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Collaboration and authority in the collective action problem

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Abstract. Direct reciprocity and contingent collaboration deter opportunistic behavior in social predicaments. However, in large collectives, these mechanisms may lose efficacy as they rely on individuals' influence. Zero-Determinant (ZD) strategies in the collective action problem reshape our understanding of individual influence. Our study introduces a theoretical framework extending these strategies to multiplayer dilemmas, offering insights into lone participants' impact. We delineate intriguing sub-classes of strategies: fair, extortionate (advantageous), and generous (disadvantageous). We explore models showcasing strategic enhancement through alignment with others. The present study elucidates the significance of individual decision-making and collective coordination as essential components contributing to favorable outcomes within expansive group settings.

Keywords: Coordination, Alliances, Cooperation, Public Goods Game.

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1 Introduction

Fostering cooperation among individuals driven by self-interest is typically challenging [6, 15, 17], and the predicament of free riders tends to escalate when groups expand in size [2, 4, 7, 21, 24, 28]. In compact communities, the establishment of cooperation is frequently upheld through various manifestations of direct and indirect reciprocation [8, 16, 27, 29]. Nevertheless, in the case of large-scale groups, there is a growing notion that these mechanisms may lose their effectiveness. This is primarily attributed to the increased complexity of tracking the reputations of numerous individuals and the diminishing impact that an individual can exert on others [4, 7, 24]. In order to mitigate "the challenges posed by the tragedy of the commons" and address the inherent limitations of fellow influence, affluent communities have implemented a pragmatic solution by establishing centralized organizations. These entities are tasked with fostering collective collaboration among community members. This approach is driven by the recognition that shared resources, when subject to unregulated exploitation, can lead to detrimental consequences for the entire community [14, 18, 26, 30].Yet,

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a recent revelation implies that our assessment might have downplayed the extent to which individual players can wield influence within repetitive game situations.

In the "iterated prisoner's dilemma" scenario, Press and Dyson [20] made a significant finding: they identified strategies. Strategies empower players to create a direct correlation between their rewards and those of their co-players, irrespective of the strategies chosen by the co-players. What's interesting is that "the set of strategies strategies" offers a wide range of possibilities and outcomes. For instance, if a player desires to guarantee that their payoff consistently aligns with that of their co-player, they can achieve this objective by employing a fair ZD strategy, such as the renowned Tit for Tat approach. Conversely, if a player aim to surpass their opponent, they can achieve this by making subtle adjustments to the Tit-for-Tat strategy, thereby creating extortionate "ZD strategies". The discovery of these strategies has sparked multiple theoretical inquiries, delving into the evolutionary patterns of diverse "ZD strategies" across a range of circumstances [1, 10, 22, 23, 25]. "ZD strategies" can be applied in various scenarios beyond the repeated prisoner's dilemma; their applicability extends to various scenarios and situations. Newly released research papers have provided evidence indicating the presence of ZD strategies in additional forms of recurring two-player contests [12] or within the arrangement of "repeated public goods games" [19]. Within this text, we aim to demonstrate the essence of such strategies across all isochronous social dilemmas, regardless of the number of participants involved. We utilize this theoretical framework to elucidate the ZD strategies capable of ensuring equitable outcomes or thwarting the dominance of free riders.

Although our discoveries are not confined solely to ZD strategies, we build upon the methodologies proposed by Press and Dyson [20], as well as Akin [1], to extend their techniques. By doing so, we establish clear criteria for identifying the circumstances in which modified propagation of "Tit-for-Tat, Win-Stay Lose-Shift" and Grim strategies foster sustainable cooperation. Through this approach, we discover that the majority of theoretical adjustments in order to iterate prisoner's dilemma is eligible to be seamlessly applied to recurring dilemmas involving any amount of participants.

In this research, we present two paradigms that delve into the ways individuals can improve their strategic decision-making through collaborative coordination. We broaden the concept of ZD strategies by incorporating subsets of players referred to as ZD alliances. These alliances involve players determining the composition of the group and adopting a shared strategy, either independently (strategic alliance) or as a unified entity with identical actions (synchronized alliance). The effectiveness of ZD alliances depends on factors such as alliance size, chosen strategy, and the nature of the social dilemma. Coordination becomes more significant as alliances grow in size, offering a wider range of strategic possibilities in synchronized alliances compared to strategic alliances.

Addressing global challenges, such as climate change, involves extensive-scale social dilemmas with multiple stakeholders. Evolutionary game theory predicts challenges in achieving mutual cooperation in multiplayer dilemmas due to limited individual control in large groups. This study expands ZD strategies to multiplayer games to identify cooperation-facilitating tactics. We introduce straightforward alliance models to address multiplayer dilemmas, where the alliance's size, employed strategies, and specific dilemma properties impact the outcome. Alliance formation becomes critical when strategic choices are limited, offering leverage and transformative effects.

2 Repeated multiplayer game

In order to obtain these findings, we investigate a scenario involving a recurring social dilemma among a group of n-players. With the progression of each round in the game, players face a choice between assistance (cooperating C) or apostasy (defecting D). The individual payoff of a player is influenced by their own decision as well as the decisions made by all other members within the group, for more information, see (Table 2).

Count of individuals represented by C among fellow participants	$n-1$	$\cdots$	m	$\cdots$	2	1	0
Payoff of cooperators	a_{n-1}	$\cdots$	a_m	$\cdots$	a_2	a_1	a_0
Payoff of defectors	b_{n-1}	$\cdots$	b_m	$\cdots$	b_2	b_1	b_0

Table 2.1: The payoff of the symmetric n-player collective action contests. The player's payoff related to the number of co-players and the decision of the focal player that cooperates.

Within a group, when j of the other members chooses to cooperate, a player who also cooperates receive a payoff represented by the parameter a_j. Conversely, a player who defects obtains a payoff denoted by the parameter b_j. We make the assumption that the payoffs adhere to three distinctive properties that are commonly associated with social dilemmas, aligning with the interpretation of altruism centered around individuals [11].

1. The situation should involve multiple individuals or parties who are interconnected or interdependent in some way.

$$a_{j+1} \geq a_j, b_{j+1} \geq b_j \quad \text{for all } j \quad \text{in the presence of} \quad 0 \leq j \leq n-1. \tag{2.1}$$

2. The individuals or parties must face a choice between pursuing their self-interests or cooperating for the greater benefit of the group.

$$b_{j+1} \geq a_j \quad \text{for all } j \text{ with} \quad 0 \leq j \leq n-1. \tag{2.2}$$

3. Each person or group is best served by pursuing their own interests, yet if all adopt this course of action, it results in a less-than-ideal outcome for the entire collective.

$$a_{n-1} > b_0. \tag{2.3}$$

Let's suppose that the social dilemma occurs in a repetitive manner, without any limit to the number of times it repeats. This supposition is being utilized to enhance the clarity of our conversation. However, it's worth noting that similar conclusions can be drawn even if we consider a finite number of rounds [9], for instance. To put it differently, the specific number of repetitions doesn't fundamentally change the results we will discuss. In iterated games, a player's decisions in each round must be shaped by the results of preceding rounds. Assuming we have knowledge of the strategies employed by all members of the group, we can represent the anticipated payoff of performer i in innings t as $\pi_i(t)$. Within the framework of the repeated game, the payoffs are determined by calculating the mediocre payoff per innings

$$\pi^i = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \pi^i(t). \tag{2.4}$$

Moving forward, we will make an assumption that these limits are present. This assumption remains valid when players make their decisions solely based on a finite number of past rounds, although the number of rounds considered can be arbitrarily large. To demonstrate the outcomes of our findings, we will provide explanations using two specific instances of multiplayer games as examples. The first one is the "public goods game" [13], specifically in the scenario where individuals cooperate, each person contributes a certain amount of money, denoted as c where $c > 0$, to a communal reserve. The aggregate contributions made by all individuals within the group are subject to multiplication strengthened by r times, where $1 < r < n$, afterward, the resultant total is evenly divided among all the members of the group. Cooperating individuals obtain a payoff of $a_j = \frac{(j+1)rc}{n} - c$, while those who defect and refrain from contributing to the common pool receive a higher payoff of $b_j = \frac{jrc}{n}$. The second one is the volunteer's dilemma; To ensure that every member of the group receives a benefit $b > c$, it is necessary for at least one member to step forward and willingly incur a cost $c > 0$. As a result, cooperative individuals, regardless of j, gain a payoff of $b - c$, where b represents the benefit received. On the other hand, defectors obtain a payoff of $b_j = b$ if their position index $j \geq 1$, and $b_0 = 0$. Both of these examples, along with various others such as the collective risk dilemma, can be categorized as straightforward illustrations of collective action problem [2,21].

