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Analyzing the fractional order T. Regge problem using the Laplace transformation method

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Abstract. This study uses the Laplace transformation method to solve the fractional-order T. Regge problem. In this paper, we develop formulations for the fractional Laplace transform applied to fractional integrals and derivatives, and we use this method to solve the T. Regge problem. Moreover, several examples are presented to demonstrate the method's value and effectiveness. Examples prove that the Laplace transformation method significantly advances the fractional computation field and can potentially solve fractional differential equations (FDEs). On the other hand, the advantages and disadvantages of the method are provided.

Keywords: Laplace transform, T.Regge problem, fractional differential equations, fractional derivatives.

2020 Mathematics Subject Classification: 44A10, 34A08, 35R11, 34K37. MSC2020

1 Introduction

Fractional calculus has emerged as a powerful tool for understanding complex behaviours of materials and processes across various scientific and technological domains [34,37]. By extending the concept of traditional calculus to non-integer order derivatives and integrals, fractional calculus offers a more flexible framework for modelling phenomena in biology, chemistry, viscoelasticity, anomalous diffusion, fluid mechanics, acoustics, and control theory [34,37]. A significant application of fractional calculus is in the study of fractional differential equations, often used to describe systems with memory effects or singularities [14,15,36].
To solve these fractional differential equations, researchers have developed a range of analytical and numerical methods [2,7,22,23]. Among these, integral transforms, especially the Laplace transform, has gained popularity due to its simplicity, precision, and ability to handle complex problems without cumbersome calculations. The Laplace transform, in particular, plays a key role in addressing scientific, engineering, and social science problems by providing efficient solutions to fractional differential equations [2,7,22,23].

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Additionally, fractional calculus has proven useful in mathematical engineering and modelling, such as in the field of chemistry. For example, in the meshless numerical solution of the fractional reaction-convection-diffusion equation has been applied to model the behaviour of proton exchange membrane (PEM) fuel cells [13,22,23,35].
The Regge issue arises in the context of quantum scattering. The SturmLiouville equation in this problem must be solved using the semiaxis after the variables have been separated and the three-dimensional Schrodinger equation with radial symmetric potential has been solved. It is comparable to the radial Schrodinger equation for S-waves in physics. See Reference [6,38] for a more comprehensive examination of the Regge issue. Ideas have been proposed on particle interactions in nuclear physics, but the details are yet unknown. Regge stated that the potential has little support and that a positive integer must satisfy the boundary requirement [6,26,38].
This article explores the solution to the fractional-order T. Regge problem. Previous studies [16, 17, 25] have addressed the existence and characteristics of this solution. This research focuses on solving the fractional-order T. Regge problem while aligning with the original functions specifications and, where suitable, applying the Laplace transformation approach [16,17,25]. Significant and beneficial properties include the existence and uniqueness of initial and boundary value problems for certain linear and nonlinear systems in both fractional and ordinary differential equations. However, these properties do not generally apply to all systems and must be assessed on a case-by-case basis [16,28,29,42,43].
Several new integral transformations have been introduced to solve fractional-order differential equations. Notable examples include Kamal's transformation [5, 10, 27] , Aboodh's method [1], Anuj's method [32], and Rishi's transformation [4,33,41], Sawi [9,19], as well as the Laplace transformation [12,39,40]. These transformations have been widely studied for their application in various problems. In particular, scholars such as Aggarwal [27,31] and others have explored many integral transformations to address significant issues in solving fractional-order differential equations. Recent studies, including those by Jwamer et al., [24], Hilmi and Jwamer [20], have applied the Rishi transformation to solve the fractional-order T. Regge problem. Similarly, the Kamal transformation method has been utilized to solve the fractional-order T. Regge problem [18,21], further advancing the theoretical foundation and numerical methods in this area. The primary objective of this study is to apply the "Laplace Transform," which possesses the necessary properties to solve fractional differential and fractional integral problems. We aim to demonstrate the effectiveness of the Laplace transformation by addressing the fractional order T. Regge problem. The proposed "Laplace transforms" are distinct from other current transforms since they produce exact outcomes without requiring intricate computations. The popular Laplace transform and the Laplace transform are examples of dualistic transformations. We will solve the fractional order T.Regge problem using the Laplace transformation to show why it is better than other transforms currently in use.

