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Taylor collocation method for high-order neutral delay Volterra integro-differential equations

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Abstract. In this paper, the Taylor collocation method is applied to numerically solve a kth-order neutral linear Volterra integro-differential equation with constant delay and variable coefficients. We also provide a rigorous analysis to estimate the difference between the exact and approximate solution and their derivatives up to order $k-1$. Numerical examples are included to prove the performance of the presented convergent algorithm.

Keywords: High-order neutral delay linear Volterra integro-differential equation, Collocation method, Taylor polynomials, Convergence analysis.

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1 Introduction

In this paper, we study a numerical method for the solution of kth-order neutral delay linear Volterra integro-differential equations with constant delay $\tau > 0$ and variable coefficients of the form:

$$x^{(k)}(t) = g(t) + \sum_{v=0}^{k-1} \left(L_v(t) x^{(v)}(t) + M_v(t) x^{(v)}(t-\tau) + (\mathcal{V}_{1,v}x)(t) + (\mathcal{V}_{2,v}x)(t) \right), \tag{1.1}$$

for $t \in [0, T]$ and $x(t) = \Phi(t)$ for $t \in [-\tau, 0)$, where $\mathcal{V}_{1,v}$ and $\mathcal{V}_{2,v}$ are the Volterra integral operators given by

$$(\mathcal{V}_{1,v}x)(t) := \int_0^t k_{1,v}(t,s) x^{(v)}(s) ds,$$
$$(\mathcal{V}_{2,v}x)(t) := \int_0^{t-\tau} k_{2,v}(t,s) x^{(v)}(s) ds,$$

g, and L_v and M_v, $v = 0, \ldots, k-1$, are real functions defined on the interval [0,T], and $k_{1,v}, k_{2,v}, v = 0, \ldots, k-1$, are a real functions defined on $D := \{(t,s); 0 \leq s, t \leq T\}$, all of them

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sufficiently smooth. Furthermore, we suppose that

$$\Phi^{(k)}(0) = g(0) + \sum_{v=0}^{k-1} L_v(0)\Phi^{(v)}(0) + \sum_{v=0}^{k-1} M_v(0)\Phi(-\tau) - \sum_{v=0}^{k-1} \int_{-\tau}^{0} k_{2,v}(0,s)\Phi^{(v)}(s)ds.$$

Existence and uniqueness results for (1.1) can be easily proved by comparison with the theory for Volterra integral equations (see for example [6,11]).

Equation (1.1) includes many important kinds of equations (see for example [4,6,12,13,15,18, 25,29]) and this method can be used to obtain numerical solutions of high order differential equations (for $\{k_{1,v}\}_{v=0}^{k-1} = \{k_{2,v}\}_{v=0}^{k-1} = \{M_v\}_{v=0}^{k-1} = 0$), high order integro-differential equations (for $\{k_{2,v}\}_{v=0}^{k-1} = \{M_v\}_{v=0}^{k-1} = 0$), high order delay differential equation (for $\{k_{1,v}\}_{v=0}^{k-1} = \{k_{2,v}\}_{v=0}^{k-1} = 0$). There are many existing numerical methods for solving Volterra integro-differential equations, such as Legendre spectral collocation method [26], Runge-Kutta method [7], spectral method [27,28], Polynomial collocation method [8–10,23], Tau method [14], operational matrices [2], Homotopy perturbation method [16,24], Haar wavelet method [21], Taylor polynomial [20,22].

The aim of the present paper is to apply a direct collocation method based on the use of Taylor polynomials so that we generalize the Taylor collocation method for delay VIEs [3], first and second order delay IDEs [4,5], and high order VIDEs [19].

This collocation method has the following advantages: it is direct, and the approximate solution is given by using explicit formulas; this method has an order of convergence; there is no algebraic system needed to be solved, which makes the proposed algorithm very effective and easy to implement.

The paper is organized as follows: In section 2, we divide the interval $[0, T]$ into subintervals, and we approximate the solution of (1.1) in each interval by a Taylor polynomial. Global convergence is established in section 3, and six numerical examples are provided in section 4.

2 Description of the Method

We suppose that $T = r\tau$, where $r \in \mathbb{N}$. Let Π_N be a uniform partition of the interval $I = [0, T]$ defined by $t_n^i = i\tau + nh$, $n = 0, 1, \ldots, N$, $i = 0, 1, \ldots, r-1$, where the stepsize is given by $h = \frac{\tau}{N}$. Define the subintervals $\sigma_n^i = [t_n^i; t_{n+1}^i), n = 0, 1, \ldots, N-1$, $i = 0, 1, \ldots, r-1$ and $\sigma_{N-1}^{r-1} = [t_{N-1}^{r-1}, t_N^{r-1}]$. Moreover, denote by π_{m+k-1} the set of all real polynomials of degree not exceeding $m + k - 1$, with $m > 1$. We define the real polynomial spline space of degree $m + k - 1$ as follows:

$$S_{m+k-1}^{(k-1)}(\Pi_N) := \{u \in C^{k-1}(I, R) : u_n^i = u|_{\sigma_n^i} \in \pi_{m+k-1}, n = 0, \ldots, N-1,\ i = 0, \ldots, r-1\}.$$

This is the space of piecewise polynomials of degree (at most) $m + k - 1$, such that $m + k > k \geq 1$. Its dimension is $rNm + k$, i.e., the same as the total number of the coefficients of the polynomials $u_n^i, n = 0, \ldots, N-1, i = 0, \ldots, r-1$. To find these coefficients, we use the Taylor polynomial on each subinterval.

First, suppose we approximate x in the interval σ_0^0 by the polynomial

$$u_0^0(t) = \sum_{j=0}^{m+k-1} \frac{x^{(j)}(0)}{j!} t^j, \quad t \in \sigma_0^0, \tag{2.1}$$

where $x^{(j)}(0), j = 0, \dots, m+k-1$ is the exact value of $x^{(j)}$ at 0.
Deriving equation (1.1) j-times, we get for $j = 0, 1, \dots, m-1$,

$$x^{(j+k)}(0) = g^{(j)}(0) + \sum_{v=0}^{k-1}\sum_{l=0}^{j}\binom{j}{l}L_v^{(j-l)}(0)x^{(l+v)}(0) + \sum_{v=0}^{k-1}\sum_{l=0}^{j}\binom{j}{l}M_v^{(j-l)}(0)\Phi^{(l+v)}(-\tau)$$
$$- \left(\sum_{v=0}^{k-1}\int_{t-\tau}^{0} k_{2,v}(t,s)\Phi^{(v)}(s)ds\right)^{(j)}(0)$$
$$+ \sum_{v=0}^{k-1}\sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\left[\partial_1^{(j-1-i)}k_{1,v}(t,t)\right]^{(i-l)}(0)x^{(v+l)}(0),$$

such that $x^{(v)}(0) = \Phi^{(v)}(0)$ for all $v = 0, \dots, k-1$.
Second, we approximate x by u_n^0, $n \in \{1, \dots, N-1\}$, on the interval σ_n^0, such that

$$u_n^0(t) = \sum_{j=0}^{m+k-1}\frac{\hat{u}_{n,0}^{(j)}(t_n^0)}{j!}(t-t_n^0)^j, \quad t \in \sigma_n^0, \tag{2.2}$$

where $\hat{u}_{n,0}$ is the exact solution of the integro-differential equation, for $t \in \sigma_n^0$

$$\hat{u}_{n,0}^{(k)}(t) = g(t) + \sum_{v=0}^{k-1}L_v(t)\hat{u}_{n,0}^{(v)}(t) + \sum_{v=0}^{k-1}M_v(t)\Phi^{(v)}(t-\tau) - \left(\sum_{v=0}^{k-1}\int_{t-\tau}^{0}K_{2,v}(t,s)\Phi^{(v)}(s)ds\right)^{(j)}(t_n^0)$$
$$+ \sum_{v=0}^{k-1}\sum_{i=0}^{j-1}\left[\partial_1^{(j-1-i)}K_{1,v}(t,t)\hat{u}_{n,0}^{(v)}(t)\right]^{(i)}(t_n^0) + \sum_{v=0}^{k-1}\sum_{i=0}^{n-1}\int_{t_i^0}^{t_{i+1}^0}\partial_1^{(j)}K_{1,v}(t_n^0,s)u_i^{0^{(v)}}(s)ds, \tag{2.3}$$