3 Zero-determinant strategies

Through our research, we have established that players possess considerable control over their payoff outcomes in repeated games by considering the results of previous rounds. Specifically, our investigation focuses on players who base their decisions solely on their past moves and the aggregate cooperators in the preceding alternation. We assume game symmetry, wherein payoffs depend on cooperation. This approach proves particularly advantageous when players have limited information about the specific actions of others and can only observe the overall game outcome. However, when considering alliances and broader player interactions, it becomes essential to explore a wider array of strategies that enable differentiation between co-players. Consequently, we will delve into the development of a more comprehensive theory concerning ZD strategies in the subsequent section. This theory will shed light on additional aspects of player decision-making and their implications in repeated game scenarios.

3.1 Memory-one strategies

A memory-one strategy entails a collection of directives that influence a player's decisions in the following round, shaped by the results of the preceding round. These tactics can be depicted using as

$$\mathbf{p} = (p_{C,n-1}, p_{C,n-2}, \cdots, p_{C,0}; p_{D,n-1}, p_{D,n-1}, \cdots, p_{D,0}), \tag{3.1}$$

the values represented by $P_{S,j}$ indicate the likelihood of a player choosing to cooperate in the subsequent round, based on their previous choices $S \in \{C, D\}$, as well as the number j of subsequent participants who decided to work together. An illustration of the performer i's repeat strategy can be expressed as $\mathbf{p}^{Rep}$, where the player consistently reproduces their own action from the preceding round. This can be mathematically represented by

$$\mathbf{p}^{Rep}_{S,i} = \begin{cases} 1 & \text{if } S_i = C \\ 0 & \text{if } S_i = D \end{cases}. \tag{3.2}$$

In the framework of strategies based on a memory of just one previous move, it is necessary to determine a probability of cooperation at the outset of the game's initial round. However, since the long-term outcome of infinitely repeated games often remains unaffected by the initial round, the initial cooperation probability is commonly disregarded or not given much importance [20,22]. In the subsequent discussion, we will only provide the initial probability of cooperation for a player if it becomes relevant or required.

When all individuals in a population employ memory-one strategies, the computation of payoffs based on Equation (2.4) becomes notably straightforward.In this scenario, we can describe the ongoing game by employing a Markov chain model, where the result of each subsequent round is solely contingent upon the result of the preceding round. While selecting memory-one strategies can simplify computations, we will subsequently demonstrate that ZD strategies showcase their unique characteristics regardless of the strategies adopted by other members within the group. Specifically, ZD players are not reliant on their fellow participants employing memory-one tactics. Now, let's examine a scenario where a "focal player", making the most of "memory-one strategy" represented by $\mathbf{p}$, engages in a repeated game alongside $n-1$ other co-players. These co-players can adopt any strategy without any specific restrictions. The expression $v_{S,t}(t)$ represents the likelihood of the result of the t-th round corresponds to (S,j). Suppose the vector $\mathbf{v}(t)$ contains a series of probabilities, denoted as $[v_{C,n-1}(t),\cdots,v_{C,0}(t)]$. A distribution limit, identified as $\mathbf{v}$, signifies a prominence that the pattern $\frac{[\mathbf{v}(1)+\cdots+\mathbf{v}(t)]}{t}$ converges as $t\to\infty$. In a limit distribution, the values of $v_{S,j}$ indicate how often the central player encounters the state (S,j) throughout the game, with each value representing the fraction of rounds in which this occurs.

3.2 Akin's Lemma

Akin's lemma [1] ensures the existence of a significant connection between $\mathbf{p}$ and $\mathbf{v}$, that is there is a compelling relationship among "the memory-one strategy" $\mathbf{p}$ along with the limit apportionment $\mathbf{v}$.

Lemma 3.1. *Consider a scenario where "the focal player" employs a "memory-one strategy" $\mathbf{p}$ without any specific constraints. In such cases, regardless of the initial round's outcome, it can be shown that for any limiting distribution $\mathbf{v}$, the following relationship holds.*

$$\left(\mathbf{p}-\mathbf{p}^{Rep}\right)\cdot\mathbf{v}=0. \tag{3.3}$$

In this context, the product refers to the standard scalar product.

Proof. Let $q_C(t)$ be the focal player cooperation probability in session t, then

$$q_C(t)=\mathbf{p}^{Rep}\mathbf{v}(t).$$

In the following round, the focal player's contribution leads to cooperation.

$$q_C(t+1)=\mathbf{p}\cdot\mathbf{v}(t),$$

by combining the preceding two equations, we obtain

$$q_C(t+1)-q_C(t)=\left(\mathbf{p}-\mathbf{p}^{Rep}\right)\cdot\mathbf{v},$$

calculating through 1 to t, attain

$$\left(\mathbf{p}-\mathbf{p}^{Rep}\right)\frac{1}{t}\left[v(1)+\cdots+v(t)\right]=\left(q_C(t+1)-q_C(t)\right)\frac{1}{t},$$

the maximum absolute value $\frac{1}{t}$, then

$$\left(\mathbf{p}-\mathbf{p}^{Rep}\right)\cdot\mathbf{v}=0,$$

which completes the proofs. □

It is important to highlight that Akin's lemma does not impose any assumptions regarding the payoff structure of the game. This means that it applies not only to games that resemble social dilemmas but also to other types of games. Furthermore, there are no limitations or constraints on the tactics adopted by the visible $n-1$ members of the group.

3.3 Zero-determinant strategies for multiplayer

By employing Akin's lemma, we can demonstrate how an individual player can acquire a surprising level of influence over the resulting payoffs in a collective action problem. In order to proceed with our analysis, it is necessary to introduce additional notation. we are considering the set of possible gains or losses that player i can experience, and these outcomes are represented as a vector $\mathbf{g}^i=\left(g_{S,j}\right)^i$ with two elements, namely $g^i_{C,j}=a_j$ and $g^i_{D,j}=b_j$. Let the average payoff correspond to the possible gains or losses of the focal players denote by $\mathbf{g}^{-i}=\left(g^{-i}_{S,j}\right)$, where $S\in\{C,D\}$, with $g^{-i}_{C,j}=\frac{\left[ja_j+(n-j-1)b_{j+1}\right]}{n-1}$ and $g^{-i}_{D,j}=\frac{\left[ja_{j-1}+(n-j-1)b_j\right]}{n-1}$. At last, Suppose $\mathbf{1}$ be the unit vector of $2n$-dimensional. Put into action of these notations, the payoff in "repeated prisoner's dilemma game" for the focal player i is like $\pi^i=\mathbf{g}^i\cdot\mathbf{v}$, along with corresponding intermediate payoff of i's co-players like $\pi^{-i}=\mathbf{g}^{-i}\cdot\mathbf{v}$. Due to $\mathbf{v}$ assign a limit distribution it is clear $\mathbf{1}\cdot\mathbf{v}=1$. By Akin's lemma, we can define Zero-Determinant strategy as follows

Definition 3.2. A strategy known as a ZD strategy is considered a memory-one approach denoted as $\mathbf{P}$ within a game involving n players. This occurs when specific constants, namely α, β, and γ, are present, with the condition that $\beta\neq 0$. This relationship is characterized by the presence of these constants in a manner that can be described as follows

$$\mathbf{p}=\mathbf{p}^{Rep}+\alpha\mathbf{g}^i+\beta\mathbf{g}^{-i}+\gamma\mathbf{1}. \tag{3.4}$$

