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A biparameterized analysis of integral inequalities for bounded and holderian mappings

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Abstract. In this study, we introduce a new parameterized identity that generates a series of Newton-Cotes formulas for one, two, three, and four points. We then derive several novel Newton-Cotes-type inequalities for functions with bounded and r-L-Hölderian derivatives. The research is finalized with numerical examples and graphical illustrations that validate the precision of our findings.

Keywords: Newton-Cotes inequalities, Lipschitzian functions, Hölderian functions, bounded functions.

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1 Introduction

In recent years, numerous works have investigated error estimations for Newton-Cotes formulas, such as Hermite-Hadamard, Ostrowski, and Simpson-type inequalities for different classes of functions, see [2,6,7,9,12,13,15,16,18,19] and the references cited therein.

However, these inequalities are seldom investigated for Lipschitzian functions. Notable works on this subject include the following:

In [5], Dragomir et al. presented some inequalities of Hadamard's type for Lipschitzian mappings. In [20], Tseng et al. established several Hadamard and Bullen-type inequalities for the same class of functions. Moreover, the authors in [10], offered various Hadamard-type inequalities for Lipschitzian functions in one and two variables. More importantly, new refinements of Hadamard's type inequalities for Lipschitzian mappings are given in [21,22]. More recently, the authors in [4] studied a parameterized three-point Newton-Cotes formula and deduced several inequalities of midpoint-, trapezium-, Bullen- and Simpson-like-type for differentiable Bounded and Lipschitzian functions.

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Parameter x	Parameter λ	$\mathcal{F}(a,x,b;\lambda)$	Formula
a	$[0,1]$	$\frac{\varphi(a)+\varphi(b)}{2}$	Trapezium
$\frac{a+b}{2}$	0	$\varphi\left(\frac{a+b}{2}\right)$	Midpoint
	1	$\frac{\varphi(a)+\varphi(b)}{2}$	Trapezium
	$1/2$	$\frac{1}{4}\varphi(a)+\frac{1}{2}\varphi\left(\frac{a+b}{2}\right)+\frac{1}{4}\varphi(b)$	Bullen
	$1/3$	$\frac{1}{6}\left(\varphi(a)+4\varphi\left(\frac{a+b}{2}\right)+\varphi(b)\right)$	Simpson
$\left[a,\frac{a+b}{2}\right]$	0	$\frac{\varphi(x)+\varphi(a+b-x)}{2}$	Companion Ostrowski
$\frac{2a+b}{3}$	$3/8$	$\frac{1}{8}\left(\varphi(a)+3\varphi\left(\frac{2a+b}{3}\right)+3\varphi\left(\frac{a+2b}{3}\right)+\varphi(b)\right)$	Simpson $3/8$

Table 1.1: Derived formulas

Drawing inspiration from the aforementioned papers and the parameterized strategies employed in [1,8,17,23,25], in this work, we focus on the general form of the 4-point Newton-Cotes rule. A two-parameter formula that enables us to recover all the famous 1, 2, 3, and 4-point formulas. Firstly, we introduce a new identity related to the formula under consideration. Based on that identity, we provide several Newton-Cotes-type inequalities for functions with bounded as well as r-L-Hölderian derivatives. The study is concluded with a series of numerical examples with 2 and 3-dimensional graphical representations.

Let us consider the following parameterized four-point quadrature rule:

$$\mathcal{F}(a,x,b;\lambda)=\frac{\lambda(x-a)}{b-a}\varphi(a)+\frac{2(1-\lambda)(x-a)+a+b-2x}{2(b-a)}\left(\varphi(x)+\varphi(a+b-x)\right)+\frac{\lambda(x-a)}{b-a}\varphi(b), \quad (1.1)$$

where $x\in[a,\frac{a+b}{2}]$ and $\lambda\in[0,1]$.

Here, we note that employing formula (1.1) along with specific values for the parameters x and λ enables us to derive all the well-known formulas, see Table 1.1

2 Main results

Lemma 2.1. *Let $\varphi: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on (a,b) with $a<b$, and $\varphi'\in L^1[a,b]$, then the following equality holds for all real number $\lambda\in[0,1]$ and $x\in\left[a,\frac{a+b}{2}\right]$*

$$\begin{aligned}\mathcal{F}(a,x,b;\lambda)-\frac{1}{b-a}\int_a^b\varphi(u)\,du=&\frac{(x-a)^2}{b-a}\int_0^1(t-\lambda)\,\varphi'((1-t)\,a+tx)\,dt\\&+\frac{(a+b-2x)^2}{4(b-a)}\int_0^1(t-1)\,\varphi'\left((1-t)\,x+t\frac{a+b}{2}\right)dt\\&+\frac{(a+b-2x)^2}{4(b-a)}\int_0^1 t\varphi'\left((1-t)\,\frac{a+b}{2}+t\,(a+b-x)\right)dt\end{aligned}$$

$$+\frac{(x-a)^2}{b-a}\int_0^1 (t-(1-\lambda))\,\varphi'\left((1-t)(a+b-x)+tb\right)dt,$$

where $\mathcal{F}(a,x,b;\lambda)$ *is defined as in* (1.1).