such that $\hat{u}_{n,0}^{(v)}(t_n^0) = u_{n-1}^{0^{(v)}}(t_n^0)$ for all $v = 0, \dots, k-1$.
Now, for all $j = 0, 1, \dots, m+k-1$ the formula for computing the values of the coefficients $\hat{u}_{n,0}^{(j)}(t_n^0)$ can be obtained by employing similar arguments to those used for obtaining the values of $x^{(j)}(0)$ above, so that we get

$$\hat{u}_{n,0}^{(j+k)}(t_n^0) = g^{(j)}(t_n^0) + \sum_{v=0}^{k-1}\sum_{l=0}^{j}\binom{j}{l}L_v^{(j-l)}(t_n^0)\hat{u}_{n,0}^{(l+v)}(t_n^0)$$
$$+ \sum_{v=0}^{k-1}\sum_{l=0}^{j}\binom{j}{l}M_v^{(j-l)}(t_n^0)\phi^{(l+v)}(t_n^0-\tau) - \left(\sum_{v=0}^{k-1}\int_{t-\tau}^{0}k_{2,v}(t,s)\Phi^{(v)}(s)ds\right)^{(j)}(t_n^0)$$
$$+ \sum_{v=0}^{k-1}\sum_{i=0}^{j-1}\left[\partial_1^{(j-1-i)}k_{1,v}(t,t)\hat{u}_{n,0}^{(v)}(t)\right]^{(i)}(t_n^0) + \sum_{v=0}^{k-1}\sum_{i=0}^{n-1}\int_{t_i^0}^{t_{i+1}^0}\partial_1^{(j)}k_{1,v}(t_n^0,s)u_i^{0^{(v)}}(s)ds$$
$$= g^{(j)}(t_n^0) + \sum_{v=0}^{k-1}\sum_{l=0}^{j}\binom{j}{l}L_v^{(j-l)}(t_n^0)\hat{u}_{n,0}^{(l+v)}(t_n^0) \tag{2.4}$$
$$+ \sum_{v=0}^{k-1}\sum_{l=0}^{j}\binom{j}{l}M_v^{(j-l)}(t_n^0)\phi^{(l+v)}(t_n^0-\tau) - \left(\sum_{v=0}^{k-1}\int_{t-\tau}^{0}k_{2,v}(t,s)\Phi^{(v)}(s)ds\right)^{(j)}(t_n^0)$$
$$+ \sum_{v=0}^{k-1}\sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\left[\partial_1^{(j-1-i)}k_{1,v}(t,t)\right]^{(i-l)}(t_n^0)\hat{u}_{n,0}^{(l+v)}(t_n^0)$$
$$+ \sum_{v=0}^{k-1}\sum_{i=0}^{n-1}\sum_{l=v}^{m+k-1}\frac{\hat{u}_{i,0}^{(l)}(t_i^0)}{(l-v)!}\int_{t_i^0}^{t_{i+1}^0}\partial_1^{(j)}k_{1,v}(t_n^0,s)(s-t_i^0)^{l-v}ds,$$

for $j = 0, \dots, m-1$ such that $\hat{u}_{n,0}^{(v)}(t_n^0) = u_{n-1}^{0^{(v)}}(t_n^0)$ for all $v = 0, 1, \dots, k-1$.
Third, we approximate x on the interval σ_n^p, $n \in \{0, \dots, N-1\}$ and $p \in \{1, \dots, r-1\}$, by u_n^p such that,

$$u_n^p(t) = \sum_{j=0}^{m+k-1} \frac{\hat{u}_{n,p}^{(j)}(t_n^p)}{j!}(t - t_n^p)^j; \quad t \in \sigma_n^p, \tag{2.5}$$

where $\hat{u}_{n,p}$ is the exact solution of the integro-differential equation:

$$\begin{aligned}
\hat{u}_{n,p}^{(k)}(t) &= g(t) + \sum_{v=0}^{k-1} L_v(t)\hat{u}_{n,p}^{(v)}(t) + \sum_{v=0}^{k-1} M_v(t) u_n^{p-1^{(v)}}(t-\tau) \\
&+ \sum_{v=0}^{k-1}\sum_{i=0}^{p-2}\sum_{d=0}^{N-1} \int_{t_d^i}^{t_{d+1}^i} k_{2,v}(t,s) u_d^{i^{(v)}}(s)ds + \sum_{v=0}^{k-1}\sum_{d=0}^{n-1} \int_{t_d^{p-1}}^{t_{d+1}^{p-1}} k_{2,v}(t,s) u_d^{p-1^{(v)}}(s)ds \\
&+ \sum_{v=0}^{k-1} \int_{t_n^{p-1}}^{t-\tau} k_{2,v}(t,s) u_n^{p-1^{(v)}}(s)ds + \sum_{v=0}^{k-1}\sum_{i=0}^{p-1}\sum_{d=0}^{N-1} \int_{t_d^i}^{t_{d+1}^i} k_{1,v}(t,s) u_d^{i^{(v)}}(s)ds \\
&+ \sum_{v=0}^{k-1}\sum_{d=0}^{n-1} \int_{t_d^p}^{t_{d+1}^p} k_{1,v}(t,s) u_d^{p^{(v)}}(s)ds + \sum_{v=0}^{k-1} \int_{t_n^p}^{t} k_{1,v}(t,s)\hat{u}_{n,p}^{(v)}(s)ds,
\end{aligned} \tag{2.6}$$

for $t \in \sigma_n^p$, $\hat{u}_{0,p}^{(v)}(t_0^p) = u_{N-1}^{p-1^{(v)}}(t_0^p)$ and $\hat{u}_{n,p}^{(v)}(t_n^p) = u_{n-1}^{p^{(v)}}(t_n^p)$ for all $v = 0, \dots, k-1$.
The coefficients $\hat{u}_{n,p}^{(j)}(t_n^p)$ for $j = 0, \dots, m-1$, are given by the following formula:

$$\begin{aligned}
\hat{u}_{n,p}^{(j+k)}(t_n^p) &= g^{(j)}(t_n^p) + \sum_{v=0}^{k-1}\sum_{l=0}^{j} \binom{j}{l} L_v^{(j-l)}(t_n^p)\hat{u}_{n,p}^{(l+v)}(t_n^p) \\
&+ \sum_{v=0}^{k-1}\sum_{l=0}^{j} \binom{j}{l} M_v^{(j-l)}(t_n^p)\hat{u}_{n,p-1}^{(l+v)}(t_n^{p-1}) \\
&+ \sum_{v=0}^{k-1}\sum_{i=0}^{p-2}\sum_{d=0}^{N-1}\sum_{l=v}^{m+k-1} \frac{\hat{u}_{d,i}^{(l)}(t_d^i)}{(l-v)!} \int_{t_d^i}^{t_{d+1}^i} \partial_1^{(j)} k_{2,v}(t_n^p,s)(s-t_d^i)^{l-v}ds \\
&+ \sum_{v=0}^{k-1}\sum_{d=0}^{n-1}\sum_{l=v}^{m+k-1} \frac{\hat{u}_{d,p-1}^{(l)}(t_d^{p-1})}{(l-v)!} \int_{t_d^{p-1}}^{t_{d+1}^{p-1}} \partial_1^{(j)} k_{2,v}(t_n^p,s)(s-t_d^{p-1})^{l-v}ds \\
&+ \sum_{v=0}^{k-1}\sum_{i=0}^{j-1}\sum_{l=0}^{i} \binom{i}{l} \left[\partial_1^{(j-1-i)} k_{2,v}(t,t-\tau)\right]^{(i-l)} (t_n^p) u_n^{p-1^{(l+v)}}(t_n^{p-1}) \\
&+ \sum_{v=0}^{k-1}\sum_{i=0}^{p-1}\sum_{d=0}^{N-1}\sum_{l=v}^{m+k-1} \frac{\hat{u}_{d,i}^{(l)}(t_d^i)}{(l-v)!} \int_{t_d^i}^{t_{d+1}^i} \partial_1^{(j)} k_{1,v}(t_n^p,s)(s-t_d^i)^{l-v}ds \\
&+ \sum_{v=0}^{k-1}\sum_{d=0}^{n-1}\sum_{l=v}^{m+k-1} \frac{\hat{u}_{d,p}^{(l)}(t_d^p)}{(l-v)!} \int_{t_d^p}^{t_{d+1}^p} \partial_1^{(j)} k_{1,v}(t_n^p,s)(s-t_d^p)^{l-v}ds \\
&+ \sum_{v=0}^{k-1}\sum_{i=0}^{j-1}\sum_{l=0}^{i} \binom{i}{l} \left[\partial_1^{(j-1-i)} k_{1,v}(t,t)\right]^{(i-l)} (t_n^p)\hat{u}_{n,p}^{(l+v)}(t_n^p),
\end{aligned} \tag{2.7}$$