2 Mathematical formulation

The T. Regge problem for fractional order with conditions is defined as [16,17,25]:

$$-{}_0^C D_x^\alpha y(x) + q(x)y(x) = \lambda^2 p(x)y(x), \quad x \in [0,a], \quad 1 < \alpha \leq 2, \tag{2.1}$$

with boundary conditions:

$$y(0) = 0, \quad y'(a) - i\lambda y(a) = 0. \tag{2.2}$$

Here, $q(x), p(x) \in L_+[0,a]$, where $L_+[0,a]$ is the set of all integrable functions $K(x)$ on $[0,a]$, satisfying $0 < m \leq K(x) \leq M < \infty$. Additionally, $y(x) \in C[0,a]$, ${}_0^C D_x^\alpha y(x) \in C^3[0,a]$, and λ is a spectral parameter. The operator ${}_0^C D_x^\alpha$ is the Caputo fractional derivative. The Caputo definition of the fractional derivative is chosen due to its compatibility with classical initial conditions and widespread use in modeling physical problems.

In this paper, we focus on solving the fractional order T. Regge problem $(1) - (2)$ using Laplace transformations. Previous studies [16, 17, 25] have demonstrated the existence and uniqueness of solutions.

2.1 Properties of fractional calculus

Fractional integrals, derivatives, and other integral transforms are used in both pure and applied mathematics to solve a wide range of differential and integral equations. We make use of crucial definitions for our research, such as:

1. **Fractional integral of order** α [3, 35, 36]: For each $\alpha > 0$ and an integrable function $y(x)$, the right fractional integral (FI) of order α is defined as:

$$ {}_aI_t^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_a^t (x-s)^{\alpha-1} y(s)\, ds, \quad -\infty \leq a < x < \infty. \tag{3}$$

2. **Riemann-Liouville derivative of order** α [11, 35]:** For every $\alpha > 0$, and $L = \lceil \alpha \rceil$, the Riemann-Liouville derivative of order α is defined as:

$$ {}_aD_t^\alpha y(x) = \frac{1}{\Gamma(L-\alpha)} \frac{d^L}{dx^L} \int_a^t (x-s)^{L-\alpha-1} y(s)\, ds. \tag{4}$$

3. **Caputo derivative of order** α [35, 36]: Let $\alpha > 0$, $L = \lceil \alpha \rceil$, and $y(x)$ be an n-times differentiable function, $x > a$. The Caputo derivative operator of order α is defined as:

$$ {}_a^C D_t^\alpha y(x) = \frac{1}{\Gamma(L-\alpha)} \int_a^t (x-j)^{L-\alpha-1} \frac{d^L}{dj^L} y(j)\, dj. \tag{5}$$

Moreover, when using these tools in our research, it's critical to remember a few key factors [3, 8, 36]:

1. The fractional operator is linear.

2. The relationship between the integration and differentiation of the Caputo operator of order α is as follows:

$$ {}^C D_a^\alpha (I_a^\alpha f(t)) = f(t). $$

3. The fractional integral of the Caputo derivative is:

$$ I_a^\alpha ({}^C D_a^\alpha f(t)) = f(t) - \sum_{n=0}^{m-1} \frac{(t-a)^n}{n!} f^{(n)}(a). \tag{6}$$

4. Note that:

$$ {}^C D_a^\alpha (I_a^\alpha f(t)) \neq I_a^\alpha ({}^C D_a^\alpha f(t)). $$

5. Additional properties:

$$I^{\alpha}t^{\epsilon} = \frac{\Gamma[1+\epsilon]}{\Gamma[1+\epsilon+\alpha]}t^{\epsilon+\alpha}, \quad \text{and} \quad D_a^{\alpha}t^{\epsilon} = {}^{C}D_a^{\alpha}t^{\epsilon} = \frac{\Gamma[1+\epsilon]}{\Gamma[1+\epsilon-\alpha]}t^{\epsilon-\alpha}.$$