In the above let $\phi=-\beta$, the mean payoff of the focal player and corresponding its co-players $s=\frac{-\alpha}{\beta}$, and the parameter $l=\frac{-\gamma}{\alpha+\beta}$, then

$$\mathbf{p}=\mathbf{p}^{Rep}+\phi\left((1-s)\left(l\mathbf{1}-\mathbf{g}^i\right)+\mathbf{g}^i-\mathbf{g}^{-i}\right). \tag{3.5}$$

Now for those who are choosing the C strategy, we have $g^i=a_j$ and $g^{-i}=\frac{ja_j+(n-j-1)b_{j+1}}{n-1}$. using these values in above we have

$$p_{C,j}=1+\phi\left[(1-s)\left(l-a_j\right)+\frac{(n-j-1)}{n-1}\left(b_{j+1}-a_j\right)\right] \tag{3.6}$$

and for those who are using D strategy we have $g^i = b_j$ and $g^{-i} = \frac{ja_{j-1}+(n-j-1)b_j}{n-1}$.,

$$p_{D,j} = \phi \left[(1-s)\left(l - b_j\right) + \frac{j}{n-1}\left(b_j - a_{j-1}\right) \right], \tag{3.7}$$

ZD strategies enable players to unilaterally determine the focal player's expected payoffs and that of their co-player.

Theorem 3.3. *(Press and Dyson) Picture the player i in the spotlight, using a "memory-one strategy" characterized by the following structure.*

$$\mathbf{p} = \mathbf{p}^{Rep} + \alpha \mathbf{g}^i + \sum_{j\neq i}^{n} \beta_j \mathbf{g}^j + \gamma \mathbf{1}, \tag{3.8}$$

accompanied by constants α, β along with γ, then, regardless of the strategy of the $n-1$ of corresponding co-players the payoffs satisfy the formula

$$\alpha \pi^i + \sum_{j\neq i}^{n} \beta_j \pi^j + \gamma \cdot 1 = 0, \tag{3.9}$$

where $\sum_{j\neq i}^{n} \beta_j = \beta$ and $\pi^j = \pi^{-i}$ and $\gamma = (1, \cdots, 1)$.

Proof. Akin's lemma gives us

$$0 = \left(\mathbf{p} - \mathbf{p}^{Rep}\right) \cdot \mathbf{v} = \left(\alpha \mathbf{g}^i + \sum_{j\neq i}^{n} \beta_j \mathbf{g}^j + \gamma \mathbf{1}\right) \cdot \mathbf{v}.$$

Then we have the formula

$$\alpha \pi^i + \sum_{j\neq i}^{n} \beta_j \pi^j + \gamma \cdot 1 = 0.$$

□

When we mention strategies of the form (3.8), we are actually referring to a particular type of strategy known as ZD strategies or ZD strategies. These strategies have distinct property of being able to enforce a specific outcome in terms of payoffs, irrespective of the choices made by the other participants. By applying zero-determinant strategies, the focal player able to impose a lineal payoff relation within her distinct payoff along with that of corresponding co-player's payoff. Furthermore, through careful selection of the aspects α, β_j, and γ, the player gains straightforward influence over the structure and nature of the payoff relationship.

we will redefine certain parameters or variables in a way that makes it more convenient and easier to work with when analyzing ZD strategies. This alternative representation allows us to manipulate the parameters in a manner that better suits our purpose

$$\phi = -\sum_{k\neq i} \beta_k, \quad \alpha = -s\phi, \quad \beta_j = w_{j\neq i}\phi, \quad \gamma = \phi l\,(1-s)\,. \tag{3.10}$$

In simpler terms, ZD strategies can be represented using the new parameters in (3.8) we get.

$$\mathbf{p} = \mathbf{p}^{Rep} + \left[s\mathbf{g}^i - \sum_{j\neq i} w_j \mathbf{g}^j + (1-s)\,\mathbf{1} \right]. \tag{3.11}$$

the strategies must adhere to specific rules: $\phi \neq 0$, and $w_i = 0$, that is (weight associated with strategy of player i) must be zero, and $\sum_{j \neq i} w_j = 1$, these limitations emerge directly from the definitions provided in (3.10). When a player i utilizes a ZD strategy of this nature, the resulting enforced relationship of payoffs, as described in Equation (3.9), can be expressed in alternative terms.

$$\pi_{-i} = s\pi_i + (1-s)\, l, \tag{3.12}$$

$\pi_{-i} = \sum_{j=1}^{n} w_j \pi^j$ denotes the average payoff of the other players, taking into account their respective weights w_j. In this context, the parameter l is commonly known as the baseline payoff of the Zero-Determinant strategy, while s represents the slope of the strategy. Additionally, the vector $\mathbf{w} = (w_j)$, corresponds to the weights associated with the strategy. The parameter ϕ does not directly impact Equation (3.12), but its value determines the rate at which the payoffs converge towards the enforced payoff relation during the course of the game [9].

3.4 ZD strategies establish a linear connection between the payoffs.

The enforcement of payoffs in Zero-Determinant strategies is determined by the parameters l baseline payoff, slope s, and weights $\mathbf{w}$. The resulting enforced payoff relation may not always follow a linear pattern but depends on the specific values assigned to these parameters.

Proposition 3.4. *If player i employs a Zero-Determinant "strategy" in support of indicators l, s, along with $\phi > 0$ in an iterated game consisting of M iterations, the payoffs can be described as*

$$|\pi_{-i} = s\pi_i + (1-s)\, l| \leq \frac{1}{\phi M}. \tag{3.13}$$

in the scenario of infinitely repeated games, where the number of rounds $M \to \infty$, the payoffs adhere to the provided condition (3.12).

Proof. Let's denote $g^i_{S,j}$ as the payoff obtained by player i in a specific round when they choose task $S \in \{C, D\}$. When a total of j co-players choose to collaborate, this means that

$$g^i_{S,j} == \begin{cases} a_j & \text{if } S = C \\ b_j & \text{if } S = D, \end{cases} \tag{3.14}$$

Likewise, let's define $g^{-i}_{S,j}$ as the average payoff earned by the other members of the group, that is

$$g^{-i}_{S,j} = \begin{cases} \frac{ja_j + (n-j-1)b_{j+1}}{n-1} & \text{if } S = C \\ \frac{ja_{j-1} + (n-j-1)b_j}{n-1} & \text{if } S = D, \end{cases} \tag{3.15}$$

Additionally, we can define the function $v_{S,j}(m)$ as the likelihood that in the mth round, player i selects strategy S, while j of their fellow players opt to cooperate. We can gather these probabilities and express them as vectors,

$$\begin{aligned} \mathbf{g}^i &= \left(g^i_{C,n-1}, \cdots, g^i_{C,0}; g^i_{D,n-1}, \cdots, g^i_{D,0}\right) \\ \mathbf{g}^{-i} &= \left(g^{-i}_{C,n-1}, \cdots, g^{-i}_{C,0}; g^{-i}_{D,n-1}, \cdots, g^{-i}_{D,0}\right) \\ \mathbf{v}(m) &= \left(v_{C,n-1}(m), \cdots, v_{C,0}(m); v_{D,n-1}(m), \cdots, v_{D,0}(m)\right), \end{aligned} \tag{3.16}$$

Using the aforementioned signage, we are in a position to represent "player i's expected payoff in round m" like $\pi_i = \mathbf{g}^i \cdot \mathbf{v}(m)$. Similarly, "expected payoff of the co-players" (excluding player i) is achievable like $\pi_{-i} = \mathbf{g}^{-i} \cdot \mathbf{v}(m)$. Let's assume that the vector $\mathbf{1}$ consists of $2n$ elements, all of which are 1. On the other hand, the vector $\mathbf{g}^0$ consists of 2n elements, with the first n elements being 1 and the last n elements being 0. Zero-Determinant strategies are characterized by the fact that the payoffs of specific players can be controlled or influenced by the decisions made by their opponents in the game,

$$\mathbf{p} = \mathbf{g}^0 + \phi \left[(1-s) \left(l\mathbf{1} - \mathbf{g}^i \right) + \mathbf{g}^i - \mathbf{g}^{-i} \right]. \tag{3.17}$$