such that $\hat{u}_{0,p}^{(v)}(t_0^p) = u_{N-1}^{p-1^{(v)}}(t_0^p)$ and $\hat{u}_{n,p}^{(v)}(t_n^p) = u_{n-1}^{p^{(v)}}(t_n^p)$ for all $v = 0, \dots, k-1$.

3 Analysis of Convergence

The following three lemmas will be used in this section.

Lemma 3.1. *(Discrete Gronwall-type inequality [6]) Let $\{k_j\}_{j=0}^{n}$ be a given non-negative sequence and the sequence $\{\varepsilon_n\}$ satisfies $\varepsilon_0 \le p_0$ and*

$$\varepsilon_n \le p_0 + \sum_{i=0}^{n-1} k_i \varepsilon_i, \quad n \ge 1,$$

with $p_0 \ge 0$. Then ε_n can be bounded by

$$\varepsilon_n \le p_0 \exp\left(\sum_{j=0}^{n-1} k_j\right), \quad n \ge 1.$$

Lemma 3.2. *(Discrete Gronwall-type inequality [1]) If $\{f_n\}_{n\ge0}$, $\{g_n\}_{n\ge0}$ and $\{\varepsilon_n\}_{n\ge0}$ are non-negative sequences and*

$$\varepsilon_n \le f_n + \sum_{i=0}^{n-1} g_i \varepsilon_i, \quad n \ge 0,$$

then,

$$\varepsilon_n \le f_n + \sum_{i=0}^{n-1} f_i g_i \exp\left(\sum_{k=0}^{n-1} g_k\right), \quad n \ge 0.$$

Lemma 3.3. *[17] Assume that the sequence $\{\varepsilon_n\}_{n\ge0}$ of non-negative numbers satisfies*

$$\varepsilon_n \le A\varepsilon_{n-1} + B\sum_{i=0}^{n-1} \varepsilon_i + K, \quad n \ge 1,$$

where A, B and K are non-negative constants, then

$$\varepsilon_n \le \frac{\varepsilon_0}{R_2 - R_1}[(R_2-1)R_2^n + (1-R_1)R_1^n] + \frac{K}{R_2-R_1}[R_2^n - R_1^n],$$

where

$$\begin{aligned} R_1 &= \left(1 + A + B - \sqrt{(1-A)^2 + B^2 + 2AB + 2B}\right)/2, \\ R_2 &= \left(1 + A + B + \sqrt{(1-A)^2 + B^2 + 2AB + 2B}\right)/2, \end{aligned}$$

therefore, $0 \le R_1 \le 1 \le R_2$.

Before starting the main result, we need the following lemma:

Lemma 3.4. *Let g, $\{k_{1,v}\}_{v=0}^{k-1}$, $\{k_{2,v}\}_{v=0}^{k-1}$, $\{L_v\}_{v=0}^{k-1}$ and $\{M_v\}_{v=0}^{k-1}$ be m-times continuously differentiable and Φ be $m+k$ times continuously differentiable on their respective domains. Then, there exists a positive number $\alpha(m)$ such that for all $n = 0, \ldots, N-1$, $p = 0, \ldots, r-1$, and $j = 0, \ldots, m+k$, it holds*

$$\|\hat{u}_{n,p}^{(j)}\|_{L^\infty(\sigma_n^p)} \le \alpha(m),$$

provided that h is sufficiently small, where $\hat{u}_{0,0}(t) = x(t)$ for $t \in \sigma_0^0$.

Proof. The proof is split into two steps.
Claim 1. There exists a positive constant $\alpha_1(m)$ such that $\|\hat{u}_{n,0}^{(j)}\|_{L^\infty(\sigma_n^0)} \le \alpha_1(m)$ for all $n =$

$0,1,\dots,N-1$ and $j=0,1,\dots,m+k$.
Let $a_n^j = \|\hat{u}_{n,0}^{(j)}\|_{L^\infty(\sigma_n^0)}$, we have for all $j=0,1,\dots,m+k$,

$$a_0^j \le \max\{\|x^{(j)}\|_{L^\infty(\sigma^0)}, j=0,1,\dots,m+k\} = \alpha_1^1(m). \tag{3.1}$$

On the other hand, for $n \ge 1$, by differentiating equation (2.3) j-times, we obtain for all $j=1,\dots,m$,

$$\begin{aligned} a_n^{j+k} &\le c_1 + L\sum_{v=0}^{k-1}\sum_{l=0}^{j} a_n^{l+v} + (mb_1^1+b_1^2)\sum_{v=0}^{k-1}\sum_{l=0}^{j-1} a_n^{l+v} + hd_1^1 \sum_{v=0}^{k-1}\sum_{r=0}^{n-1}\sum_{l=v}^{m+k-1} a_r^l \\ &\le c_1 + \underbrace{\left(kL + mkb_1^1 + kb_1^2\right)}_{b_1} \sum_{l=0}^{j+k-1} a_n^l + h\underbrace{kd_1^1}_{d_1} \sum_{r=0}^{n-1}\sum_{l=0}^{m+k-1} a_r^l, \end{aligned} \tag{3.2}$$

where c_1, L, b_1^1, b_1^2 and d_1^1 are positive numbers.
Now, for each fixed $n \ge 1$, we consider the sequence $y_j = a_n^{j+k}$ for $j=0,\dots,m$, then, from (3.2), the sequence (y_j) satisfies for $j=1,\dots,m$

$$y_j \le c_1 + b_1\sum_{v=0}^{k-1} a_n^v + b_1\sum_{l=0}^{j-1} y_l + hd_1\sum_{r=0}^{n-1}\sum_{l=0}^{m+k-1} a_r^l,$$

and for $j=0$, we get from (2.3),

$$\begin{aligned} y_0 = a_n^k &\le c_1 + (L+b_1^2)\sum_{v=0}^{k-1} a_n^v + hd_1\sum_{r=0}^{n-1}\sum_{l=0}^{m+k-1} a_r^l \\ &\le c_1 + b_1\sum_{v=0}^{k-1} a_n^v + hd_1\sum_{r=0}^{n-1}\sum_{l=0}^{m+k-1} a_r^l. \end{aligned}$$