Definition: The two-parameter Mittag-Leffler function is defined as:

$$E_{a,b}(s) = \sum_{n=0}^{\infty} \frac{s^n}{\Gamma(an+b)}, \quad a > 0, b > 0, s \in \mathbb{C}.$$

The inverse Laplace transform of a related expression is given by:

$$\mathcal{L}^{-1}\left[\frac{s^{-(\beta-\alpha)}}{s^{\alpha}-a}\right] = t^{\beta-1}E_{\alpha,\beta}(at^{\alpha}), \quad \alpha, \beta > 0, s^{\alpha} > |a|.$$

3 Fundamentals of Laplace transform

Let $s \in \mathbb{C}$, then the Laplace transformation (LT) is defined as:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st}f(t)\,dt, \quad s \neq 0, t \in (0,\infty).$$

The Laplace transformation has two key propositions related to fundamental functions. These propositions help us understand how the Laplace transformation and its inverse work, along with a noteworthy convolution property.

Convolution Property [4]: If $\mathcal{L}\{y(x)\} = F_1(s)$ and $\mathcal{L}\{z(x)\} = F_2(s)$, then:

$$\mathcal{L}\{y(x) * z(x)\} = F_1(s)F_2(s),$$

where $*$ denotes the convolution of y and z, defined as:

$$y(x) * z(x) = \int_0^t y(x-u)z(u)\,du.$$

In [4,33], fundamental functions, along with their Laplace transforms and inverse Laplace transforms, have been presented.

4 Laplace transform for fractional derivatives

The Laplace transform for derivatives [39] is:

$$\mathcal{L}\{y^{(n)}(x)\} = s^nY(s) - \sum_{k=0}^{n-1} s^k y^{(n-1-k)}(0).$$

We use the above formula to derive the Laplace transform for fractional derivatives for both types (Riemann and Caputo).

Proposition 3: If $Y(s)$ is the Laplace transform of the order-α fractional derivative of Riemann-Liouville, and $y(x)$ is a function, then:

$$\mathcal{L}\{D_t^{\alpha}y(x)\} = s^{\alpha}Y(s) - \sum_{k=0}^{m-1} s^k D_t^{\alpha-k-1}Y(0).$$

Proposition 4: If $Y(s)$ is the Laplace transform of the order-α fractional derivative of Caputo, and $y(x)$ is a function, then:

$$\mathcal{L}\{{}^{c}D_{a}^{\alpha}y(x)\} = s^{\alpha}Y(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1}y^{(k)}(0).$$

Proof: From the relationship between the fractional integral and the Caputo fractional derivative, we have:

$$\mathcal{L}\{D^{\alpha}y(x)\} = \mathcal{L}\{I^{n-\alpha}D^{n}y(x)\}.$$

Thus,

$$\mathcal{L}\{{}^{c}D_{a}^{\alpha}y(x)\} = \mathcal{L}\{J_{t}^{n-\alpha}\,{}^{c}D_{t}^{n}y(x)\}.$$

Expanding this:

$$= s^{-(n-\alpha)}\mathcal{L}\{y^{(n)}(x)\}.$$

Using the Laplace transform for derivatives:

$$\mathcal{L}\{{}^{c}D_{a}^{\alpha}y(x)\} = s^{-(n-\alpha)}\left[s^{n}Y(s) - \sum_{k=0}^{m-1} s^{n-k-1}y^{(k)}(0)\right].$$

Simplify the expression:

$$\mathcal{L}\{{}^{c}D_{a}^{\alpha}y(x)\} = s^{\alpha}Y(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1}y^{(k)}(0).$$

5 Solution of fractional order T. Regge problem by Laplace transformation method

This article uses Laplace's transformation approach to solve the T.Regge problem of fractional order (1)(2). We will look at two scenarios: any continuous function $q(x)$ and $p(x) = 1$, and $q(x) = M = p(x)$. Additionally, a thorough description of how this approach functions in each scenario is provided below.