Proof. Let

$$I=\frac{(x-a)^2}{b-a}I_1+\frac{(a+b-2x)^2}{4(b-a)}I_2+\frac{(a+b-2x)^2}{4(b-a)}I_3+\frac{(x-a)^2}{b-a}I_4, \tag{2.1}$$

where

$$I_1=\int_0^1 (t-\lambda)\,\varphi'\left((1-t)a+tx\right)dt,$$

$$I_2=\int_0^1 (t-1)\,\varphi'\left((1-t)x+t\tfrac{a+b}{2}\right)dt,$$

$$I_3=\int_0^1 t\varphi'\left((1-t)\tfrac{a+b}{2}+t(a+b-x)\right)dt$$

and

$$I_4=\int_0^1 (t-(1-\lambda))\,\varphi'\left((1-t)(a+b-x)+tb\right)dt.$$

Integrating by parts I_1, we get

$$\begin{aligned}I_1&=\int_0^1 (t-\lambda)\,\varphi'\left((1-t)a+tx\right)dt\\&=\frac{1}{x-a}(t-\lambda)\,\varphi\left((1-t)a+tx\right)\Big|_{t=0}^{t=1}-\frac{1}{x-a}\int_0^1\varphi\left((1-t)a+tx\right)dt\\&=\frac{1-\lambda}{x-a}\varphi(x)+\frac{\lambda}{x-a}\varphi(a)-\frac{1}{(x-a)^2}\int_a^x\varphi(u)\,du.\end{aligned} \tag{2.2}$$

Similarly, we get

$$\begin{aligned}I_2&=\int_0^1 (t-1)\,\varphi'\left((1-t)x+t\tfrac{a+b}{2}\right)dt\\&=\frac{2}{a+b-2x}(t-1)\,\varphi\left((1-t)x+t\tfrac{a+b}{2}\right)\Big|_{t=0}^{t=1}-\frac{2}{a+b-2x}\int_0^1\varphi\left((1-t)x+t\tfrac{a+b}{2}\right)dt\\&=\frac{2}{a+b-2x}\varphi(x)-\frac{4}{(a+b-2x)^2}\int_x^{\frac{a+b}{2}}\varphi(u)\,du,\end{aligned} \tag{2.3}$$

$$I_3=\int_0^1 t\varphi'\left((1-t)\tfrac{a+b}{2}+t(a+b-x)\right)dt$$

$$= \tfrac{2}{a+b-2x} t\varphi\left(\left(1-t\right)\tfrac{a+b}{2}+t\left(a+b-x\right)\right)\Big|_{t=0}^{t=1} - \tfrac{2}{a+b-2x}\int_0^1 \varphi\left(\left(1-t\right)\tfrac{a+b}{2}+t\left(a+b-x\right)\right)dt$$

$$= \tfrac{2}{a+b-2x}\varphi\left(a+b-x\right) - \tfrac{4}{\left(a+b-2x\right)^2}\int_{\frac{a+b}{2}}^{a+b-x} \varphi\left(u\right)du \tag{2.4}$$

and

$$I_4 = \int_0^1 \left(t-\left(1-\lambda\right)\right)\varphi'\left(\left(1-t\right)\left(a+b-x\right)+tb\right)dt$$

$$= \tfrac{1}{x-a}\left(t-\left(1-\lambda\right)\right)\varphi\left(\left(1-t\right)\left(a+b-x\right)+tb\right)\Big|_{t=0}^{t=1} - \tfrac{1}{x-a}\int_0^1 \varphi\left(\left(1-t\right)\left(a+b-x\right)+tb\right)dt$$

$$= \tfrac{\lambda}{x-a}\varphi\left(b\right) + \tfrac{1-\lambda}{x-a}\varphi\left(a+b-x\right) - \tfrac{1}{\left(x-a\right)^2}\int_{a+b-x}^{b} \varphi\left(u\right)dt. \tag{2.5}$$

Substituting (2.2)-(2.5) in (2.1), we get the desired result. □

Theorem 2.2. *Let $\varphi : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) such that $\varphi' \in L^1[a,b]$ with $0 \le a < b$. If there exist constants $-\infty < m < M < +\infty$ such that $m \le \varphi'(x) \le M$ for all $x \in [a,b]$, then the following inequality holds*

$$\left|\tfrac{\lambda(x-a)}{b-a}\varphi\left(a\right) + \tfrac{2(1-\lambda)(x-a)+a+b-2x}{2(b-a)}\left(\varphi\left(x\right)+\varphi\left(a+b-x\right)\right) + \tfrac{\lambda(x-a)}{b-a}\varphi\left(b\right) - \tfrac{1}{b-a}\int_a^b \varphi\left(u\right)du\right|$$
$$\le \tfrac{(M-m)(x-a)^2}{2(b-a)}\left(\lambda^2+\left(1-\lambda\right)^2\right) + \tfrac{(M-m)(a+b-2x)^2}{8(b-a)}.$$