Hence, by Lemma 3.1, for all $j=0,\dots,m$

$$\begin{aligned} y_j &\le \underbrace{c_1\exp(b_1m)}_{c_2} + \underbrace{b_1\exp(b_1m)}_{b_2}\sum_{v=0}^{k-1} a_n^v + h\underbrace{d_1\exp(b_1m)}_{d_2}\sum_{r=0}^{n-1}\sum_{l=0}^{m+k-1} a_r^l \\ &\le c_2 + b_2\sum_{v=0}^{k-1} a_n^v + hd_2\sum_{r=0}^{n-1}\sum_{l=0}^{k-1} a_r^l + hd_2\sum_{r=0}^{n-1}\sum_{l=k}^{m+k} a_r^l. \end{aligned} \tag{3.3}$$

Next, we consider the sequence $z_n = \sum_{j=k}^{m+k} a_n^j$ for $n=0,\dots,N-1$.
Then, from (3.3), z_n satisfies for $n=1,\dots,N-1$,

$$\begin{aligned} z_n = \sum_{j=0}^{m} y_j &\le \underbrace{(m+1)c_2}_{c_3^1} + \underbrace{(m+1)b_2}_{b_3}\sum_{v=0}^{k-1} a_n^v + h\underbrace{(m+1)d_2}_{d_3}\sum_{r=0}^{n-1}\sum_{l=0}^{k-1} a_r^l + h\underbrace{(m+1)d_2}_{d_3}\sum_{r=0}^{n-1} z_r \\ &\le c_3^1 + b_3\sum_{v=0}^{k-1} a_n^v + hd_3\sum_{r=0}^{n-1}\sum_{l=0}^{k-1} a_r^l + hd_3\sum_{r=0}^{n-1} z_r. \end{aligned}$$

Moreover, from (3.1), we have $z_0 \leq (m+1)\alpha_1^1(m) = c_3^2$.
Let $c_3 = \max(c_3^1, c_3^2)$, we deduce, by Lemma 3.2, that for all $n = 0, \dots, N-1$,

$$\begin{aligned} z_n &\leq c_3 + b_3 \sum_{v=0}^{k-1} a_n^v + hd_3 \sum_{r=0}^{n-1}\sum_{l=0}^{k-1} a_r^l \\ &\quad + hd_3 \exp(\tau d_3) \sum_{r=0}^{n-1} \left(c_3 + b_3 \sum_{v=0}^{k-1} a_r^v + hd_3 \sum_{s=0}^{r-1}\sum_{v=0}^{k-1} a_s^v \right) \\ &\leq c_4 + b_3 \sum_{v=0}^{k-1} a_n^v + hd_4 \sum_{r=0}^{n-1}\sum_{l=0}^{k-1} a_r^l, \end{aligned} \tag{3.4}$$

where c_4 and d_4 are positive numbers. On the other hand, we integrate $\hat{u}_{n,0}^k$ in (2.3) l-times, $l = 1, 2, \dots, k$ from t_n^0 to $t \in \sigma_0^0$, to get for all $i = 0, 1, \dots, k-1$,

$$a_n^i \leq h^{k-i} \left(c + (b_1^2 + L) \sum_{v=0}^{k-1} a_n^v + hd \sum_{r=0}^{n-1} \sum_{l=0}^{m+k-1} a_r^l \right) + \sum_{j=i}^{k-1} h^{j-i} |u_{n-1}^{0(j)}(t_n^0)|, \tag{3.5}$$

where c and d are positive numbers. Moreover, from (2.2) and by differentiating u_{n-1}^0 j-times, $j = 0, 1, \dots, k-1$, we obtain

$$|u_{n-1}^{0(j)}(t_n^0)| \leq \sum_{l=j}^{k-1} h^{l-j} a_{n-1}^l + h \sum_{v=k}^{m+k-1} a_{n-1}^v.$$

Hence

$$a_n^i \leq h^{k-i} \left(c + \underbrace{(b_1^2 + L)}_{\widetilde{L}} \sum_{v=0}^{k-1} a_n^v + hd \sum_{r=0}^{n-1} \sum_{l=0}^{m+k-1} a_r^l \right) + \sum_{j=i}^{k-1} h^{j-i} \left(\sum_{l=j}^{k-1} h^{l-j} a_{n-1}^l + h \sum_{v=k}^{m+k-1} a_{n-1}^v \right).$$

This implies that

$$\begin{aligned}
\sum_{i=0}^{k-1} a_n^i &\leq \sum_{i=0}^{k-1} h^{k-i}\left(c + \widetilde{L}\sum_{v=0}^{k-1} a_n^v + hd\sum_{r=0}^{n-1}\sum_{l=0}^{m+k-1} a_r^l\right)\\
&+ \sum_{i=0}^{k-1}\sum_{j=i}^{k-1} h^{j-i}\left(\sum_{l=j}^{k-1} h^{l-j} a_{n-1}^l + h\sum_{v=k}^{m+k-1} a_{n-1}^v\right)\\
&\leq h\sum_{i=0}^{k-1}\left(c + \widetilde{L}\sum_{v=0}^{k-1} a_n^v + hd\left(\sum_{r=0}^{n-1}\sum_{l=0}^{k-1} a_r^l + \sum_{r=0}^{n-1}\sum_{l=k}^{m+k} a_r^l\right)\right)\\
&+ \sum_{i=0}^{k-1}\sum_{j=i}^{k-1} h^{j-i}\left(\left(a_{n-1}^j + h\sum_{l=j+1}^{k-1} a_{n-1}^l\right) + h\sum_{v=k}^{m+k} a_{n-1}^v\right)\\
&\leq hk\left[c + \widetilde{L}\sum_{v=0}^{k-1} a_n^v + hd\sum_{r=0}^{n-1}\sum_{l=0}^{k-1} a_r^l + hd\sum_{r=0}^{n-1} z_r\right]\\
&+ \sum_{i=0}^{k-1}\left[a_{n-1}^i + h\sum_{l=i+1}^{k-1} a_{n-1}^l + h\sum_{j=i+1}^{k-1}\left(a_{n-1}^j + h\sum_{l=j+1}^{k-1} a_{n-1}^l\right) + h(k-i)z_{n-1}\right]\\
&\leq hk\left[c + \widetilde{L}\sum_{v=0}^{k-1} a_n^v + hd\sum_{r=0}^{n-1}\sum_{l=0}^{k-1} a_r^l + hd\sum_{r=0}^{n-1} z_r\right]\\
&+ \sum_{i=0}^{k-1} a_{n-1}^i + hk\sum_{l=1}^{k-1} a_{n-1}^l + h\sum_{i=0}^{k-1}\left(\sum_{j=i+1}^{k-1} a_{n-1}^j + h(k-i-1)\sum_{l=i+2}^{k-1} a_{n-1}^l\right) + hk^2 z_{n-1}\\
&\leq hk\left[c + \widetilde{L}\sum_{v=0}^{k-1} a_n^v + hd\sum_{r=0}^{n-1}\sum_{l=0}^{k-1} a_r^l + hd\sum_{r=0}^{n-1} z_r\right]\\
&+ (1+hk)\sum_{i=0}^{k-1} a_{n-1}^i + hk\sum_{j=1}^{k-1} a_{n-1}^j + h^2k(k-1)\sum_{l=2}^{k-1} a_{n-1}^l + hk^2 z_{n-1}\\
&\leq hkc + hk\widetilde{L}\sum_{v=0}^{k-1} a_n^v + h^2kd\sum_{r=0}^{n-1}\sum_{l=0}^{k-1} a_r^l + h^2kd\sum_{r=0}^{n-1} z_r\\
&+ (1+2hk+h^2k(k-1))\sum_{i=0}^{k-1} a_{n-1}^i + hk^2 z_{n-1},
\end{aligned}$$

