccos\left(\frac{x_0+r_0}{r_0}\right)} (1-\cos\theta) \sum_{i=1}^{2} c_i(-(x_0 + r_0(1-\cos\theta))^i d\theta \qquad (3.8)\\
& + \int_{\arccos\left(\frac{x_0+r_0}{r_0}\right)}^{\pi} (1-\cos\theta) \sum_{i=1}^{2} c_i (x_0 + r_0(1-\cos\theta)^i d\theta,
\end{aligned}
$$

and

$$f_1(x_0,r_0) = \int_0^{\pi} \cos\theta \sum_{i=1}^{2} c_i |x_0 + r_0(1-\cos\theta)|^i d\theta$$
$$= \int_0^{\arccos\left(\frac{x_0+r_0}{r_0}\right)} \cos\theta \sum_{i=1}^{2} c_i (-(x_0 + r_0(1-\cos\theta))^i d\theta \tag{3.9}$$
$$+ \int_{\arccos\left(\frac{x_0+r_0}{r_0}\right)}^{\pi} \cos\theta \sum_{i=1}^{2} c_i (x_0 + r_0(1-\cos\theta)^i d\theta.$$

Using this equality $\arccos\theta + \arcsin\theta = \pi/2$ and taking $x_0 = -r_0 + Xr_0$, we get

$$f_1(x_0,r_0) = 2\,r_0 \left(c_1\,(X+2)\,\sqrt{-X^2+1} + c_1\,(2\,X+1)\arcsin(X) + c_2 r_0 \pi\,\left(X^2+X+1/2\right)\right),$$
$$f_2(x_0,r_0) = r_0 \left(-2\,c_1 X\,\sqrt{-X^2+1} - 2\,\pi\,X c_2 r_0 - 2\,c_1 \arcsin(X)\right).$$

Now, we should solve the system $f_1(x_0,r_0) = 0$, $f_2(x_0,r_0) = 0$. From the second equation, it follows that $X = 0$, and from the first equation that $r_0^* = -\frac{4c_1}{\pi c_2}$. Therefore, we have the solution $(x_0,r_0) = \left(\frac{4c_1}{\pi c_2}, -\frac{4c_1}{\pi c_2}\right)$. Since r_0 must be positive, then c_1 and c_2 must have different sign. The Jacobian of the map (f_1,f_2) evaluated at $(x_0,r_0) = \left(\frac{4c_1}{\pi c_2}, \frac{-4c_1}{\pi c_2}\right)$ is $16c_1^2$. It follows from Theorem 2.1 and for ε a non-zero real parameter sufficiently small that system (4.3) has a periodic solution $\phi(\theta,\varepsilon) = (x(\theta,\varepsilon), r(\theta,\varepsilon)) = \left(\frac{4c_1}{\pi c_2} + o(\varepsilon), -\frac{4c_1}{\pi c_2} + o(\varepsilon)\right)$. In addition, the eigenvalues of the Jacobian matrix of the map (f_1,f_2) evaluated at the solution $\left(\frac{4c_1}{\pi c_2}, \frac{-4c_1}{\pi c_2}\right)$ are $\pm 4c_1$, so the periodic solution is linearly stable.

Now we must identify the periodic solution of system (1.4) which corresponds to the periodic solution found. Going back to system (1.4) with the independent variable t, we obtain the periodic solution:

$$\big(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon)\big) = \left(\frac{4c_1}{\pi\,c_2}, -\frac{4c_1}{\pi\,c_2}\cos t, -\frac{4c_1}{\pi\,c_2}\sin t\right) + o(\varepsilon).$$

This completes the Proof the statement (a) of Theorem 1.1.

Proof of statement (b) of Theorem 1.1

If $\left|\frac{x_0+r_0}{r_0}\right| < 1$, then the unique zero of the averaged function $\mathcal{F}(x_0,r_0)$ is $(x_0,r_0) = \left(\frac{4c_1}{\pi c_2}, -\frac{4c_1}{\pi c_2}\right)$. So for c_1 and c_2 have the same sign, then $r_0 < 0$. Because of r_0 must be positive, then the averaged function $\mathcal{F}(x_0,r_0)$ has no zero in this case. This completes the Proof the statement (b) of Theorem 1.1.

4 Example

In this section, we provide an application of Theorem 1.

Consider the perturbed system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -y + \varepsilon\big(-\pi c_2/4|x| + c_2 x^2\big), \end{aligned} \tag{4.1}$$