Case 1: Constant coefficients

In this case, we suppose that $q(x) = M = p(x)$. Taking Laplace transformation for both sides, we get:

$$-{}_{0}^{C}D_{x}^{\alpha}y(x) + My(x) = \lambda^{2}My(x),$$

$${}_{0}^{C}D_{x}^{\alpha}y(x) + M(\lambda^{2}-1)y(x) = 0,$$

$$L\{{}_{0}^{C}D_{x}^{\alpha}y(x)\} + L\{M(\lambda^{2}-1)y(x)\} = L\{0\}.$$

Let $L\{y(x)\} = \int_{0}^{\infty} e^{-st}y(t)\,dt = Y(s)$ and $L\{0\} = 0$. From the properties of Laplace transformation, we have:

$$L[{}_{0}^{C}D_{x}^{\alpha}f(t)] = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}f^{(k)}(0), \quad \alpha \in (n-1, n].$$

For $\alpha \in (1,2]$:

$$L[({}_0^C D_x^\alpha y(x))] = s^\alpha Y(s) - s^{\alpha-1} y(0) - s^{\alpha-2} y'(0).$$

From the boundary condition $y(0) = 0$:

$$L[D^\alpha y(x)] = s^\alpha Y(s) - s^{\alpha-2} y'(0).$$

Thus:

$$L\{{}_0^C D_x^\alpha y(x)\} + L\{M(\lambda^2 - 1) y(x)\} = L\{0\}$$

then

$$s^\alpha Y(s) - s^{\alpha-2} y'(0) + M(\lambda^2 - 1) Y(s) = 0,$$

$$Y(s)\big(s^\alpha + M(\lambda^2 - 1)\big) = A s^{\alpha-2}, \quad A = y'(0).$$

Hence:

$$Y(s) = \frac{A s^{\alpha-2}}{s^\alpha - M(1 - \lambda^2)}.$$

The corresponding inverse Laplace transform is:

$$f(t) = \frac{1}{2\pi i} \lim_{t\to\infty} \int_{\gamma - it}^{\gamma + it} F(s) e^{st}\, dt = L^{-1}[F(s)], \quad \gamma \in \mathbb{R},\, i = \sqrt{-1}.$$

Taking the inverse Laplace transform:

$$y(x) = L^{-1}[Y(s)] = L^{-1}\left(\frac{A s^{\alpha-2}}{s^\alpha - M(1 - \lambda^2)}\right).$$

Using the Mittag-Leffler function, we get:

$$y(x) = A x E_{\alpha,2}(M(1 - \lambda^2) x^\alpha), \quad A \text{ is constant, } \alpha \in (1,2],\ \lambda \text{ is a complex number.}$$

It exists if $\alpha, \beta > 0, s^\alpha > |a|$, but in our case $\beta = 2 > 0$ and $\alpha \in (1,2]$. So $\alpha > 0$, and the only condition is $s^\alpha > M|1 - \lambda^2|$ for the existence of the above inverse Laplace transformation.

Now, the solution of the fractional boundary value problem given by (1) and (2) is:

$$y(x) = A x E_{\alpha,2}\left(M(1 - \lambda^2) x^\alpha\right), \quad x \in [0, a], \quad \alpha \in (1,2].$$

Here, $A = y'(0)$ is any non-zero constant.

Here, for $\alpha = 2$:

$$y(x) = A x E_{2,2}(M(1 - \lambda^2) x^2) = \frac{A}{\sqrt{M(1 - \lambda^2)}} \sinh\left(\sqrt{M(1 - \lambda^2)}\, x\right).$$

For $\alpha = 1$:

$$y(x) = A x E_{1,2}(M(1 - \lambda^2) x) = A x \sum_{n=0}^{\infty} \frac{\left(M(1 - \lambda^2) x^2\right)^n}{\Gamma(n+2)} = A x \frac{e^{M(1-\lambda^2)x} - 1}{M(1 - \lambda^2) x} = A \frac{e^{M(1-\lambda^2)x} - 1}{M(1 - \lambda^2)}.$$

Case 2: Variable coefficients

In this case, we assume that the weight function $p(x) = 1$ and $q(x)$ is any continuous function. Let $q(x)y(x) = f(x)$. Applying the Laplace transform to both sides:

$$-{}_0^C D_x^\alpha y(x) + f(x) = \lambda^2 y(x) \implies ({}_0^C D_x^\alpha y(x)) + \lambda^2 y(x) = f(x).$$