Using (3.4), we deduce that

$$\begin{aligned}
\sum_{i=0}^{k-1} a_n^i &\leq hkc + hk\widetilde{L}\sum_{v=0}^{k-1} a_n^v + h^2kd\sum_{r=0}^{n-1}\sum_{l=0}^{k-1} a_r^l + h^2kd\sum_{r=0}^{n-1}\left(c_4 + b_3\sum_{v=0}^{k-1} a_r^v + hd_4\sum_{s=0}^{r-1}\sum_{l=0}^{k-1} a_s^l\right)\\
&+ (1+2hk+h^2k(k-1))\sum_{i=0}^{k-1} a_{n-1}^i + hk^2\left(c_4 + b_3\sum_{v=0}^{k-1} a_{n-1}^v + hd_4\sum_{r=0}^{n-2}\sum_{l=0}^{k-1} a_r^l\right)\\
&\leq hc_5 + hk\widetilde{L}\sum_{v=0}^{k-1} a_n^v + (1+hb_4+h^2b_5)\sum_{i=0}^{k-1} a_{n-1}^i + h^2d_5\sum_{r=0}^{n-1}\sum_{l=0}^{k-1} a_r^l,
\end{aligned}$$

where b_4, b_5, c_5 and d_5 are positive numbers.
This implies that

$$(1-hk\widetilde{L})\sum_{i=0}^{k-1} a_n^i \leq hc_5 + (1+hb_4+h^2b_5)\sum_{i=0}^{k-1} a_{n-1}^i + h^2d_5\sum_{r=0}^{n-1}\sum_{l=0}^{k-1} a_r^l,$$

Hence, for all $h \in (0, \frac{1}{k\widetilde{L}})$, it holds

$$\sum_{i=0}^{k-1} a_n^i \leq \frac{1 + hb_4 + h^2 b_5}{1 - hk\widetilde{L}} \sum_{i=0}^{k-1} a_{n-1}^i + \frac{h^2 d_5}{1 - hk\widetilde{L}} \sum_{r=0}^{n-1} \sum_{l=0}^{k-1} a_r^l + \frac{hc_5}{1 - hk\widetilde{L}}.$$

Then, by Lemma 3.3, we obtain for all $n \in \{0, 1, \ldots, N-1\}$,

$$\begin{aligned} \sum_{i=0}^{k-1} a_n^i &\leq \frac{\sum_{i=0}^{k-1} a_0^i}{R_2 - R_1} [(R_2 - 1) R_2^n + (1 - R_1) R_1^n] + \frac{hc_5 [R_2^n - R_1^n]}{(R_2 - R_1)(1 - hk\widetilde{L})} \\ &\leq \frac{k\alpha_1^1(m)}{R_2 - R_1} [(R_2 - 1) R_2^n + (1 - R_1) R_1^n] + \frac{hc_5 [R_2^n - R_1^n]}{(R_2 - R_1)(1 - hk\widetilde{L})}, \end{aligned} \tag{3.6}$$

where

$$R_1 = \left(1 + \frac{1 + (b_4 - \sqrt{\zeta})h + (b_5 + d_5)h^2}{1 - hk\widetilde{L}}\right)/2,$$
$$R_2 = \left(1 + \frac{1 + (b_4 + \sqrt{\zeta})h + (b_5 + d_5)h^2}{1 - hk\widetilde{L}}\right)/2,$$

with $\zeta = 4d_5 + (k\widetilde{L} + b_4)^2 + 2h(k\widetilde{L}(b_5 - d_5) + d_4(b_5 + d_5)) + h^2(b_5 + d_5)^2$.
Hence, there exists $h_1 \in (0, \frac{1}{k\widetilde{L}})$, such that for all $h \in (0, h_1]$, it holds

$$R_1^n \leq 1 \leq R_2^n \leq R_2^N = R_2^{\frac{\tau}{h}}, n = 0, 1, \ldots, N-1,$$

having taken into account that $0 \leq R_1 \leq 1 \leq R_2$. From (3.6), this implies that

$$\sum_{i=0}^{k-1} a_n^i \leq k\alpha_1^1(m) \frac{(R_2 - 1) R_2^{\frac{\tau}{h}} + (1 - R_1)}{R_2 - R_1} + c_5 \frac{hR_2^{\frac{\tau}{h}}}{(R_2 - R_1)(1 - hk\widetilde{L})}.$$

Therefore for all $h \in (0, h_1]$, there exists $\alpha_1^2(m) > 0$ such that,

$$\sum_{i=0}^{k-1} a_n^i \leq \alpha_1^2(m), n = 0, 1, \ldots, N-1.$$

Hence, from (3.4), for all $j = k, \ldots, m+k$ and $n = 0, 1, \ldots, N-1$, it holds

$$a_n^j \leq z_n \leq c_4 + b_3 \alpha_1^2(m) + \tau d_4 \alpha_1^2(m) = \alpha_1^3(m).$$

Then, the first step is completed by setting,

$$\alpha_1(m) = \max(\alpha_1^2(m), \alpha_1^3(m)).$$

Claim 2. There exists a positive constant $\alpha(m)$ such that $\|\hat{u}_{n,p}^{(j)}\|_{L^\infty(\sigma_n^p)} \leq \alpha(m)$ for all $n = 0, 1, \ldots, N-1$, $j = 0, 1, \ldots, m+k$ and $p = 1, \ldots, r-1$.
Let $a_{n,p}^j = \|\hat{u}_{n,p}^{(j)}\|_{L^\infty(\sigma_n^p)}$ and $\xi_p = \max\{a_{i,p}^j, j = 0, \ldots, m+k, i = 0, \ldots, N-1\}$ for $p = 0, \ldots, r-1$.
Similarly to Claim 1, by differentiating equation (2.6) for $n = 0$, j-times, we obtain for all $j = 0, \ldots, m$,

$$a_{0,p}^{j+k} \leq c_1 + b_1 \sum_{i=0}^{p-1} \xi_i + d_1 \sum_{l=0}^{j+k-1} a_{0,p}^l,$$

where c_1, b_1 and d_1 are positive numbers.
On the other hand, by integrating (2.6) for $n=0$, l-times, $l=1,\dots,k-1$ from t_0^p to $t\in\sigma_0^p$, we get,

$$a_{0,p}^{k-l} \le C_{k-l} + B_{k-l}\sum_{i=0}^{p-1}\xi_i + D_{k-l}\sum_{v=0}^{k-l-1} a_{0,p}^{v},$$

where C_{k-l}, B_{k-l} and D_{k-l} are positive numbers.
Then, we integrate (2.6) for $n=0$, k-times from t_0^p to $t\in\sigma_0^p$, to get

$$a_{0,p}^{0} \le C_0 + B_0\sum_{i=0}^{p-1}\xi_i + h^k D_0 a_{0,p}^{0},$$

where C_0, B_0 and D_0 are positive numbers.
Hence, there exists $h_2\in(0,h_1]$ and positive numbers c_2, b_2, d_2 such that for all $h\in(0,h_2]$, we have

$$a_{0,p}^{j} \le c_2 + b_2\sum_{i=0}^{p-1}\xi_i + d_2\sum_{l=0}^{j-1} a_{0,p}^{l},$$

for all $j\in\{0,1,\dots,m+k\}$.
Then, by Lemma 3.1, for all $j\in\{0,1,\dots,m+k\}$ it follows that

$$a_{0,p}^{j} \le \underbrace{c_2\exp(d_2(m+k))}_{c_3^1} + \underbrace{b_2\exp(d_2(m+k))}_{b_3}\sum_{i=0}^{p-1}\xi_i,$$

Hence, for $c_3=\max(\alpha_1(m), c_3^1)$, we get for all $p=0,1,\dots,r-1$, $j\in\{0,1,\dots,m+k\}$

$$a_{0,p}^{j} \le c_3 + b_3\sum_{i=0}^{p-1}\xi_i. \tag{3.7}$$

Next, by differentiating (2.6) j-times, we obtain for all $n=1,\dots,N-1$ and $j=0,\dots,m$,

$$a_{n,p}^{j+k} \le c_4 + b_4\sum_{i=0}^{p-1}\xi_i + e_4\sum_{l=0}^{j+k-1} a_{n,p}^{l} + hd_4\sum_{i=0}^{n-1}\sum_{l=0}^{m+k-1} a_{i,p}^{l},$$

where c_4, b_4, e_4 and d_4 are positive numbers.
Then, by Lemma 3.1, for all $j\in\{0,\dots,m\}$

$$\begin{aligned} a_{n,p}^{j+k} \le{}& \underbrace{c_4\exp((m+k)e_4)}_{c_5} + \underbrace{b_4\exp((m+k)e_4)}_{b_5}\sum_{i=0}^{p-1}\xi_i + \underbrace{e_4\exp((m+k)e_4)}_{e_5}\sum_{l=0}^{k-1} a_{n,p}^{l} \\ &+ h\underbrace{d_4\exp((m+k)e_4)}_{d_5}\sum_{i=0}^{n-1}\sum_{l=0}^{m+k} a_{i,p}^{l}. \end{aligned}$$