Proof. From Lemma 2.1, we have

$$\tfrac{\lambda(x-a)}{b-a}\varphi\left(a\right) + \tfrac{2(1-\lambda)(x-a)+a+b-2x}{2(b-a)}\left(\varphi\left(x\right)+\varphi\left(a+b-x\right)\right) + \tfrac{\lambda(x-a)}{b-a}\varphi\left(b\right) - \tfrac{1}{b-a}\int_a^b \varphi\left(u\right)du$$

$$= \tfrac{(x-a)^2}{b-a}\int_0^1 \left(t-\lambda\right)\left(\varphi'\left(\left(1-t\right)a+tx\right) - \tfrac{m+M}{2} + \tfrac{m+M}{2}\right)dt$$

$$+ \tfrac{(a+b-2x)^2}{4(b-a)}\int_0^1 \left(t-1\right)\left(\varphi'\left(\left(1-t\right)x+t\tfrac{a+b}{2}\right) - \tfrac{m+M}{2} + \tfrac{m+M}{2}\right)dt$$

$$+ \tfrac{(a+b-2x)^2}{4(b-a)}\int_0^1 t\left(\varphi'\left(\left(1-t\right)\tfrac{a+b}{2}+t\left(a+b-x\right)\right) - \tfrac{m+M}{2} + \tfrac{m+M}{2}\right)dt$$

$$+ \tfrac{(x-a)^2}{b-a}\int_0^1 \left(t-\left(1-\lambda\right)\right)\left(\varphi'\left(\left(1-t\right)\left(a+b-x\right)+tb\right) - \tfrac{m+M}{2} + \tfrac{m+M}{2}\right)dt$$

$$= \tfrac{(x-a)^2}{b-a}\int_0^1 \left(t-\lambda\right)\left(\varphi'\left(\left(1-t\right)a+tx\right) - \tfrac{m+M}{2}\right)dt$$

$$+\tfrac{(a+b-2x)^2}{4(b-a)}\int_0^1 (t-1)\left(\varphi'\left((1-t)\,x+t\tfrac{a+b}{2}\right)-\tfrac{m+M}{2}\right)dt$$
$$+\tfrac{(a+b-2x)^2}{4(b-a)}\int_0^1 t\left(\varphi'\left((1-t)\,\tfrac{a+b}{2}+t\,(a+b-x)\right)-\tfrac{m+M}{2}\right)dt$$
$$+\tfrac{(x-a)^2}{b-a}\int_0^1 (t-(1-\lambda))\left(\varphi'\left((1-t)\,(a+b-x)+tb\right)-\tfrac{m+M}{2}\right)dt$$
$$+\tfrac{(m+M)(x-a)^2}{2(b-a)}\int_0^1 (t-\lambda)\,dt+\tfrac{(m+M)(x-a)^2}{2(b-a)}\int_0^1 (t-1)\,dt$$
$$+\tfrac{(m+M)(a+b-2x)^2}{8(b-a)}\int_0^1 t dt+\tfrac{(m+M)(x-a)^2}{2(b-a)}\int_0^1 (t-(1-\lambda))\,dt$$
$$=\tfrac{(x-a)^2}{b-a}\int_0^1 (t-\lambda)\left(\varphi'\left((1-t)\,a+tx\right)-\tfrac{m+M}{2}\right)dt$$
$$+\tfrac{(a+b-2x)^2}{4(b-a)}\int_0^1 (t-1)\left(\varphi'\left((1-t)\,x+t\tfrac{a+b}{2}\right)-\tfrac{m+M}{2}\right)dt$$
$$+\tfrac{(a+b-2x)^2}{4(b-a)}\int_0^1 t\left(\varphi'\left((1-t)\,\tfrac{a+b}{2}+t\,(a+b-x)\right)-\tfrac{m+M}{2}\right)dt$$
$$+\tfrac{(x-a)^2}{b-a}\int_0^1 (t-(1-\lambda))\left(\varphi'\left((1-t)\,(a+b-x)+tb\right)-\tfrac{m+M}{2}\right)dt, \tag{2.6}$$

where we use the fact that

$$\tfrac{(m+M)(x-a)^2}{2(b-a)}\int_0^1 (t-\lambda)\,dt+\tfrac{(m+M)(x-a)^2}{2(b-a)}\int_0^1 (t-(1-\lambda))\,dt$$
$$+\tfrac{(m+M)(a+b-2x)^2}{8(b-a)}\int_0^1 t dt+\tfrac{(m+M)(x-a)^2}{2(b-a)}\int_0^1 (t-1)\,dt$$
$$=\tfrac{(m+M)(x-a)^2}{2(b-a)}\int_0^1 (2t-1)\,dt+\tfrac{(m+M)(x-a)^2}{2(b-a)}\int_0^1 (2t-1)\,dt$$
$$=\left(\tfrac{(m+M)(x-a)^2}{2(b-a)}+\tfrac{(m+M)(x-a)^2}{2(b-a)}\right)\int_0^1 (2t-1)\,dt=0.$$