Taking the Laplace transform:

$$\mathcal{L}\{({}_0^C D_x^\alpha y(x))\} + \mathcal{L}\{\lambda^2 y(x)\} = \mathcal{L}\{f(x)\}.$$

Let $\mathcal{L}\{y(x)\} = \int_0^\infty e^{-st} y(t)\, dt = Y(s)$ and $\mathcal{L}\{f(x)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$. From the properties of the Laplace transform for $\alpha \in (1, 2]$, we have:

$$\mathcal{L}[{}_0^C D_x^\alpha y(x)] = s^\alpha Y(s) - \sum_{k=0}^{1} s^{\alpha-k-1} y^{(k)}(0) = s^\alpha Y(s) - s^{\alpha-1} y(0) - s^{\alpha-2} y'(0).$$

Using the boundary condition $y(0) = 0$, it simplifies to:

$$\mathcal{L}[{}_0^C D_x^\alpha y(x)] = s^\alpha Y(s) - s^{\alpha-2} y'(0).$$

Then,

$$\mathcal{L}\{{}_0^C D_x^\alpha y(x)\} + \mathcal{L}\{\lambda^2 y(x)\} = \mathcal{L}\{f(x)\} \implies s^\alpha Y(s) - s^{\alpha-2} y'(0) + \lambda^2 Y(s) = F(s).$$

Thus,

$$Y(s)(s^\alpha + \lambda^2) = c_1 s^{\alpha-2} + F(s),$$

where $c_1 = y'(0)$. Therefore,

$$Y(s) = \frac{c_1 s^{\alpha-2}}{s^\alpha + \lambda^2} + \frac{F(s)}{s^\alpha + \lambda^2}.$$

If it exists, taking the inverse Laplace transform of both sides, we get:

$$\mathcal{L}\{y(x)\} = Y(s) \implies y(x) = \mathcal{L}^{-1}\{Y(s)\}, \quad \mathcal{L}\{f(x)\} = F(s) \implies f(x) = \mathcal{L}^{-1}\{F(s)\}.$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{c_1 s^{\alpha-2}}{s^\alpha + \lambda^2}\right\} + \mathcal{L}^{-1}\left\{\frac{F(s)}{s^\alpha + \lambda^2}\right\}.$$

From the properties of the Laplace transform:

$$\mathcal{L}^{-1}\left[\frac{s^{\alpha-2}}{s^\alpha + \lambda^2}\right] = x E_{\alpha,2}(-\lambda^2 x^\alpha), \quad \text{where } s^\alpha > |\lambda^2|,$$

and

$$\mathcal{L}^{-1}\left[\frac{1}{s^\alpha + \lambda^2}\right] = x^{\alpha-1} E_{\alpha,\alpha}(-\lambda^2 x^\alpha).$$

It implies that:

$$y(x) = c_1 x E_{\alpha,2}(-\lambda^2 x^\alpha) + x^{\alpha-1} E_{\alpha,\alpha}(-\lambda^2 x^\alpha) f(x).$$

Using the properties of the Mittag-Leffler function $E_{\alpha,\alpha}(s) = \frac{1}{s} E_{\alpha,0}(s)$, we have:

$$x^{\alpha-1} E_{\alpha,\alpha}(-\lambda^2 x^\alpha) = x^{\alpha-1} \frac{1}{-\lambda^2 x^\alpha} E_{\alpha,0}(-\lambda^2 x^\alpha) = \frac{-1}{\lambda^2 x} E_{\alpha,0}(-\lambda^2 x^\alpha).$$

Thus,

$$y(x) = c_1 x E_{\alpha,2}(-\lambda^2 x^\alpha) - \frac{1}{\lambda^2 x} E_{\alpha,0}(-\lambda^2 x^\alpha) f(x).$$

Using the convolution property, the general solution can be written as:

$$y(x) = c_1 x E_{\alpha,2}(-\lambda^2 x^\alpha) - \int_0^x \frac{E_{\alpha,0}(-\lambda^2 (x-s)^\alpha)}{\lambda^2 (x-s)} q(s) y(s)\, ds,$$

which is the Volterra integral equation equivalent of (1) and (2).