Lastly, we introduce the notation $q_C(m)$ to represent the probability that player i chooses to cooperate in round m. We are capable of express $q_C(m) = \mathbf{g}^0 \cdot \mathbf{v}(m)$. Similarly, the probability of player i cooperating within the next session, represented as $q_C(m+1)$, is feasible as $q_C(m+1) = p \cdot \mathbf{v}(m)$. We can write $w(m) = q_C(m+1) - q_C(m)$ and express the relationship or conclusion as follows

$$w(m) = \left(p - \mathbf{g}^0\right) \cdot \mathbf{v}(m) = \phi \left[(1-s) \left(l\mathbf{1} - \mathbf{g}^i \right) + \mathbf{g}^i - \mathbf{g}^{-i} \right] \cdot \mathbf{v}(m). \tag{3.18}$$

That is

$$w(m) = \phi \left[s\pi_i(m) + (1-s) l - \pi_{-i}(m) \right]. \tag{3.19}$$

By considering the definition of $w(m)$, computing the aggregate for different values of m, and taking the limit as $M \to \infty$ approaches infinity, we can obtain the following conclusion.

$$\frac{1}{M} \sum_{m=1}^{M} w(m) = \frac{1}{M} \sum_{m=1}^{M} q_C(m+1) - q_C(m) = \frac{q_C(M+1) - q_C(0)}{M} = \frac{q_C(M+1) - p_0}{M}. \tag{3.20}$$

On the other hand, from equation (3.19) we have

$$\frac{1}{M} \sum_{m=1}^{M} w(m) = \frac{\phi}{M} \sum_{m=1}^{M} s\pi_i(m) + (1-s) l - \pi_{-i}(m) = \phi \left[s\pi_i + (1-s) l - \pi_{-i} \right]. \tag{3.21}$$

In order for the two limits to coincide, $|q_C(M+1) - p_0| \leq 1$, and the result. □

Proposition 3.5. *(Essential prerequisites for legally binding payoff relationships). Any payoff relationship* $(l, s, \mathbf{w})$ *that can be enforced must meet condition* $-\frac{1}{n-1} \leq s \leq 1$. *Additionally, if* $s < 1$, *then condition* $b_0 \leq l \leq a_{n-1}$ *must also be satisfied. Furthermore, the selection of aspect* $\phi \neq 0$ *should meet the condition that* $\phi > 0$ *is fulfilled.*

Furthermore, one can provide a description of all potential payoff arrangements, in addition to these essential prerequisites.

Proof. By definition of ZD strategies, mutual cooperation and mutual defection gives

$$\begin{aligned} p_{C,n-1} &= 1 + \phi (1-s) (l - a_{n-1}), \\ p_{D,n-1} &= \phi (1-s) (l - b_0), \end{aligned} \tag{3.22}$$

Henceforth

$$\begin{aligned} \phi (1-s) (l - a_{n-1}) &\leq 0, \\ 0 &\leq \phi (1-s) (l - b_0). \end{aligned} \tag{3.23}$$

Adding these together yields $\phi (1-s) (b_0 - a_{n-1}) \leq 0$, this implies that

$$\phi (1-s) \geq 0. \tag{3.24}$$

Analogously, for p_σ, where σ is contrary to conditions $C, C, \cdots, C$ and $D, D, \cdots, D$, in which case

$$p_\sigma = \begin{cases} 1+\phi\left[sa_{n-2}-\left(1-\frac{j}{n-1}\right)a_{n-2}-\frac{j}{n-1}b_{n-1}+(1-s)\,l\right] & \text{the defector is a coplayer} j \neq i \\ \phi\left[sb_{n-1}-a_{n-2}+(1-s)\,l\right] & \text{if the defector is a player } i \end{cases} \tag{3.25}$$

since $p_{S,j} \in [0,1]$ then

$$\begin{aligned} \phi\left[sa_{n-2}-\left(1-\frac{j}{n-1}\right)a_{n-2}-\frac{j}{n-1}b_{n-1}+(1-s)\,l\right] \leq 0 \\ 0 \leq \phi\left[sb_{n-1}-a_{n-2}+(1-s)\,l\right]. \end{aligned} \tag{3.26}$$

By adding this two we have $\phi\left(s+\frac{j}{n-1}\right)(b_{n-1}-a_{n-2}) \geq 0$, for all $j \neq i$ this implies that

$$\phi\left(s+\frac{j}{n-1}\right) \geq 0 \quad \text{for all} \quad j \neq i, \tag{3.27}$$

combining $\phi\,(1-s) \geq 0$ and $\phi\left(s+\frac{j}{n-1}\right) \geq 0 \quad$ for all$j \neq i$, then yields

$$\phi\left(1+\frac{j}{n-1}\right) \geq 0 \quad \text{for all} \quad j \neq i, \tag{3.28}$$

from this it confirms $\phi \geq 0$. The constraint $\phi \neq 0$ henceforth conveys $\phi > 0$. from $\phi\,(1-s) \geq 0$ and $\phi\left(s+\frac{j}{n-1}\right) \geq 0$ without exclusion $j \neq i$, we possess $-\min_{j\neq i}\frac{j}{n-1} \leq s \leq 1$. Since $-\min_{j\neq i}\frac{j}{n-1} \leq \frac{1}{n-1}$, it follows that $-\frac{1}{n-1} \leq s \leq 1$. □

Proposition 3.6. *(Enforceable Payoff Relations).* $\left(l, s, \left(\frac{j}{n-1}\right)\right)$ *"is enforceable if and only if either* $s = 1$ *or*

$$\max_{0\leq j\leq n-1}\left\{b_j-\frac{j}{n-1}\frac{b_j-a_{j-1}}{1-s}\right\} \leq l \leq \min_{0\leq j\leq n-1}\left\{a_j+\frac{n-j-1}{n-1}\frac{b_{j+1}-a_j}{1-s}\right\}". \tag{3.29}$$

Proof. A zero-determinant strategy can be written as

$$p_{S,j} = p^{Rep}+\phi\left[(1-s)\left(l-g^i_{S,j}\right)+\sum_{j\neq i}\frac{j}{n-1}\left(g^i_{S,j}-g^j_{S,j}\right)\right],$$

Since $p_{S,j} \in [0,1]$ then the following holds

$$\begin{aligned} (1-s)\left(l-a_{j-1}\right)-\sum_{j\in\sigma^D}\frac{j}{n-1}\left(b_j-a_{j-1}\right) \leq 0 \quad \text{if} \quad S_i = C, \\ (1-s)\left(l-b_j\right)-\sum_{j\in\sigma^C}\frac{j}{n-1}\left(b_j-a_{j-1}\right) \geq 0 \quad \text{if} \quad S_i = D. \end{aligned}$$

For $s = 1$ there is nothing to prove. Suppose $s < 1$, dividing the above inequality by $1-s$ we have

$$\begin{aligned} a_{j-1}-\frac{\sum_{j\in\sigma^D}\frac{j}{n-1}\left(b_j-a_{j-1}\right)}{1-s} \geq l \quad \text{if} \quad S_i = C, \\ b_j-\frac{\sum_{j\in\sigma^C}\frac{j}{n-1}\left(b_j-a_{j-1}\right)}{1-s} \leq l \quad \text{if} \quad S_i = D. \end{aligned}$$

this implies that

$$\max_{0\le j\le n-1}\left\{b_j-\frac{j}{n-1}\frac{b_j-a_{j-1}}{1-s}\right\}\le l\le \min_{0\le j\le n-1}\left\{a_j+\frac{n-j-1}{n-1}\frac{b_{j+1}-a_j}{1-s}\right\}.$$

□

4 Zero-determinant alliances

In our previous study of repeated social dilemmas, we considered how individual actions limited strategic options as the group size grew larger. To overcome this limitation and gain more strategic influence, We presented the notion of alliances with ZD properties. These alliances are subgroups created to coordinate strategies collaboratively. We explored two models to evaluate the effectiveness of ZD alliances based on their coordination methods. The first model, known as strategy alliances, involves alliance parts agreeing on a common ZD strategy solitarily, without any further communication during gameplay. This type of coordination requires minimal effort. In contrast, we also investigated synchronized alliances, where alliance members coordinate their actions in each round by collectively deciding to either cooperate or defect. Synchronized alliances act as a unified entity, granting them greater leverage than individual players. Achieving this higher level of coordinated play demands significant cooperation among alliance members.