Using the absolute value on both sides of (2.6), we get

$$\left|\tfrac{\lambda(x-a)}{b-a}\varphi\,(a)+\tfrac{2(1-\lambda)(x-a)+a+b-2x}{2(b-a)}\left(\varphi\,(x)+\varphi\,(a+b-x)\right)+\tfrac{\lambda(x-a)}{b-a}\varphi\,(b)-\tfrac{1}{b-a}\int_a^b\varphi\,(u)\,du\right|$$

$$\leq \frac{(x-a)^2}{b-a}\int_0^1 |t-\lambda| \left|\varphi'((1-t)a+tx)-\frac{m+M}{2}\right| dt$$
$$+\frac{(a+b-2x)^2}{4(b-a)}\int_0^1 (1-t)\left|\varphi'\left((1-t)x+t\frac{a+b}{2}\right)-\frac{m+M}{2}\right| dt$$
$$+\frac{(a+b-2x)^2}{4(b-a)}\int_0^1 t\left|\varphi'\left((1-t)\frac{a+b}{2}+t(a+b-x)\right)-\frac{m+M}{2}\right| dt$$
$$+\frac{(x-a)^2}{b-a}\int_0^1 |t-(1-\lambda)| \left|\varphi'((1-t)(a+b-x)+tb)-\frac{m+M}{2}\right| dt. \tag{2.7}$$

Since $m \leq \varphi'(u) \leq M$ for all $u \in [a,b]$, then we have $\left|\varphi'(u)-\frac{m+M}{2}\right| \leq \frac{M-m}{2}$ for all $u \in [a,b]$.

So, from (2.7) we have

$$\left|\frac{\lambda(x-a)}{b-a}\varphi(a)+\frac{2(1-\lambda)(x-a)+a+b-2x}{2(b-a)}(\varphi(x)+\varphi(a+b-x))+\frac{\lambda(x-a)}{b-a}\varphi(b)-\frac{1}{b-a}\int_a^b \varphi(u)\,du\right|$$
$$\leq \frac{(M-m)(x-a)^2}{2(b-a)}\int_0^1 |t-\lambda|\,dt+\frac{(M-m)(x-a)^2}{2(b-a)}\int_0^1 |t-(1-\lambda)|\,dt$$
$$+\frac{(M-m)(a+b-2x)^2}{8(b-a)}\int_0^1 (1-t)\,dt+\frac{(M-m)(a+b-2x)^2}{8(b-a)}\int_0^1 t dt$$
$$=\frac{(M-m)(x-a)^2}{2(b-a)}\left(\int_0^\lambda (\lambda-t)\,dt+\int_\lambda^1 (t-\lambda)\,dt+\int_0^{1-\lambda} (1-\lambda-t)\,dt+\int_{1-\lambda}^1 (t-(1-\lambda))\,dt\right)$$
$$+\frac{(M-m)(a+b-2x)^2}{8(b-a)}\int_0^1 dt$$
$$=\frac{(M-m)(x-a)^2}{2(b-a)}\left(\frac{1}{2}\lambda^2+\frac{1}{2}(1-\lambda)^2+\frac{1}{2}(1-\lambda)^2+\frac{1}{2}(1-(1-\lambda))^2\right)+\frac{(M-m)(a+b-2x)^2}{8(b-a)}$$
$$=\frac{(M-m)(x-a)^2}{2(b-a)}\left(\lambda^2+(1-\lambda)^2\right)+\frac{(M-m)(a+b-2x)^2}{8(b-a)}.$$

The proof is completed. □

Theorem 2.3. *Let $\varphi : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) such that $\varphi' \in L^1[a,b]$ with $0 \leq a < b$. If φ' is r-L-Hölderian function on $[a,b]$ (i.e. there exist $L > 0$ and $0 < r \leq 1$ such that $|\varphi'(x)-\varphi'(y)| \leq L|x-y|^r$), then the following inequality holds*

$$\left|\frac{\lambda(x-a)}{b-a}\varphi(a)+\frac{2(1-\lambda)(x-a)+a+b-2x}{2(b-a)}(\varphi(x)+\varphi(a+b-x))+\frac{\lambda(x-a)}{b-a}\varphi(b)-\frac{1}{b-a}\int_a^b \varphi(u)\,du\right|$$
$$\leq \left(\frac{\left(2\left(\lambda^{2+r}+(1-\lambda)^{2+r}\right)+r\right)(x-a)^{2+r}}{(1+r)(2+r)}+\frac{\left|(1-\lambda)^2-\lambda^2\right|(x-a)^2(b-x)^r}{2}+\frac{(3+r)(a+b-2x)^{2+r}}{2^{3+r}(1+r)}\right)\frac{L}{b-a}.$$