Remark

If $\alpha \to 2$, the integral equation (2.6) becomes:

$$y(x) = c_1 x E_{2,2}\left(-\lambda^2 x^2\right) - \int_0^x \frac{E_{2,0}\left(-\lambda^2 (x-s)^2\right)}{\lambda^2 (x-s)} q(s) y(s)\, ds.$$

Then,

$$y(x) = \frac{c_1}{i|\lambda|} \sinh(i|\lambda| x) - \int_0^x \frac{\sin\left(i|\lambda|(x-s)\right)}{i|\lambda|} q(s) y(s)\, ds.$$

6 Illustrative examples

Four problems that demonstrate the use of the Laplace transformation are listed in this section. In both the Liouville-Caputo and Riemann-Liouville senses, these examples show how to precisely solve linear fractional differential equations of multi-high order. The aforementioned problems were initially tackled by Laplace transformation methods and other techniques in references [16, 26]. In this study, we present a novel approach that uses the Laplace transformation method to address these issues and yields the same outcomes.
Furthermore, case 1 in the first example has $M = 1/10$ and $\alpha = 4/3$. Examples 2 and 3 relate to situation 2, where $\lambda = i$ and $\lambda = 1$, respectively, differ while $\alpha = 3/2$ and $p(x) = q(x) = 1$. In the final example, a linear differential equation for fractional derivatives of Liouville-Caputo is examined.

Example 1 Consider the fractional boundary value problem (1)-(2) in case 1, where $M = \frac{1}{6}$, $a = 1$, and $\alpha = \frac{4}{3}$:

$$-{}_0^C D_x^{4/3} y(x) + \frac{1}{6} y(x) = \lambda^2 \frac{1}{6} y(x), \quad x \in [0, 1], \quad y(0) = 0, \quad y'(1) - i\lambda y(1) = 0.$$

Solution: We have $M = \frac{1}{6}$. Now

$$-{}_0^C D_x^{4/3} y(x) + \frac{1}{6} y(x) = \frac{1}{6} \lambda^2 y(x).$$

$${}_0^C D_x^{4/3} y(x) = \frac{1}{6}(1 - \lambda^2) y(x).$$

Taking the Laplace transform of the above equation:

$$L\{{}_0^C D_x^{4/3} y(x)\} = L\left\{\frac{1}{6}(1 - \lambda^2) y(x)\right\}.$$

Using Proposition 1 and Proposition 5, the Laplace transformation of the Caputo fractional derivative is:

$$L\{{}^C D_a^{\alpha} y(x)\} = s^{\alpha} Y(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} y^{(k)}(0).$$

For $\alpha = \frac{4}{3}$:

$$L\{{}_0^C D_x^{4/3} y(x)\} = s^{4/3} Y(s) - \sum_{k=0}^{1} s^{1/3-k} y^{(k)}(0),$$

Then

$$L\{{}_0^C D_x^{4/3} y(x)\} = s^{4/3} Y(s) - s^{-2/3} y'(0) - s^{1/3} y(0).$$

Since $y(0) = 0$:

$$L\{{}_0^C D_x^{4/3} y(x)\} = s^{4/3} Y(s) - s^{-2/3} y'(0).$$

Thus:

$$s^{4/3} Y(s) - s^{-2/3} y'(0) = \frac{1}{6}(1 - \lambda^2) Y(s),$$

$$(s^{4/3} + \frac{1}{6}(\lambda^2 - 1)) Y(s) = s^{-2/3} y'(0).$$

Let $g = y'(0)$. Then:

$$Y(s) = \frac{g s^{-2/3}}{s^{4/3} + \frac{1}{6}(\lambda^2 - 1)}.$$

Taking the inverse Laplace transform:

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{g s^{-2/3}}{s^{4/3} + \frac{1}{6}(\lambda^2 - 1)}\right\}.$$

Using references [16,30] for the inverse Laplace transform:

$$y(x) = g x E_{\frac{4}{3},2}\left(\frac{1}{6}(1 - \lambda^2) x^{3/2}\right).$$

On the other hand, we can derive some results easily by using the fractional integral operator before applying the Laplace transform. We explain this below:

$$-{}_0^C D_x^{4/3} y(x) + \frac{1}{6} y(x) = \lambda^2 \frac{1}{6} y(x) \implies {}_0^C D_x^{4/3} y(x) = \frac{1}{6}(1 - \lambda^2) y(x).$$