Consider the sequence $y_n=\sum_{j=k}^{m+k} a_{n,p}^{j}$, $n=0,1,\dots,N-1$. By the above inequality, the sequence (y_n) satisfies for all $n=1,\dots,N-1$,

$$\begin{aligned} y_n \le{}& \underbrace{(m+1)c_5}_{c_6^1} + \underbrace{(m+1)b_5}_{b_6^1}\sum_{i=0}^{p-1}\xi_i + \underbrace{(m+1)e_5}_{e_6}\sum_{l=0}^{k-1} a_{n,p}^{l} \\ &+ h\underbrace{(m+1)d_5}_{d_6}\sum_{i=0}^{n-1}\sum_{l=0}^{k-1} a_{i,p}^{l} + h\underbrace{(m+1)d_5}_{d_6}\sum_{i=0}^{n-1} y_i. \end{aligned} \tag{3.8}$$

Moreover, from (3.7), we obtain,

$$y_0 \le \underbrace{(m+1)c_3}_{c_6^2} + \underbrace{(m+1)b_3}_{b_6^2} \sum_{i=0}^{p-1} \xi_i. \tag{3.9}$$

Let $c_6 = \max\{c_6^1, c_6^2\}$ and $b_6 = \max\{b_6^1, b_6^2\}$.
Then, from (3.8) and (3.9), we get for all $n = 0, 1, \dots, N-1$,

$$y_n \le c_6 + b_6 \sum_{i=0}^{p-1} \xi_i + e_6 \sum_{l=0}^{k-1} a_{n,p}^l + hd_6 \sum_{i=0}^{n-1} \sum_{l=0}^{k-1} a_{i,p}^l + hd_6 \sum_{i=0}^{n-1} y_i,$$

hence, by Lemma 3.2, we obtain

$$y_n \le c_7 + b_7 \sum_{i=0}^{p-1} \xi_i + e_6 \sum_{l=0}^{k-1} a_{n,p}^l + hd_7 \sum_{i=0}^{n-1} \sum_{l=0}^{k-1} a_{i,p}^l, \tag{3.10}$$

where c_7, b_7 and d_7 are positive numbers.
On the other hand, by integrating (2.6) l-times, $l = 1, 2, \dots, k$ from t_n^p to $t \in \sigma_n^p$, we get for all $i = 0, 1, \dots, k-1$,

$$a_{n,p}^i \le h^{k-i} \left(c + b_1^2 a_{n,p}^0 + L \sum_{v=0}^{k-1} a_{n,p}^v + e \sum_{i=0}^{p-1} \xi_i + hd \sum_{r=0}^{n-1} \sum_{l=0}^{m+k-1} a_{r,p}^l \right) + \sum_{j=i}^{k-1} h^{j-i} |u_{n-1}^{p^{(j)}}(t_n^p)|,$$

where c, e and d are positive numbers. Moreover, from (2.5) and by differentiating u_{n-1}^p j-times, $j = 0, 1, \dots, k-1$, we get

$$|u_{n-1}^{p^{(j)}}(t_n^p)| \le \sum_{l=j}^{k-1} h^{l-j} a_{n-1,p}^l + h \sum_{v=k}^{m+k-1} a_{n-1,p}^v.$$

Hence

$$a_{n,p}^i \le h^{k-i} \left(c + \underbrace{(b_1^2 + L)}_{\widetilde{L}} \sum_{v=0}^{k-1} a_{n,p}^v + e \sum_{i=0}^{p-1} \xi_i + hd \sum_{r=0}^{n-1} \sum_{l=0}^{m+k-1} a_{r,p}^l \right)$$
$$+ \sum_{j=i}^{k-1} h^{j-i} \left(\sum_{l=j}^{k-1} h^{l-j} a_{n-1,p}^l + h \sum_{v=k}^{m+k-1} a_{n-1,p}^v \right).$$

This implies that

$$
\begin{aligned}
\sum_{i=0}^{k-1} a_{n,p}^{i} \leq & \sum_{i=0}^{k-1} h^{k-i}\left(c+\widetilde{L}\sum_{v=0}^{k-1} a_{n,p}^{v}+e\sum_{i=0}^{p-1}\xi_i+hd\sum_{r=0}^{n-1}\sum_{l=0}^{m+k-1} a_{r,p}^{l}\right)\\
&+\sum_{i=0}^{k-1}\sum_{j=i}^{k-1} h^{j-i}\left(\sum_{l=j}^{k-1} h^{l-j}a_{n-1,p}^{l}+h\sum_{v=k}^{m+k-1} a_{n-1,p}^{v}\right)\\
&\leq h\sum_{i=0}^{k-1}\left(c+\widetilde{L}\sum_{v=0}^{k-1} a_{n,p}^{v}+e\sum_{i=0}^{p-1}\xi_i+hd\left(\sum_{r=0}^{n-1}\sum_{l=0}^{k-1} a_{r,p}^{l}+\sum_{r=0}^{n-1}\sum_{l=k}^{m+k} a_{r,p}^{l}\right)\right)\\
&+\sum_{i=0}^{k-1}\sum_{j=i}^{k-1} h^{j-i}\left(\left(a_{n-1,p}^{j}+h\sum_{l=j+1}^{k-1} a_{n-1,p}^{l}\right)+h\sum_{v=k}^{m+k} a_{n-1,p}^{v}\right)\\
&\leq hk\left[c+\widetilde{L}\sum_{v=0}^{k-1} a_{n,p}^{v}+e\sum_{i=0}^{p-1}\xi_i+hd\sum_{r=0}^{n-1}\sum_{l=0}^{k-1} a_{r,p}^{l}+hd\sum_{r=0}^{n-1} y_r\right]\\
&+\sum_{i=0}^{k-1}\left[a_{n-1,p}^{i}+h\sum_{l=i+1}^{k-1} a_{n-1,p}^{l}+h\sum_{j=i+1}^{k-1}\left(a_{n-1,p}^{j}+h\sum_{l=j+1}^{k-1} a_{n-1,p}^{l}\right)+h(k-i)y_{n-1}\right]\\
&\leq hk\left[c+\widetilde{L}\sum_{v=0}^{k-1} a_{n,p}^{v}+e\sum_{i=0}^{p-1}\xi_i+hd\sum_{r=0}^{n-1}\sum_{l=0}^{k-1} a_{r,p}^{l}+hd\sum_{r=0}^{n-1} y_r\right]\\
&+\sum_{i=0}^{k-1} a_{n-1,p}^{i}+hk\sum_{l=1}^{k-1} a_{n-1,p}^{l}+h\sum_{i=0}^{k-1}\left(\sum_{j=i+1}^{k-1} a_{n-1,p}^{j}+h(k-i-1)\sum_{l=i+2}^{k-1} a_{n-1,p}^{l}\right)\\
&+hk^2 y_{n-1}\\
&\leq hk\left[c+\widetilde{L}\sum_{v=0}^{k-1} a_{n,p}^{v}+e\sum_{i=0}^{p-1}\xi_i+hd\sum_{r=0}^{n-1}\sum_{l=0}^{k-1} a_{r,p}^{l}+hd\sum_{r=0}^{n-1} y_r\right]\\
&+(1+hk)\sum_{i=0}^{k-1} a_{n-1,p}^{i}+hk\sum_{j=1}^{k-1} a_{n-1,p}^{j}+h^2k(k-1)\sum_{l=2}^{k-1} a_{n-1,p}^{l}+hk^2 y_{n-1}\\
&\leq hkc+hk\widetilde{L}\sum_{v=0}^{k-1} a_{n,p}^{v}+hke\sum_{i=0}^{p-1}\xi_i+h^2kd\sum_{r=0}^{n-1}\sum_{l=0}^{k-1} a_{r,p}^{l}+h^2kd\sum_{r=0}^{n-1} y_r\\
&+(1+2hk+h^2k(k-1))\sum_{i=0}^{k-1} a_{n-1,p}^{i}+hk^2 y_{n-1}.
\end{aligned}
$$