Proof. From Lemma 2.1, we have

$$\frac{\lambda(x-a)}{b-a}\varphi(a)+\frac{2(1-\lambda)(x-a)+a+b-2x}{2(b-a)}(\varphi(x)+\varphi(a+b-x))+\frac{\lambda(x-a)}{b-a}\varphi(b)-\frac{1}{b-a}\int_a^b \varphi(u)\,du$$

$$
\begin{aligned}
&= \tfrac{(x-a)^2}{b-a} \int_0^1 (t-\lambda)\left(\varphi'((1-t)a+tx) - \varphi'(a) + \varphi'(a)\right) dt \\
&+ \tfrac{(a+b-2x)^2}{4(b-a)} \int_0^1 (t-1)\left(\varphi'\left((1-t)x + t\tfrac{a+b}{2}\right) - \varphi'(x) + \varphi'(x)\right) dt \\
&+ \tfrac{(a+b-2x)^2}{4(b-a)} \int_0^1 t\left(\varphi'\left((1-t)\tfrac{a+b}{2} + t(a+b-x)\right) - \varphi'\left(\tfrac{a+b}{2}\right) + \varphi'\left(\tfrac{a+b}{2}\right)\right) dt \\
&+ \tfrac{(x-a)^2}{b-a} \int_0^1 (t-(1-\lambda))\left(\varphi'((1-t)(a+b-x)+tb) - \varphi'(a+b-x) + \varphi'(a+b-x)\right) dt \\
&= \tfrac{(x-a)^2}{b-a} \int_0^1 (t-\lambda)\left(\varphi'((1-t)a+tx) - \varphi'(a)\right) dt \\
&+ \tfrac{(a+b-2x)^2}{4(b-a)} \int_0^1 (t-1)\left(\varphi'\left((1-t)x + t\tfrac{a+b}{2}\right) - \varphi'(x)\right) dt \\
&+ \tfrac{(a+b-2x)^2}{4(b-a)} \int_0^1 t\left(\varphi'\left((1-t)\tfrac{a+b}{2} + t(a+b-x)\right) - \varphi'\left(\tfrac{a+b}{2}\right)\right) dt \\
&+ \tfrac{(x-a)^2}{b-a} \int_0^1 (t-(1-\lambda))\left(\varphi'((1-t)(a+b-x)+tb) - \varphi'(a+b-x)\right) dt \\
&+ \tfrac{(x-a)^2}{b-a} \left(\varphi'(a)\int_0^1 (t-\lambda)\,dt + \varphi'(a+b-x)\int_0^1 (t-(1-\lambda))\,dt\right) \\
&+ \tfrac{(a+b-2x)^2}{4(b-a)} \left(\varphi'(x)\int_0^1 (t-1)\,dt + \varphi'\left(\tfrac{a+b}{2}\right)\int_0^1 tdt\right) \\
&= \tfrac{(x-a)^2}{b-a} \int_0^1 (t-\lambda)\left(\varphi'((1-t)a+tx) - \varphi'(a)\right) dt \\
&+ \tfrac{(a+b-2x)^2}{4(b-a)} \int_0^1 (t-1)\left(\varphi'\left((1-t)x + t\tfrac{a+b}{2}\right) - \varphi'(x)\right) dt \\
&+ \tfrac{(a+b-2x)^2}{4(b-a)} \int_0^1 t\left(\varphi'\left((1-t)\tfrac{a+b}{2} + t(a+b-x)\right) - \varphi'\left(\tfrac{a+b}{2}\right)\right) dt \\
&+ \tfrac{(x-a)^2}{b-a} \int_0^1 (t-(1-\lambda))\left(\varphi'((1-t)(a+b-x)+tb) - \varphi'(a+b-x)\right) dt \\
&+ \tfrac{\left((1-\lambda)^2-\lambda^2\right)(x-a)^2}{2(b-a)} \left(\varphi'(a) - \varphi'(a+b-x)\right) + \tfrac{(a+b-2x)^2}{8(b-a)} \left(\varphi'\left(\tfrac{a+b}{2}\right) - \varphi'(x)\right). \qquad (2.8)
\end{aligned}
$$

Applying the absolute value on both sides of (2.8), and then using the fact that φ' is r-L-