Applying the fractional integral operator to both sides, we get:

$$I^{4/3} {}_0^C D_x^{4/3} y(x) = y(x) - \sum_{k=0}^{1} \frac{x^k}{k!} y^{(k)}(0) = y(x) - y(0) - x y'(0) = y(x) - x y'(0).$$

Thus:

$$y(x) - x y'(0) = \frac{1}{6}(1 - \lambda^2) I^{4/3}(y(x)).$$

Taking the Laplace transform of both sides:

$$\mathcal{L}\{y(x)\} - g\mathcal{L}\{x\} = \frac{1}{6}(1 - \lambda^2)\mathcal{L}\{I^{4/3}(y(x))\}.$$

Using the Laplace transform properties, this becomes:

$$Y(s) - g\frac{1}{s^2} = \frac{1}{6}(1 - \lambda^2)\frac{Y(s)}{s^{4/3}}.$$

Rearranging, we obtain:

$$Y(s) = \frac{g s^{-2/3}}{1 + \frac{1}{6}(1 - \lambda^2) s^{4/3}}.$$

The solution is:

$$y(x) = g x E_{4/3,2}\left(\frac{1}{6}(1 - \lambda^2) x^{3/2}\right),$$

where $E_{4/3,2}$ is the Mittag-Leffler function.

Example 2 The Liouville-Caputo fractional derivative's linear fractional differential equation can be found as follows:

$$ {}_0^C D_t^{1.4} y(x) + {}_0^C D_t^{0.4} y(x) + y(x) = \frac{\Gamma(2.3)}{\Gamma(1.1)} x^{0.1} + \frac{\Gamma(2.3)}{\Gamma(2.1)} x^{1.2} + \frac{\Gamma(2.3)}{\Gamma(2.5)} x^{1.5}, $$

under the initial conditions:

$$ y(0) = 0, \quad y'(0) = 0. $$

Solution: For both sides, apply the Laplace transform using Proposition (5), we obtain:

$$ \mathcal{L}\{ {}_0^C D_t^{1.4} y(x) + {}_0^C D_t^{0.4} y(x)\} + \mathcal{L}\{y(x)\} = \mathcal{L}\left\{ \frac{\Gamma(2.5)}{\Gamma(1.1)} x^{0.1} + \frac{\Gamma(2.5)}{\Gamma(2.1)} x^{1.1} + x^{1.5} \right\}. $$

By using Propositions (1) and (5), we get:

$$ \begin{aligned} s^{1.4} Y(s) - \sum_{k=0}^{1} s^{1.4-k} y^{(k)}(0) + s^{0.4} Y(s) - \sum_{k=0}^{0} s^{0.4-k} y^{(k)}(0) + Y(s) \\ = \frac{\Gamma(2.5)}{\Gamma(1.1)} \frac{1}{s^{1.1}} \Gamma(1.1) + \frac{\Gamma(2.5)}{\Gamma(2.1)} \frac{1}{s^{2.1}} \Gamma(2.1) + \frac{\Gamma(2.5)}{\Gamma(2.5)} \frac{1}{s^{2.5}}. \end{aligned} $$

Simplify the statement:

$$ s^{1.4} Y(s) + s^{0.4} Y(s) + Y(s) = \Gamma(2.5) \left(\frac{1}{s^{1.1}} + \frac{1}{s^{2.1}} + \frac{1}{s^{2.5}} \right). $$

We obtain:

$$ Y(s)(s^{1.4} + s^{0.4} + 1) = \Gamma(2.5) \left(\frac{1}{s^{1.1}} + \frac{1}{s^{2.1}} + \frac{1}{s^{2.5}} \right), $$

suggesting that:

$$ Y(s) = \frac{\Gamma(2.5)}{s^{2.5}}. $$

When we now apply the inverse Laplace transform to both sides, we obtain:

$$ \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{ \frac{\Gamma(2.5)}{s^{2.5}} \right\}. $$