4.1 Violation of coordination

Suppose there is n-player in "iterated prisoner's dilemma", let for the baseline l and slope s and $\phi > 0$ be given, let the measure $m_j = \frac{1}{n-1}$ for all $j \neq i$ and assume all of the players coordinate to apply a ZD strategy, and let $n-k$, where $1 \leq k < n-1$, of the players deviate from ZD strategy after some rounds, to clarify the matter, pay attention to the figure

$$\begin{array}{r} CCCCC\ldots \\ CCCCC\ldots \\ CCCCD\ldots \\ \vdots \\ CCCCD\ldots \end{array}$$

Each ally i where $i \in \{1,2,\cdots,k\}$ in the union maintains a relationship where their enforcement is based on the formula $\pi_{-i} = s\pi_i + (1-s)\,l$ then we can write

$$\frac{k\hat{\pi} + (n-1-k)\,\pi}{n-1} = s\pi + (1-s)\,l,$$

after calculation we have

$$\hat{\pi} = \frac{(n-1)\,s-(n-1-k)}{k}\pi + \left(1-\frac{(n-1)\,s-(n-1-k)}{k}\right)l,$$

if we take $\hat{s} = \frac{(n-1)s-(n-1-k)}{k}$ then

$$\hat{\pi} = \hat{s}\pi + (1 - \hat{s})\, l. \tag{4.1}$$

What this show is that, ZD player "enforce a linear relation" between the payoff of $n - 2$ deviated player and their own payoff with new slop $\hat{s}$.

Example 4.1. Suppose there are three players in an iterated multi-player prisoner's dilemma game, let the baseline l and slope s and $\phi > 0$ be given, let the measure $m_j = \frac{1}{2}$ for all $j \neq i$ and assume all of the players coordinate to apply a ZD strategy, and let one of the players deviate from ZD strategy after some rounds.

We see that one of the players deviates from the ZD strategy. What this shows is that the ZD player "enforces a linear relation between the" payoff of deviated player along with their own payoff with new slop $\hat{s}$ as follows.

$$\frac{\hat{\pi} + \pi}{2} = s\pi + (1 - s)\, l,$$

then

$$\hat{\pi} = \hat{s}\pi + (1 - \hat{s})\, l,$$

where $\hat{s} = (2s - 1)$.

Example 4.2. For the four-player also if after some rounds two of them deviate from the ZD strategy then we can write

$$\frac{2\hat{\pi} + \pi}{3} = s\pi + (1 - s)\, l,$$

after calculation we have

$$\hat{\pi} = \frac{3s - 1}{2}\pi + \left(1 - \frac{3s - 1}{2}\right) l,$$

if we take $\hat{s} = \frac{3s-1}{2}$ then then

$$\hat{\pi} = \hat{s}\pi + (1 - \hat{s})\, l.$$

Deviation of coordination can lead to sub-optimal outcomes for the group as a whole. It highlights the challenges of achieving cooperation in multi-player scenarios, where individual incentives often conflict with collective interests. Game theorists and researchers study various strategies and mechanisms to encourage cooperation and mitigate the deviation of coordination in multiplayer prisoner's dilemma games.

4.2 Noise in a group of ZD strategy

Suppose in a group of ZD strategy players $\mathbf{p}$ with baseline l, slop s, constant $\phi > 0$, and $m = \frac{1}{n-1}$ for every $j \neq i$, and assume nth player deviate, call it noisy player and for the rest of $(n - 1)$ players in the group call quite players. Let π be the payoff of the quiet players and $\hat{\pi}$ be the payoff of the noisy player, since quiet players adopt ZD strategy the formula $\pi^{-i} = s\pi^i + (1 - s)\, l$, gives the result

$$\frac{n-2}{n-1}\pi + \frac{1}{n-1}\hat{\pi} = s\pi + (1 - s)\, l, \tag{4.2}$$

that is,

$$\hat{\pi} = s_Q\pi + (1 - s_Q)\, l, \tag{4.3}$$

with

$$s_Q = s\,(n-1) - (n-2)\,. \tag{4.4}$$

Quiet players concertedly enforce a linear relation on noisy player with the baseline l and slop s_Q. Similarly, if a quiet players deviate, the noises yield the payoff $\hat{\pi} = \pi$, for $S_Q = 0$ then equation $\hat{\pi} = s_Q\pi + (1-s_Q)\,l$ implies that $\hat{\pi} = \pi = l$. for $s_Q = 1$ the equation $\hat{\pi} = s_Q\pi + (1-s_Q)\,l$ implies that $\hat{\pi} = \pi$.

4.3 Coordination and ZD Strategies

Extensions of ZD strategies to multiplayer IPD involve strategies that attempt to establish cooperative alliances or exploit the behaviors of other players. These strategies can aim to form coalitions with specific players, punish defectors or exploiters or encourage cooperation within a subset of players. In this part, we are looking up to answer the question of whether can individuals enhance control by making coordination of their intentions with other group members. Let $n_u < n$ denote the number of players who consent to a unison. We consider the coordinated individuals as allies and the rest of the individuals as outsiders. Coordination can be done in different ways, we investigate two types of coordination in this section:

4.3.1 Strategic Alliances

Strategic alliances can be formed in multiplayer "prisoner's dilemma" games to enhance cooperation in the crowd of a subset of players and increase their collective payoff. These alliances aim to establish trust, coordinate strategies, and deter defection within the group.

In *strategic alliances*, the set of members and a ZD strategy is agreed upon between the players, during the game, the allies have no connection with each other, which means that a minimum level of coordination is needed. This type of alliance takes place when the possibility of communication is zero on during the play of the game for the allies, or when coordination in each round of the game is expensive.

Let π_u denote the payoff of an ally and π_{-u} denote the average payoff of the outsiders. Here we auscultate the ZD strategies in terms of what kind of linear payoff relations it imposes and how the strategic power of actors depends on coordination.

Let $1 \leq n_u < n$ denote a group of unions, assume that unions use the same ZD strategy, and the parameters l, s, and $\phi > 0$ are given and let $\mathbf{M} = (m_j)$ denote the measure of the ZD strategy depending on the players inside the union U. This means that for the player i inside the union

$$m_{j\neq i} = \begin{cases} m_u & \text{if } j \in U \\ m_{-u} & \text{if } j \notin U, \end{cases} \tag{4.5}$$

with $m_u \geq 0$ and $m_{-u} \geq 0$, such that the measues satisfies

$$(n_u - 1)\,m_u + (n - n_u)\,m_{-u} = 1. \tag{4.6}$$

Since all members of the union use the same strategy, then all get the same payoff. Morever, by equation $\pi^{-i} = s\pi^i + (1-s)\,l$, each members of the union enforces the payoff relationship

$$(n_u - 1)\,m_u\pi_u + (n - n_u)\,m_{-u}\pi_{-u} = s\pi_u + (1-s)\,l. \tag{4.7}$$

After the calculation it can be written as

$$\pi_{-u} = s_u\pi_u + (1-s_u)\,l, \tag{4.8}$$

such that

$$s_u = \frac{s - (n_u - 1)\, m_u}{1 - (n_u - 1)\, m_u}, \tag{4.9}$$

is the effective slope of the strategy alliance, For $m_u = \frac{1}{n-1}$ we retrieve the motive for the parallel measure of all populations.