Hölderian function, we obtain

$$\left|\frac{\lambda(x-a)}{b-a}\varphi(a)+\frac{2(1-\lambda)(x-a)+a+b-2x}{2(b-a)}\left(\varphi(x)+\varphi(a+b-x)\right)+\frac{\lambda(x-a)}{b-a}\varphi(b)-\frac{1}{b-a}\int_a^b\varphi(u)\,du\right|$$

$$\begin{aligned}
&\leq\frac{(x-a)^2}{b-a}\int_0^1|t-\lambda|\left|\varphi'((1-t)a+tx)-\varphi'(a)\right|dt\\
&\quad+\frac{(a+b-2x)^2}{4(b-a)}\int_0^1(1-t)\left|\varphi'\left((1-t)x+t\tfrac{a+b}{2}\right)-\varphi'(x)\right|dt\\
&\quad+\frac{(a+b-2x)^2}{4(b-a)}\int_0^1 t\left|\varphi'\left((1-t)\tfrac{a+b}{2}+t(a+b-x)\right)-\varphi'\left(\tfrac{a+b}{2}\right)\right|dt\\
&\quad+\frac{(x-a)^2}{b-a}\int_0^1|t-(1-\lambda)|\left|\varphi'((1-t)(a+b-x)+tb)-\varphi'(a+b-x)\right|dt\\
&\quad+\frac{\left|(1-\lambda)^2-\lambda^2\right|(x-a)^2}{2(b-a)}\left|\varphi'(a)-\varphi'(a+b-x)\right|+\frac{(a+b-2x)^2}{8(b-a)}\left|\varphi'\left(\tfrac{a+b}{2}\right)-\varphi'(x)\right|\\
&\leq\frac{(x-a)^{2+r}}{b-a}\left(\int_0^1|t-\lambda|\,t^r dt+\int_0^1|t-(1-\lambda)|\,t^r dt\right)L\\
&\quad+\frac{(a+b-2x)^{2+r}}{2^{2+r}(b-a)}\left(\int_0^1(1-t)\,t^r dt+\int_0^1 t^{1+r}dt\right)L\\
&\quad+\left(\frac{\left|(1-\lambda)^2-\lambda^2\right|(x-a)^2(b-x)^r}{2(b-a)}+\frac{(a+b-2x)^{2+r}}{2^{3+r}(b-a)}\right)L\\
&=\left(\frac{\left(2\left(\lambda^{2+r}+(1-\lambda)^{2+r}\right)+r\right)(x-a)^{2+r}}{(1+r)(2+r)}+\frac{\left|(1-\lambda)^2-\lambda^2\right|(x-a)^2(b-x)^r}{2}+\frac{(3+r)(a+b-2x)^{2+r}}{2^{3+r}(1+r)}\right)\frac{L}{b-a}
\end{aligned}$$

The proof is completed. □

Corollary 2.4. *Under the assumptions of Theorem 2.3, if φ' is a Lipschitzian function, we have*

$$\left|\frac{\lambda(x-a)}{b-a}\varphi(a)+\frac{2(1-\lambda)(x-a)+a+b-2x}{2(b-a)}\left(\varphi(x)+\varphi(a+b-x)\right)+\frac{\lambda(x-a)}{b-a}\varphi(b)-\frac{1}{b-a}\int_a^b\varphi(u)\,du\right|$$
$$\leq\left(\frac{\left(2\left(\lambda^3+(1-\lambda)^3\right)+1\right)(x-a)^3}{6}+\frac{\left|(1-\lambda)^2-\lambda^2\right|(x-a)^2(b-x)}{2}+\frac{(a+b-2x)^3}{8}\right)\frac{L}{b-a}.$$

3 Examples

In this section, to visually validate and evaluate the accuracy of the obtained results, we present some examples with 2D and 3D graphical representations. The functional graphs of the left (red), middle (green), and right (blue) sides were plotted using MATLAB software.

Example 3.1. Let us consider the function $\varphi : [0,1] \to \mathbb{R}$ defined by $\varphi(u) = \frac{2}{3}u^{\frac{3}{2}}$. Then, $\varphi'(u) = \sqrt{u}$ is bounded on $[0,1]$, with $m = 0$ and $M = 1$.

In Theorem 2.2, we will first fix one of the two parameters x and λ at a time and plot the curves of the three inequality terms with respect to the remaining parameter. Next, we will plot the surfaces of the three inequality terms as functions of both variables x and λ.

Case 1. If we choose to fix $x = \frac{1}{2}$, we obtain the following result with respect to $\lambda \in [0,1]$ (see Figure 3.1):

$$-\frac{\lambda^2+(1-\lambda)^2}{8} \leq \frac{1+\lambda(\sqrt{2}-1)}{3\sqrt{2}} - \frac{4}{15} \leq \frac{\lambda^2+(1-\lambda)^2}{8}.$$

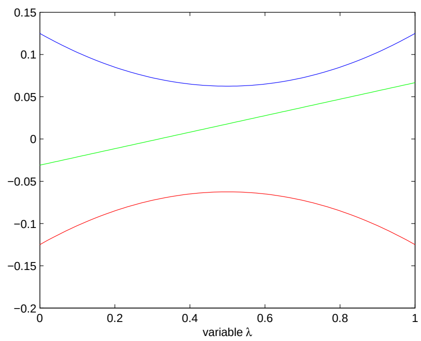


Figure 3.1: The function curve for $x = \frac{1}{2}$

Case 2. If we choose to fix $\lambda = 0$, we obtain the following result with respect to $x \in \left[0, \frac{1}{2}\right]$ (see Figure 3.2):

$$-\frac{4x^2+(1-2x)^2}{8} \leq \frac{x^{\frac{3}{2}}+(1-x)^{\frac{3}{2}}}{3} - \frac{4}{15} \leq \frac{4x^2+(1-2x)^2}{8}.$$