Therefore, the exact solution is:

$$ y(x) = x\sqrt{x}. $$

Example 3

If $\lambda = i$ and applying Case 1 (i.e., $p(x) = 1 = q(x)$) for the fractional boundary value problem (1)-(2):

$$ x \in [0, a = 1], \quad \alpha \in (1, 2], $$

$$ -{}_0^C D_x^{3/2} y(x) + y(x) = \lambda^2 y(x), \quad x \in [0, 1], \quad y(0) = 0, \quad y'(1) - i\lambda y(1) = 0. $$

Solution: If we have $\lambda = i$, then:

$$ -{}_0^C D_x^{3/2} y(x) + y(x) = -y(x) \implies {}_0^C D_x^{3/2} y(x) = 2y(x). $$

Since $1 < \alpha \leq 2$, using the Laplace transform (LT), we have:

$$\mathcal{L}\{D^\alpha y(t)\} = 2\mathcal{L}\{y(t)\} \implies s^\alpha Y - s^{\alpha-1}y(0) - s^{\alpha-2}y'(0) = 2Y.$$

Thus:

$$Y = \frac{c_1}{2}\frac{s^{\alpha-2}}{s^\alpha - 1} \implies Y = \frac{c_1}{2s^2}\frac{1}{1-1/s^\alpha},$$

where $y'(0) = c_1$. Using the identity:

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots, \quad |u| < 1,$$

we have:

$$Y = \frac{c_1}{2}\left(\frac{1}{s^2} + \frac{1}{s^{\alpha+2}} + \frac{1}{s^{2\alpha+2}} + \cdots\right).$$

The solution becomes:

$$y(t) = \frac{c_1}{2}\left(\frac{t}{\Gamma(2)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \cdots\right).$$

Note that:

$$\lim_{\alpha\to 2} y(t) = \frac{c_1}{2}\left(\frac{t}{1!} + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right) = \frac{c_1}{2}\sinh(t).$$

Example 4

The problem (1)-(2) in this example changes to $\lambda = 1$ and $p(x) = 1 = q(x)$, which is Case 1:

$$x \in [0, a = 1], \quad \alpha \in (1, 2],$$

$$-{}_0^C D_x^{3/2} y(x) + y(x) = \lambda^2 y(x), \quad x \in [0,1], \quad y(0) = 0, \quad y'(1) - i\lambda y(1) = 0.$$

Solution: If we have $\lambda = 1$, then:

$$-{}_0^C D_x^{3/2} y(x) + y(x) = -y(x) \implies {}_0^C D_x^{3/2} y(x) = 0.$$

Using Laplace's transformation on both sides:

$$\mathcal{L}\{{}_0^C D_x^{3/2} y(t)\} = \mathcal{L}\{0\}.$$

$$\mathcal{L}\{{}_0^C D_x^{3/2} y(x)\} = s^{3/2}Y(s) - \sum_{k=0}^{1} s^{0.5-k} y^{(k)}(0) = s^{3/2}Y(s) - y'(0)s^{0.5} - y(0)s^{-0.5} = s^{3/2}Y(s) - y'(0)s^{0.5}.$$

Now:

$$s^{3/2}Y(s) - y'(0)s^{0.5} = 0 \implies Y(s) = \frac{a}{s}, \quad \text{where } a = y'(0).$$

Taking the inverse Laplace transform:

$$\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{a}{s}\right),$$

we find the solution:

$$y(x) = ax.$$

7 Conclusion

Conclusion The Laplace transformation method is a useful technique for solving fractional differential equations (FDEs), and it is the subject of this paper's investigation, specially fractional order T.Regge problem. Moreover, they present advantages of reducing equation difficulty, simplifying their forms, and managing FDEs with constant coefficients. Research has shown that Laplace's approach is a useful tool for many FDEs. Though it has the same limits as any other approach, it offers an alternative to existing FDE methods and has the potential to make a major contribution to fractional computation. Simple and promising results are obtained from this study's original application of the Laplace method to the Regge fractional order problem. In conclusion, the Laplace transformation method provides a practical strategy for solving FDEs. Further research and development could make it an even more useful tool for managing increasingly complex FDEs.

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Conflict of interest

The authors have no conflicts of interest to declare.

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