Example 4.3. Suppose there are 10 players "in the repeated prisoner's dilemma game" and let 6 of them use C strategy, then four of them use D strategy then we have the equation as

$$\hat{\pi} = \frac{9s-5}{4}\pi + \left(1 - \frac{9s-5}{4}\right) l. \tag{4.10}$$

Solution: Here the five player agrees with the focal player which $n_u = 6$ and $n_u - 1 = 5$ so we have

$$\frac{(n - n_u)\, \hat{\pi} + (n_u - 1)\, \pi}{n-1} = s\pi + (1-s)\, l,$$

after the calculation we have

$$\hat{\pi} = \frac{9s-5}{4}\pi + \left(1 - \frac{9s-5}{4}\right) l.$$

The payoff relation $\pi_{-u} = s_u \pi_u + (1 - s_u)\, l$ with parameter l, s^u is valid if if we find $\phi > 0$ and $0 \leq m_u < \frac{1}{n_u - 1}$ in a fashion where every entries of the yielding ZD strategy conforming to $\mathbf{P} = \mathbf{P}^{Rep} + \phi \left[s\mathbf{g}^i - \sum_{j \neq i} m_j \mathbf{g}^j + (1-s)\, l\mathbf{1} \right]$ are in the unit interval. For more information we can illustrate the following proposition.

Proposition 4.4. *The payoff relation l, s_u can be enforced by strategy alliance "if and only if either" $s_u = 1$ or $s_u < 1$ and*

$$\max_{0 \leq j \leq n - n_u} \left\{ b_j - \frac{j}{n - n_u} \frac{b_j - a_{j-1}}{1 - s_u} \right\} \leq l \leq \min_{n_u - 1 \leq j \leq n-1} \left\{ a_j + \frac{n-j-1}{n-n_u} \frac{b_{j+1} - a_j}{1 - s_u} \right\}. \tag{4.11}$$

Furthermore, if $n_u \leq \frac{n}{2}$, in that case $-1 \leq -\frac{n_u}{n - n_u} \leq s_u \leq 1$.

Proof. Let every member of allies adopt a ZD strategy $\mathbf{p}$ with baseline l, slope s, ϕ and measure $\mathbf{m}$. To perform an impressive slope s^u, equation $s_u = \frac{s-(n_u-1)m_u}{1-(n_u-1)m_u}$, transmit that s must be such that

$$s = s_u + (n_u - 1)\, m_u\, (1 - s_u)\,. \tag{4.12}$$

Fro $s_u = 1$, we get $s = 1$, and the payoff relationship is enforceable. Let $s_u < 1$,The alliance ensures that the payoff relationship l, s_u holds true only when the conditions are met such that $\mathbf{m} = (m_j)$

$$\max_{0 \leq j \leq n-1} \left\{ b_j - \frac{\hat{m}_j}{1 - (n_u - 1)\, m_u} \frac{b_j - a_{j-1}}{1 - s_u} \right\} \leq l \leq \min_{0 \leq j \leq n-1} \left\{ a_j + \frac{\hat{m}_{n-j-1}}{1 - (n_u - 1)\, m_u} \frac{b_{j+1} - a_j}{1 - s_u} \right\}. \tag{4.13}$$

Here, $\hat{m}_j$ is the sum f the j smallest entries in $\mathbf{m}$. Therefore we can write

$$\hat{m} = \begin{cases} jm_u & \text{if } m_u \leq m_{-u}, j \leq n_u - 1 \\ (n_u - 1)\, m_u + (j - n_u + 1)\, m_{-u} & \text{if } m_u \leq m_{-u}, j > n_u - 1 \\ jm_{-u} & \text{if } m_u > m_{-u}, j \leq n - n_u \\ (j + n_u - n)\, m_u + (n - n_u)\, m_{-u} & \text{if } m_u > m_{-u}, j > n - n_u. \end{cases} \tag{4.14}$$

Restrictions $(n_u - 1)\, m_u + (n - n_u)\, m_{-u} = 1$ gives that $m_{-u} = \frac{1-(n_u-1)m_u}{(n-n_u)}$, from the above equation we have

$$\frac{\hat{m}}{1-(n_u-1)\, m_u} = \begin{cases} \frac{jm_u}{1-(n_u-1)m_u} & \text{if } m_u \leq \frac{1}{n-1}, j \leq n_u - 1 \\ \frac{1}{1-(n_u-1)m_u} - \frac{n-j-1}{n-n_u} & \text{if } m_u \leq \frac{1}{n-1}, j > n_u - 1 \\ \frac{j}{n-n_u} & \text{if } m_u > \frac{1}{n-1}, j \leq n - n_u \\ \frac{1-(n-j-1)m_u}{1-(n_u-1)m_u} & \text{if } m_u > \frac{1}{n-1}, j > n - n_u. \end{cases} \tag{4.15}$$

From (4.13) we get that the space of achievable relationships reaches its maximum extent when we choose m_u in a way that $\frac{\hat{m}_j}{1-(n_u-1)m_u}$ becomes maximal. The equation (4.15) proposes that $\frac{\hat{m}_j}{1-(n_u-1)m_u}$ progresses uniformly in m_u. Then, for constraint $0 \leq m_u < \frac{1}{n_u-1}$, the highest point is reached with $m_u \to \frac{1}{n_u-1}$, along with $m_u > \frac{1}{n_u-1}$. Then we have

$$\lim_{m_u \to \frac{1}{n_u-1}} \frac{\hat{m}_j}{1-(n_u-1)\, m_u} = \begin{cases} \frac{j}{n-n_u} & \text{if } j \leq n - n_u \\ \infty & \text{if } j > n - n_u. \end{cases} \tag{4.16}$$

Therefore, for m_u adequately close to $\frac{1}{n_u-1}$ condition (4.13) is satisfied if and only if

$$\max_{0 \leq j \leq n-1} \left\{ b_j - \frac{j}{n-n_u} \frac{b_j - a_{j-1}}{1-s_u} \right\} \leq l \leq \min_{0 \leq j \leq n-1} \left\{ a_j + \frac{n-j-1}{n-n_u} \frac{b_{j+1} - a_j}{1-s_u} \right\}. \tag{4.17}$$

which is coinside in general condition. Additionaly, if $n_u \leq \frac{n}{2}$, we will pick out j with $n_u - 1 < j \leq n - n_u$ in such a way that the above equation hold then

$$b_j - \frac{j}{n-n_u} \frac{b_j - a_{j-1}}{1-s_u}. \tag{4.18}$$

Similarly, for $j-1$ it gives that

$$l \leq a_{j-1} + \frac{j}{n-n_u} \frac{b_{j+1} - a_j}{1-s_u}. \tag{4.19}$$

After summing the inequalities we have $s^u \geq -\frac{n_u}{n-n_u}$. □

From the above proposition we see that large groups lose their strategic power. Alliance increases the players' advantages. But the ally should not be big enough to include all the members of the game. In alliance where the actions of outsiders have little effect, they form the most vigorous alliances. For instance, suppose the size of alliance is n_u such that $b_{n-n_u} \leq a_{n_u-1}$, since b_{n-n_u} is upper bound and a_{n_u-1} is lower bound then "any" l alongside $b_{n-n_u} \leq l \leq a_{n_u-1}$ "can be enforced", regardless involving the amount $s_u < l$.

4.3.2 Synchronized alliance

A synchronized alliance in multiplayer prisoner's dilemma refers to a cooperative arrangement when a collective of participants collaborates to synchronize their activities and make simultaneous decisions to maximize their joint payoff. In this type of alliance, players agree to act in unison and select the same strategy at each round of the game.

In each game stage, if the players behave in such a way that the same action performed by each members of allies, we call this "*synchronized alliance*", in fact, it is a coordinated alliance that treats similar to a new entity and can have a preferable weighted lever than any actor. In the former framework, we supposed that each of the allies resolves freely, to cooperate in a certain period.