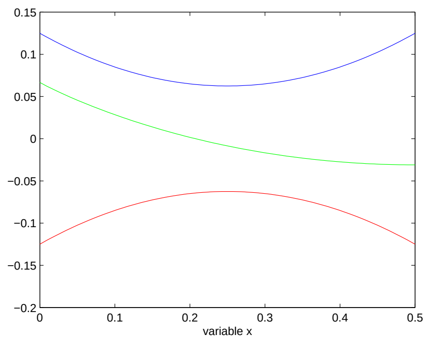
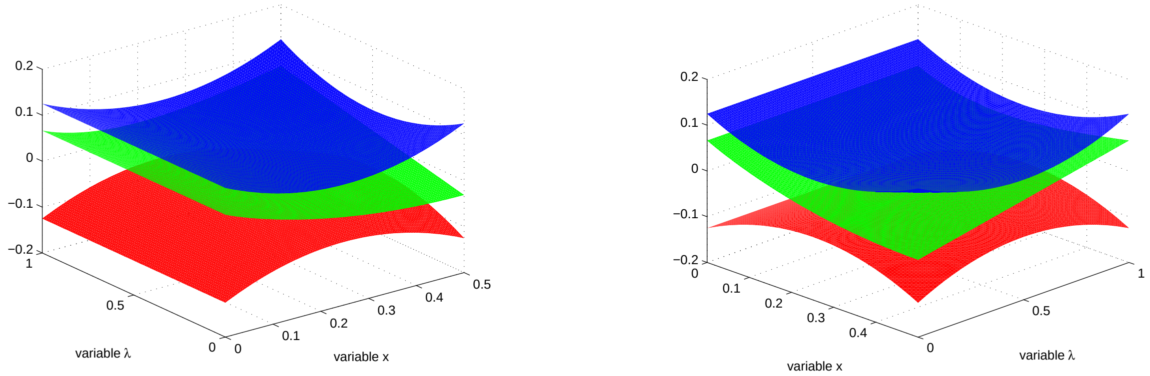


Figure 3.2: The function curve for $\lambda = 0$

Case 3. If we consider both x and λ as variables, we obtain the following result for $(x,\lambda) \in \left[0, \frac{1}{2}\right] \times [0,1]$ (refer to Figure 3.3 from different perspectives):

$$-\frac{x^2\left(\lambda^2+(1-\lambda)^2\right)}{2} - \frac{(1-2x)^2}{8} \leq \left(x^{\frac{3}{2}} + (1-x)^{\frac{3}{2}}\right)\frac{1-2\lambda x}{3} + \frac{10\lambda x-4}{15} \leq \frac{x^2\left(\lambda^2+(1-\lambda)^2\right)}{2} + \frac{(1-2x)^2}{8}.$$



(a) View No.1. (b) View No.2.

Figure 3.3: The function surface

Example 3.2. Let us consider the function $\varphi : [0,1] \to \mathbb{R}$ defined by $\varphi(u) = \frac{2}{3}u^{\frac{3}{2}}$. Then, $\varphi'(u) = \sqrt{u}$ is r-L-Hölderian on $[0,1]$, with $r = \frac{1}{2}$ and $L = 1$.

In Theorem 2.3, similar to the previous example, we will begin by fixing one of the parameters x and λ and plotting the corresponding curves with respect to the other variable. Then, we will generate plots of the surfaces parameterized with respect to the two variables x and λ.

Case 1. If we choose to fix $x = \frac{1}{2}$, we obtain the following result with respect to $\lambda \in [0,1]$ (see Figure 3.4):

$$-\frac{2\left(\lambda^{\frac{5}{2}}+(1-\lambda^{\frac{5}{2}}\right)+\frac{1}{2}}{15\sqrt{2}}-\frac{|1-2\lambda|}{8\sqrt{2}} \le \frac{1+\lambda(\sqrt{2}-1)}{3\sqrt{2}}-\frac{4}{15} \le \frac{2\left(\lambda^{\frac{5}{2}}+(1-\lambda^{\frac{5}{2}}\right)+\frac{1}{2}}{15\sqrt{2}}+\frac{|1-2\lambda|}{8\sqrt{2}}.$$

Case 2. If we choose to fix $\lambda = 0$, we obtain the following result with respect to x in $x \in \left[0, \frac{1}{2}\right]$