Using (3.10), we deduce that

$$
\begin{aligned}
\sum_{i=0}^{k-1} a_{n,p}^{i} \leq & hkc+hk\widetilde{L}\sum_{v=0}^{k-1} a_{n,p}^{v}+hke\sum_{i=0}^{p-1}\xi_i+h^2kd\sum_{r=0}^{n-1}\sum_{l=0}^{k-1} a_{r,p}^{l}\\
&+h^2kd\sum_{r=0}^{n-1}\left(c_7+b_7\sum_{i=0}^{p-1}\xi_i+e_6\sum_{l=0}^{k-1} a_{r,p}^{l}+hd_7\sum_{s=0}^{r-1}\sum_{l=0}^{k-1} a_{s,p}^{l}\right)+(1+2hk+h^2k(k-1))\sum_{i=0}^{k-1} a_{n-1,p}^{i}\\
&+hk^2\left(c_7+b_7\sum_{i=0}^{p-1}\xi_i+e_6\sum_{l=0}^{k-1} a_{n-1,p}^{l}+hd_7\sum_{i=0}^{n-2}\sum_{l=0}^{k-1} a_{i,p}^{l}\right),\\
&\leq hc_8+hk\widetilde{L}\sum_{v=0}^{k-1} a_{n,p}^{v}+hb_8\sum_{i=0}^{p-1}\xi_i+(1+he_7+h^2e_8)\sum_{i=0}^{k-1} a_{n-1,p}^{i}+h^2d_8\sum_{r=0}^{n-1}\sum_{l=0}^{k-1} a_{r,p}^{l},
\end{aligned}
$$

where b_8, c_8, d_8, e_7 and e_8 are positive numbers.
This implies that

$$(1-hk\widetilde{L})\sum_{i=0}^{k-1} a_{n,p}^i \le hc_8 + hb_8\sum_{i=0}^{p-1}\xi_i + (1+he_7+h^2e_8)\sum_{i=0}^{k-1} a_{n-1,p}^i + h^2 d_8 \sum_{r=0}^{n-1}\sum_{l=0}^{k-1} a_{r,p}^l,$$

Hence, for all $h \in (0, \frac{1}{k\widetilde{L}})$, we have

$$\sum_{i=0}^{k-1} a_{n,p}^i \le \frac{1+he_7+h^2e_8}{1-hk\widetilde{L}}\sum_{i=0}^{k-1} a_{n-1,p}^i + \frac{h^2d_8}{1-hk\widetilde{L}}\sum_{r=0}^{n-1}\sum_{l=0}^{k-1} a_{r,p}^l + \frac{h\left(c_8+b_8\sum_{i=0}^{p-1}\xi_i\right)}{1-hk\widetilde{L}}.$$

Then, by Lemma 3.3, we obtain for all $n \in \{0,1,\dots,N-1\}$,

$$\sum_{i=0}^{k-1} a_{n,p}^i \le \frac{\sum_{i=0}^{k-1} a_{0,p}^i}{R_2-R_1}[(R_2-1)R_2^n + (1-R_1)R_1^n] + \frac{h\left(c_8+b_8\sum_{i=0}^{p-1}\xi_i\right)[R_2^n-R_1^n]}{(R_2-R_1)(1-hk\widetilde{L})},$$

where

$$R_1 = \left(1+\frac{1+(e_7-\sqrt{\zeta})h+(e_8+d_8)h^2}{1-hk\widetilde{L}}\right)/2,$$
$$R_2 = \left(1+\frac{1+(e_7+\sqrt{\zeta})h+(e_8+d_8)h^2}{1-hk\widetilde{L}}\right)/2,$$

where
$\zeta = 4d_8 + (k\widetilde{L}+e_7)^2 + 2h\left(k\widetilde{L}(e_8-d_8)+e_7(e_8+d_8)\right) + h^2(e_8+d_8)^2.$
Hence, similar as in (3.6), there exist $\overline{R} > 0$ such that for all $h \in (0, \frac{1}{k\widetilde{L}})$, we have

$$\sum_{i=0}^{k-1} a_{n,p}^i \le \sum_{i=0}^{k-1} a_{0,p}^i \overline{R} + (c_8 + b_8\sum_{i=0}^{p-1}\xi_i)\overline{R},$$

which implies, by using (3.7), that for all $n \in \{0,1,\dots,N-1\}$ and $p \in \{0,1,\dots,r-1\}$,

$$\sum_{i=0}^{k-1} a_{n,p}^i \le \underbrace{(kc_3+c_8)\overline{R}}_{c_9} + \underbrace{(kb_3+b_8)\overline{R}}_{b_9}\sum_{i=0}^{p-1}\xi_i.$$

Then, from (3.10), we get for all $n \in \{0,1,\dots,N-1\}$, $j \in \{k,\dots,m+k\}$ and $p \in \{0,1,\dots,r-1\}$,

$$a_{n,p}^j \le y_n \le \underbrace{(c_7+e_6c_9+\tau d_7c_9)}_{c_{10}^1} + \underbrace{(b_7+e_6b_9+\tau d_7b_9)}_{b_{10}^1}\sum_{i=0}^{p-1}\xi_i.$$

Let $c_{10} = \max(c_9, c_{10}^1)$ and $b_{10} = \max(b_9, b_{10}^1)$.
We deduce that, for all $p \in \{0,1,\dots,r-1\}$,

$$\xi_p \le c_{10} + b_{10}\sum_{i=0}^{p-1}\xi_i.$$

Then, by Lemma 3.1, we get for all $p \in \{0,1,\dots,r-1\}$, $n \in \{0,1,\dots,N-1\}$ and $j \in \{0,1,\dots,m+k\}$,

$$a_{n,p}^j \le \xi_p \le c_{10}\exp(rb_{10}) = \alpha(m).$$

This completes the proof of Lemma 3.4. □

The following provides the order of convergence of the method.

Theorem 3.5. *Let* $g, \{K_{1,v}\}_{v=0}^{k-1}, \{K_{2,v}\}_{v=0}^{k-1}, \{L_v\}_{v=0}^{k-1}$ *and* $\{M_v\}_{v=0}^{k-1}$ *be m-times continuously differentiable on their respective domains. Assume that* $u \in S_{m+k-1}^{(k-1)}(\Pi_N)$ *in equations* (2.1), ..., (2.7) *defines a unique approximate solution u and* $u^{(v)}$ *(vth-order derivative) for* $v = 1, \ldots, k-1$. *Then the resulting errors functions* $e^{(v)} := x^{(v)} - u^{(v)}$, $v = 0, \ldots, k-1$, *satisfy*

$$\|e^{(v)}\|_{L^\infty(I)} \leq Ch^m;$$

for all $v = 0, \ldots, k-1$, *and C is a finite constant independent of h.*

Proof. The proof is split into two steps.
Claim 1. There exists a constant C_0 independent of h such that for all $v = 0, \ldots, k-1$:

$$\|e^{0^{(v)}}\|_{L^\infty(\sigma^0)} \leq C_0 h^m,$$

where the error $e^{0^{(v)}} = e^{(v)}|_{\sigma^0}$ is defined on σ_n^0 by $e^{0^{(v)}}(t) = e_n^{0^{(v)}}(t) = x^{(v)}(t) - u_n^{0^{(v)}}(t)$ for all $n \in \{0, 1, \ldots, N-1\}$ and $v \in \{0, \ldots, k-1\}$.
Let $t \in \sigma_0^0$, from Lemma 3.4, for sufficiently small h and $v \in \{0, \ldots, k-1\}$, we have

$$|e_0^{0^{(v)}}(t)| = |x^{(v)}(t) - u_0^{0^{(v)}}(t)| \leq \frac{\|x^{(m+k)}\|_{L^\infty(\sigma_0^0)}}{(m+k-v)!} h^{m+k-v} \leq \frac{\alpha(m)}{(m+k-v)!} h^{m+k-v}.$$