Now let's move on to a framework where the allies meet after each round to resolve what action to take together on the subsequent alternation. Relative to a consequence, alliance subjects function as a new individual entity in the game, with $n - n_u$ co-players. A memory-one strategy $\mathbf{p} = (p_{S,j})$ where $S \in \{C, D\}$ and each j of the non-alliance cooperates in the previous round used to determine the probability of cooperation by allies or defect all allies in next round. The synchronized alliance is given by

$$p_{S,j}^{Rep} = \begin{cases} 1 & \text{if } S = C \\ 0 & \text{if } S = D, \end{cases} \tag{4.20}$$

as alliance adopts a memory strategy of one $\mathbf{p}$, then for "corresponding limit distribution" $\mathbf{v}$ we have $(\mathbf{p} - \mathbf{p}^{Rep}) \cdot \mathbf{v} = 0$. The possible payoff in a given round $\mathbf{g}^u$ has the entries

$$\mathbf{g}_{S,j}^{u} = \begin{cases} a_{n_u+j-1} & \text{if } S = C \\ b_j & \text{if } S = D, \end{cases} \tag{4.21}$$

and the average payoff of outsiders $\mathbf{g^{-u}}$ takes the form

$$\mathbf{g}_{S,j}^{-u} = \begin{cases} \frac{j a_{n_u+j-1} + (n-n_u-j) b_{n_u+j}}{n-n_u} & \text{if } S = C \\ \frac{j a_{j-1} + (n-n_u-j) b_j}{n-n_u} & \text{if } S = D, \end{cases} \tag{4.22}$$

using $\mathbf{g}^u$ and $\mathbf{g}^{-u}$ we have $\pi_u = \mathbf{g^u} \cdot \mathbf{v}$ and $\pi_{-u} = \mathbf{g}^{-u} \cdot \mathbf{v}$. Similarly to the case of individual players, Press and Dyson equations for ZD strategies yields

$$\mathbf{p} = \mathbf{p}^{Rep} + \alpha \mathbf{g}^u + \beta \mathbf{g}^{-u} + \gamma \mathbf{1}, \tag{4.23}$$

here $\mathbf{1}$ is the strategy tht all entries are 1 and α, β and γ are real numbers and $\beta \neq 0$, Akins limma gives that

$$\alpha \pi_u + \beta \pi_{-u} + \gamma = 0. \tag{4.24}$$

The parameter transformation $l = -\frac{\gamma}{(\alpha+\beta)}, s_u = -\frac{\alpha}{\beta}$, and $\phi = -\beta$ gives

$$\mathbf{P} = \mathbf{P}^{Rep} + \phi \left[s_u \mathbf{g}^u - \mathbf{g}^{-u} + (1 - s_u) l \mathbf{1} \right]. \tag{4.25}$$

Now, according to the equation (4.24) we have

$$\pi_{-u} = s_u \pi_u + (1 - s_u) l. \tag{4.26}$$

The synchronized alliance aims to create a cooperative bloc within the larger group of players, thereby increasing the chances of mutual cooperation and obtaining higher payoffs collectively. By aligning their actions, the members of the synchronized alliance send a strong signal to the other players, encouraging them to cooperate or face the consequences of collective punishment.

Proposition 4.5. *The payoff relation l, s_u can be enforced by synchronized alliance if and only if either $s_u = 1$ or $s_u \neq 1$ and*

$$\max_{0 \le j \le n-n_u} \left\{ b_j - \frac{j}{n-n_u} \frac{b_j - a_{j-1}}{1 - s_u} \right\} \le l \le \min_{n_u - 1 \le j \le n-1} \left\{ a_j + \frac{n-j-1}{n-n_u} \frac{b_{j+1} - a_j}{1 - s_u} \right\}. \tag{4.27}$$

Proof. Acording to the definition of synchronized alliance a ZD strategy is of the form

$$p_{S,j} = \begin{cases} 1 + \phi \left[(1 - s_u) \left(l - a_{n_u+j-1} \right) - \frac{n-n_u-j}{n-n_u} \left(b_{n_u+j-a_{n_u+j-1}} \right) \right] & \text{if } S = C \\ \phi \left[(1 - s_u) \left(l - b_j \right) + \frac{j}{n-n_u} \left(b_j - a_{j-1} \right) \right] & \text{if } S = D, \end{cases} \tag{4.28}$$

for $0 \leq j \leq n - n_u$. The state $0 \leq p_{s,j} \leq 1$ gives the following equations

$$0 \geq \phi \left[(1 - s_u) \left(l - a_{n_u+j-1} \right) - \frac{n - n_u - j}{n - n_u} \left(b_{n_u+j-a_{n_u+j-1}} \right) \right] \tag{4.29}$$

$$0 \leq \phi \left[(1 - s_u) \left(l - b_j \right) + \frac{j}{n - n_u} \left(b_j - a_{j-1} \right) \right], \tag{4.30}$$

for every $0 \leq j \leq n - n_u$. Taking $j = n - n_u$ in the equation (4.29) yields

$$\phi \left(1 - s_u\right) \left(l - a_{n-1}\right) \leq 0, \tag{4.31}$$

and taking $j = 0$ in the equation (4.30) yields

$$0 \leq \phi \left(1 - s_u\right) \left(l - b_0\right). \tag{4.32}$$

Summing the equations (4.29) and (4.30) we have

$$\phi \left(1 - s_u\right) \geq 0. \tag{4.33}$$

Taking $\phi > 0$ adequately. Suppose $s_u \neq 1$, then we have $\phi \left(1 - s_u\right) > 0$. By dividing the sign constraints in the equation (4.29) by $\phi \left(1 - s_u\right)$ we have

$$\max_{0 \leq j \leq n - n_u} \left\{ b_j - \frac{j}{n - n_u} \frac{b_j - a_{j-1}}{1 - s_u} \right\} \leq l \leq \min_{n_u - 1 \leq j \leq n-1} \left\{ a_j + \frac{n - j - 1}{n - n_u} \frac{b_{j+1} - a_j}{1 - s_u} \right\}. \tag{4.34}$$

Conversely, suppose the last equation holds for some baseline l and slop s with $s_u \neq 1$, for every choice of ϕ such that $\phi \left(1 - s_u\right) > 0$, the entries $p_{s,j}$ satisfy $0 \leq p_{s,j} \leq 1$, for every j. Taking ϕ adequately adjacent to 0, the payoff relationship l and s_u is enforceable. Eventually, let $n_u \leq \frac{n}{2}$, taking $j = 0$ in the branches of the equation (4.29) yields

$$\phi \left[\left(1 - s_u\right) \left(l - a_{n_u-1}\right) - \left(b_{n_u} - a_{n_u-1}\right) \right] \leq 0, \tag{4.35}$$

and taking $j = n_u$ then

$$0 \leq \phi \left[\left(1 - s_u\right) \left(l - b_{n_u}\right) + \frac{n_u}{n - n_u} \left(b_{n_u} - a_{n_u-1}\right) \right]. \tag{4.36}$$

Summing the equations (4.35) and (4.36) we have

$$\phi \left(s_u + \frac{n_u}{n - n_u} \right) \geq 0, \tag{4.37}$$

we see that $\phi > 0$, which in turn gives $-\left[\frac{n_u}{n-n_u}\right] \leq s_u \leq 1$. □

Strategic alliances require less coordination. But they have the strategic power of synchronized alliances. In general, synchronized alliances are stronger than strategic alliances if the alliance reaches the size of n_u such that $b_{n-n_u} < a_{n_u-1}$.

Creating a synchronized alliance in multiplayer prisoner's dilemma games can be challenging due to the complexity of coordinating actions among multiple players. It requires a high level of trust, effective communication channels, and a strong commitment to cooperation. Additionally, the success of a synchronized alliance may depend on the specific game dynamics, the number of players involved, and the strategies employed by other participants. It's important to note that in some multiplayer prisoner's dilemma scenarios, coordinated strategies like the synchronized alliance may face difficulties due to the potential for defection or the lack of enforce-ability. Players may have incentives to defect from the alliance if they believe they can achieve higher individual payoffs.

5 Application

In this section, we consider one example of multi-player social dilemmas. For convenience, we focus on symmetric payoff games, which solely pertain to the size of cooperators, but not to the recognition and identity of cooperators.