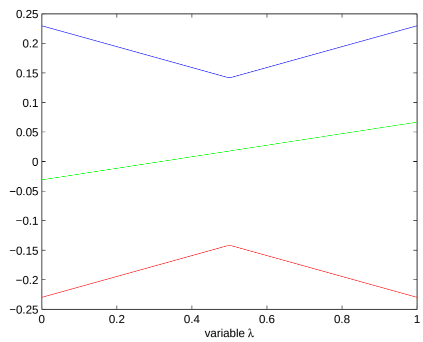


Figure 3.4: The function curve for $x = \frac{1}{2}$

(see Figure 3.5):

$$-\frac{7}{6}\left(x^{\frac{5}{2}}+\frac{(1-2x)^{\frac{5}{2}}}{8\sqrt{2}}\right) \le \frac{x^{\frac{3}{2}}+(1-x)^{\frac{3}{2}}}{3}-\frac{4}{15} \le \frac{7}{6}\left(x^{\frac{5}{2}}+\frac{(1-2x)^{\frac{5}{2}}}{8\sqrt{2}}\right).$$

Case 3. If we consider both x and λ as variables, we obtain the following result for $(x,s) \in$

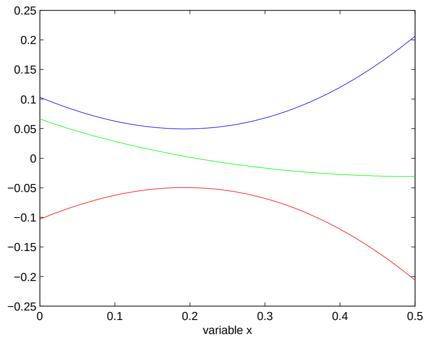
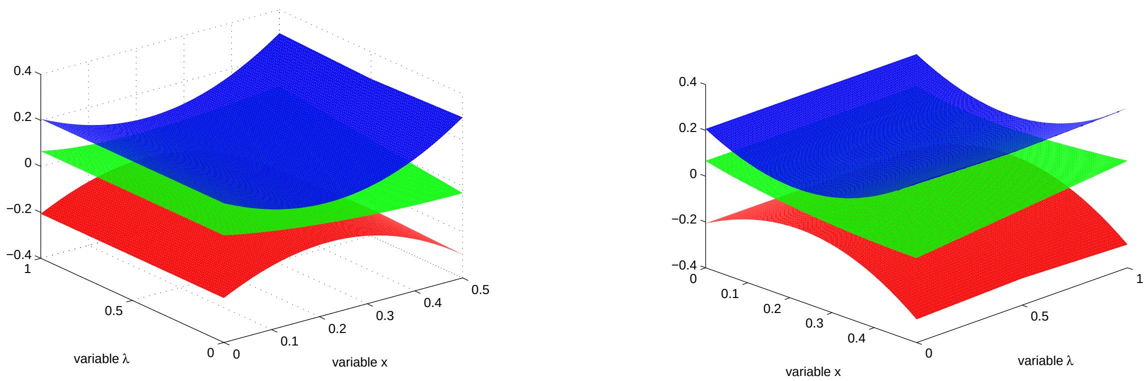


Figure 3.5: The function curve for $\lambda = 0$

$[0, \frac{1}{2}] \times [0, 1]$ (refer to Figure 3.6 from different perspectives):

$$
\begin{aligned}
&-\left(\frac{8\left(\lambda^{\frac{5}{2}}+(1-\lambda^{\frac{5}{2}}\right)+2}{15}+\frac{\left|(1-\lambda)^2-\lambda^2\right|}{8\sqrt{2}}\right)x^{\frac{5}{2}}-\frac{7}{24\sqrt{2}}(1-2x)^{\frac{5}{2}} \\
&\leq \left(x^{\frac{3}{2}}+(1-x)^{\frac{3}{2}}\right)\frac{1-2\lambda x}{3}+\frac{2\lambda x}{3}-\frac{4}{15} \\
&\leq \left(\frac{8\left(\lambda^{\frac{5}{2}}+(1-\lambda^{\frac{5}{2}}\right)+2}{15}+\frac{\left|(1-\lambda)^2-\lambda^2\right|}{8\sqrt{2}}\right)x^{\frac{5}{2}}+\frac{7}{24\sqrt{2}}(1-2x)^{\frac{5}{2}}.
\end{aligned}
$$



(a) View No.1. (b) View No.2.

Figure 3.6: The function surface

4 Conclusion

In this study, we introduced a novel parameterized identity pertaining to the general form of the 4-point Newton-Cotes formula which recovers the most famous 1, 2, 3 and 4-point formulas. Then, we provided numerous new integral inequalities for functions with bounded derivatives as well as r-L-Hölderian derivatives. Our research has expanded on the current literature surrounding integral inequalities and holds significant implications for the fields of mathematics and physics. Furthermore, we have identified specific cases and presented some examples with visual representations to illustrate their accuracy. In summary, we are confident that our work will inspire further investigation in this area and pave the way for fresh developments in the study of integral inequalities.

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Competing interests The authors declare that they have no competing interests.

Availability of data and materials Not applicable.

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