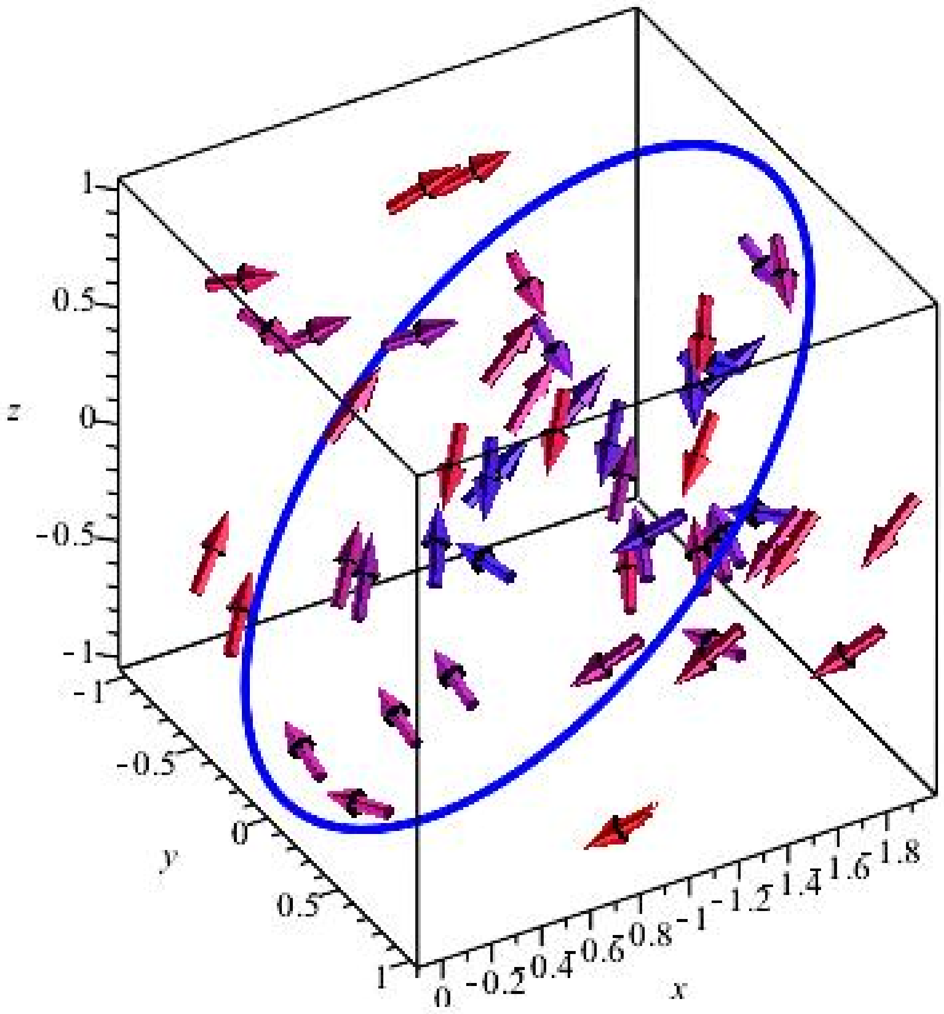

Figure 4.1: The periodic orbit of the differential system (4.1) with $c_2 = 1$ and $\varepsilon = 10^{-4}$ using the initial conditions $x_0 = -0.25,\ y_0 = -z_0 = 0.7$.

with ε a nonzero real parameter sufficiently small, c_2 a nonzero real number.

System (4.1) in the previous cylindrical coordinates can be written as

$$\begin{aligned} \dot{x} &= r\sin\theta, \\ \dot{r} &= \varepsilon c_2 \cos\theta\big(-\pi/4|x| + x^2\big), \\ \dot{\theta} &= 1 - \frac{\varepsilon c_2}{r}\sin\theta\big(-\pi/4|x| + x^2\big). \end{aligned} \tag{4.2}$$

Taking θ as a new independent variable, system (4.2) is equivalent with the following system

$$\begin{aligned} x' &= r\sin\theta + \varepsilon c_2 \sin^2\theta\big(-\pi/4|x| + x^2\big) + o(\varepsilon^2), \\ r' &= \varepsilon c_2 \cos\theta\big(-\pi/4|x| + x^2\big) + o(\varepsilon^2), \end{aligned} \tag{4.3}$$

The unperturbed system and the fundamental matrix associated with it are given in (3.4) and (3.3), respectively. For system (4.1) and by taking $x_0 = -r_0 + r_0 X$, the averaged function is given by

$$\mathcal{F}(x_0, r_0) = \frac{\pi c_2}{4} r_0 \big(f_1(x_0, r_0), f_2(x_0, r_0)\big),$$

with

$$\begin{aligned} f_1(x_0, r_0) = \; & 4\, r_0 \left(2\, X^2 + 2\, X + 1\right) - 2\, \left(X + 2\right) \sqrt{-X^2 + 1} - 2\, \left(2\, X + 1\right) \arcsin\left(X\right), \\ f_2(x_0, r_0) = \; & 2\, X \sqrt{-X^2 + 1} - 8\, X r_0 + 2\, \arcsin\left(X\right). \end{aligned}$$

It is clearly that $X = 0$ is the only solution of the equation $f_2(X, r_0) = 0$. Substituting in $f_1(X, r_0) = 0$, we obtain

$$\pi\, c_2 r_0 \left(-1 + r_0\right).$$

As consequence, the $f_1(x_0, r_0)$ only vanishes at $r_0 = 1$. So the unique zero of the averaged function $\mathcal{F}(x_0, r_0)$ is $(-1, 1)$. Moreover, the Jacobian matrix of $\mathcal{F}(x_0, r_0)$ evaluated at the zero $(-1, 1)$ is

$$\pi c_2 \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix},$$

and its determinant is $(\pi c_2)^2$. Thus from Theorem 2.1, system(4.1) has the periodic solution (see Figure 4.1).

$$\begin{aligned} x(t, \varepsilon) &= -1 + o(\varepsilon), \\ y(t, \varepsilon) &= \cos\theta + o(\varepsilon), \\ z(t, \varepsilon) &= \sin\theta + o(\varepsilon). \end{aligned}$$

Also we have that the eigenvalues of Jacobian matrix of $\mathcal{F}(x_0, r_0)$ are $\pm i\pi c_2$. As a result this periodic solution is linearly stable.

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