5.1 Public goods games

In a public goods dilemma, individuals must decide how much to contribute to a common pool that benefits all members of the group. Each individual faces the temptation to free-ride, enjoying the benefits of the public good without contributing their fair share. When individuals opt to engage in free-riding, it can result in insufficient provision of the public good, ultimately leading to a less-than-ideal sequel for the entire cluster.

In a "public goods game" we know that every player of a population makes a donation of cost $c > 0$ to a common good. Whole contributors are propagated by a ratio r where $1 < r < n$ and fairly apportioned amid each member of the group. Then, the payoff in public goods game is

$$a_j = \frac{(j+1)\,rc}{n} - c \quad \text{and} \quad b_j = \frac{jrc}{n}. \tag{5.1}$$

In the public goods game ZD strategies are of the form

$$p_{S,j} = \begin{cases} 1 + \phi\left[(1-s)\left(l - \frac{(j+1)rc}{n}\right) - \frac{n-j-1}{n-1}c\right] & \text{if } S = C \\ \phi\left[(1-s)\left(l - \frac{jrc}{n}\right) - \frac{j}{n-1}c\right] & \text{if } S = D. \end{cases} \tag{5.2}$$

To know which baseline l and slop s for if an individual can enforce, we consider the harbor conditions $j = 0$ and $j = n-1$ in the relation

$$\max_{0\le j\le n-1}\left\{b_j - \frac{j}{n-1}\frac{b_j - a_{j-1}}{1-s}\right\} \le l \le \min_{0\le j\le n-1}\left\{a_j + \frac{n-j-1}{n-1}\frac{b_{j+1}-a_j}{1-s}\right\},$$

we have that

$$0 \le l \le rc - c, \tag{5.3}$$

$$\frac{(n-1)\,rc}{n} - \frac{c}{1-s} \le l \le \frac{(r-n)\,c}{n} + \frac{c}{1-s}. \tag{5.4}$$

For "the existence of ZD strategies" we reach the result that it depends of the size n of the group. If $l = 0 = b_0$ then the ZD strategies are extortion strategies, from the equation (5.4) we have that the slope $s \geq \frac{r-1}{r}$ can ever be enforced regardless of the amount n, and the slopes $s < \frac{r-1}{r}$ are only enforceable if n is adequately inconsiderable

$$n \leq \frac{r\,(1-s)}{r\,(1-s)-1}. \tag{5.5}$$

Therefore, if $n \to \infty$ then only "extortionate strategies" with $s < \frac{r-1}{r}$ are practical. If $l = a_{n-1} = rc - c$ then the ZD strategies are generous which enforce the analogous slopes as "extortionate players". If $s = 0$ then the ZD strategies are equalizers, in this case

1. Presuming $n \leq \frac{r}{r-1}$ for every $0 \leq l \leq rc - c$, equalizers can be enforced.
2. Presuming $\frac{r}{r-1} < n \leq \frac{2r}{r-1}$ then the fraction of the original $0 < l < rc - c$, "can be enforced"; and
3. Presuming $n > \frac{2r}{r-1}$, there are no means to make things equal.

For $r > 1$ we conclude that the set of equalizer strategies disappear when n becomes large.

Lemma 5.1. *"For the public goods game the baseline l determined by*

$$\max\left\{0, \frac{rc\,(n-1)}{n} - \frac{c}{1-s}\right\} \leq l \leq \min\left\{\frac{rc}{n} - c + \frac{c}{1-s}, rc - c\right\}, \tag{5.6}$$

with at least one strict inequality".

Proof. Consider the single round payoff a_j and b_j in the bounds by inequalities of

$$\max_{0\leq j\leq n-1}\left\{b_j - \frac{j}{n-1}\frac{b_j - a_{j-1}}{1-s}\right\} \leq l \leq \min_{0\leq j\leq n-1}\left\{a_j + \frac{n-j-1}{n-1}\frac{b_{j+1} - a_j}{1-s}\right\},$$

and because the bounds are linear in j. Then the lemma. □

5.1.1 Alliances for public goods game

According to proposition 4.5, a linear relationship with parameters l and s^u can be enforced by a coalition of n_u alliance members if and only if one of the following conditions is met: either s^u is equal to 1, or the inequalities hold true

$$0 \leq l \leq rc - c$$
$$\frac{(n-n_u)\,rc}{n} - \frac{c}{1-s^u} \leq l \leq \frac{(rn_u - n)\,c}{n} + \frac{c}{1-s^u}. \tag{5.7}$$

For the extortion, generous, and equalizers strategies we derive the following results:

1. Extortion strategies $l = b_0 = 0$. For a certain slope s^u inequalities (5.7) become

$$\frac{n_u}{n} \geq \frac{r\,(1-s^u)-1}{r\,(1-s^u)}. \tag{5.8}$$

   In particular if $s^u \to 0$ then $\frac{n_u}{n} \geq \frac{r-1}{r}$. The satisfaction of this condition belongs to $n_n = n-1$, is remains always extortionate strategy.

2. "Generous strategies" $l = a_{n-1} = rc - c$, inequalities 5.7 is the same as the case of extortion strategies.

3. Equalizer strategies $s = 0$ inequalities 5.7 implies $0 \leq l \leq rc - c$, then the allies need the inequality

$$\frac{n_u}{n} \geq \frac{r-1}{r}. \tag{5.9}$$

However, for the payoff of outsiders for $0 \leq l \leq rc - c$ members of comrades needs no more than to surpass

$$\frac{n_u}{n} \geq \frac{(n-2)(r-1)}{n+(n-2)r}. \tag{5.10}$$

6 Conclusion

Press and Dyson's identification of a novel category of strategies, known as ZD strategies, in the context of the repeated Prisoner's Dilemma, came as a remarkable and unexpected advancement. This discovery was particularly astonishing considering the considerable amount of previous research conducted on the dilemma. However, the situation is different for persistent multiplayer dilemmas, where there has been a lack of theoretical study on cooperation in large groups. The complexities of recurring n-player dilemmas and exponential payoff computations have hindered research. Our demonstration shows that insights from the repeated prisoner's dilemma are directly applicable to broader social issues involving any number of participants. The combination of Akin's lemma and Press and Dyson's theory has laid the groundwork for a significant advancement, establishing a fresh framework in the field. This new framework holds great promise for further developments and applications.

Our approach has contributed three important dimensions to the theory of recurring multiplayer dilemmas. Our initial finding reveals the universality of ZD strategies across various social problems, regardless of the particular issue, the number of participants involved, or the countermeasures implemented by other individuals in the group. In essence, ZD strategies transcend the specifics of the situation and apply broadly in diverse social contexts. These tactics allow players to unilaterally impose linear payment relationships. We demonstrated the applicability of fair, greedy, and extortionate strategies in various social situations. Extortionate strategies provide relative advantages to a player, fair strategies mitigate inequality within a group, and generous strategies facilitate the resumption of mutual cooperation after unintentional defections. Each strategy class offers distinct advantages. Furthermore, ZD tactics are remarkably simple and straightforward in their implementation.

Our second approach involved investigating the efficacy of both pure memory-one strategies and ZD strategies concerning their ability to maintain cooperation in dilemmas involving multiple participants. We sought to understand how these two types of strategies performed in sustaining collaborative behaviors within the context of multiplayer scenarios. We identified that ZD strategies must strike a balance between being liberal (to avoid surpassing others) and cautious (to avoid being excessively generous). The optimal level of generosity is determined by the group size rather than the specific social dilemma at hand. Generally, we found that participants need to display less generosity in larger groups to maintain cooperation successfully.

In our study, we took the idea of ZD strategies and expanded it to incorporate groups of players known as ZD alliances. The alliances were established with two coordination levels:

strategy alliances, where members adopt the same ZD strategy without coordinating individual moves, and synchronized alliances, where members act as a unified entity, collectively deciding to cooperate or defect in each round. The impact of these alliances on the game's outcomes varied significantly based on these distinct elements.

Declarations

Availability of data and materials

Data sharing is not applicable to this paper

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Conflict of interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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