In general for $n = 1, 2, \ldots, N-1$ and $t \in \sigma_n^0$, we have from (2.3),

$$\begin{aligned} x^{(k)}(t) - \hat{u}_{n,0}^{(k)}(t) = &\sum_{v=0}^{k-1} L_v(t)(x^{(v)}(t) - \hat{u}_{n,0}^{(v)}(t)) + \sum_{v=0}^{k-1}\sum_{d=0}^{n-1} \int_{t_d^0}^{t_{d+1}^0} k_{1,v}(t,s)(x^{(v)}(s) - u_i^{0^{(v)}}(s))ds \\ &+ \sum_{v=0}^{k-1} \int_{t_n^0}^{t} k_{1,v}(t,s)(x^{(v)}(s) - \hat{u}_{n,0}^{(v)}(s))ds, \end{aligned}$$

which implies that

$$\|x^{(k)} - \hat{u}_{n,0}^{(k)}\|_{L^\infty(\sigma_n^0)} \leq (hk_1 + L) \sum_{v=0}^{k-1} \|x^{(v)} - \hat{u}_{n,0}^{(v)}\|_{L^\infty(\sigma_n^0)} + hk_1 \sum_{v=0}^{k-1}\sum_{d=0}^{n-1} \|e_d^{0^{(v)}}\|_{L^\infty(\sigma_d^0)}, \tag{3.11}$$

where $k_1 = \max\left\{\|K_{1,v}\|_{L^\infty(\sigma^0\times\sigma^0)}, v = 0, \ldots, k-1\right\}$ and $L = \max\{\|L_v\|_{L^\infty(\sigma^0)}, v = 0, \ldots, k-1\}$.
On the other hand, for all $v \in \{1, \ldots, k\}$, we have

$$\begin{aligned} x^{(k-v)}(t) - \hat{u}_{n,0}^{(k-v)}(t) &= \sum_{i=1}^{v} (t - t_n^0)^{(v-i)} \left(x^{(k-i)}(t_n^0) - \hat{u}_{n,0}^{(k-i)}(t_n^0)\right) \\ &\quad + \int_{t_n^0}^{t}\int_{t_n^0}^{s_v} \ldots \int_{t_n^0}^{s_2} \left(x^{(k)}(s_1) - \hat{u}_{n,0}^{(k)}(s_1)\right) ds_1 ds_2 \ldots ds_v \\ &= \sum_{i=1}^{v} h^{(v-i)} e_{n-1}^{0^{(k-i)}}(t_n^0) + \int_{t_n^0}^{t}\int_{t_n^0}^{s_v} \ldots \int_{t_n^0}^{s_2} \left(x^{(k)}(s_1) - \hat{u}_{n,0}^{(k)}(s_1)\right) ds_1 ds_2 \ldots ds_v. \end{aligned}$$

It follows that

$$\|x^{(k-v)} - \hat{u}_{n,0}^{(k-v)}\|_{L^\infty(\sigma_n^0)} \leq \sum_{i=1}^{v} h^{v-i} \|e_{n-1}^{0^{(k-i)}}\|_{L^\infty(\sigma_{n-1}^0)} + h^v \|x^{(k)} - \hat{u}_{n,0}^{(k)}\|_{L^\infty(\sigma_n^0)}. \tag{3.12}$$

This implies that

$$\begin{aligned}
\sum_{v=1}^{k} \|x^{(k-v)} - \hat{u}_{n,0}^{(k-v)}\|_{L^\infty(\sigma_n^0)} &\leq \sum_{v=1}^{k}\sum_{i=1}^{v} h^{v-i}\|e_{n-1}^{0^{(k-i)}}\|_{L^\infty(\sigma_{n-1}^0)} + \sum_{v=0}^{k} h^{v+1}\|x^{(k)} - \hat{u}_{n,0}^{(k)}\|_{L^\infty(\sigma_n^0)} \\
&\leq \|e_{n-1}^{0^{(k-1)}}\| + \sum_{v=2}^{k}\sum_{i=1}^{v} h^{v-i}\|e_{n-1}^{0^{(k-i)}}\|_{L^\infty(\sigma_{n-1}^0)} + h\sum_{v=1}^{k}\|x^{(k)} - \hat{u}_{n,0}^{(k)}\|_{L^\infty(\sigma_n^0)} \\
&\leq \|e_{n-1}^{0^{(k-1)}}\| + \sum_{v=2}^{k}\left(\sum_{i=1}^{v-1} h^{v-i}\|e_{n-1}^{0^{(k-i)}}\|_{L^\infty(\sigma_{n-1}^0)} + \|e_{n-1}^{0^{(k-v)}}\|_{L^\infty(\sigma_{n-1}^0)}\right) + hk\|x^{(k)} - \hat{u}_{n,0}^{(k)}\|_{L^\infty(\sigma_n^0)} \\
&\leq \|e_{n-1}^{0^{(k-1)}}\| + h\sum_{v=2}^{k}\sum_{i=1}^{v-1}\|e_{n-1}^{0^{(k-i)}}\|_{L^\infty(\sigma_{n-1}^0)} + \sum_{i=0}^{k-2}\|e_{n-1}^{0^{(i)}}\|_{L^\infty(\sigma_{n-1}^0)} + hk\|x^{(k)} - \hat{u}_{n,0}^{(k)}\|_{L^\infty(\sigma_n^0)} \\
&\leq hk\sum_{i=1}^{k-1}\|e_{n-1}^{0^{(i)}}\|_{L^\infty(\sigma_{n-1}^0)} + \sum_{i=0}^{k-1}\|e_{n-1}^{0^{(i)}}\|_{L^\infty(\sigma_{n-1}^0)} + hk\|x^{(k)} - \hat{u}_{n,0}^{(k)}\|_{L^\infty(\sigma_n^0)} \\
&\leq (1+hk)\sum_{i=0}^{k-1}\|e_{n-1}^{0^{(i)}}\|_{L^\infty(\sigma_{n-1}^0)} + hk\|x^{(k)} - \hat{u}_{n,0}^{(k)}\|_{L^\infty(\sigma_n^0)}.
\end{aligned}$$

Therefore, by (3.11), we have

$$\begin{aligned}
\sum_{v=0}^{k-1}\|x^{(v)} - \hat{u}_{n,0}^{(v)}\|_{L^\infty(\sigma_n^0)} &\leq \frac{1+hk}{1-hk(hk_1+L)}\sum_{v=0}^{k-1}\|e_{n-1}^{0^{(v)}}\|_{L^\infty(\sigma_{n-1}^0)} \\
&\quad + \frac{h^2kk_1}{1-hk(hk_1+L)}\sum_{v=0}^{k-1}\sum_{i=0}^{n-1}\|e_i^{0^{(v)}}\|_{L^\infty(\sigma_i^0)}.
\end{aligned}$$

Then, by Lemma 3.4, we deduce that

$$\begin{aligned}
\sum_{v=0}^{k-1}\|e_n^{0^{(v)}}\|_{L^\infty(\sigma_n^0)} &\leq \sum_{v=0}^{k-1}\left(\|x^{(v)} - \hat{u}_{n,0}^{(v)}\|_{L^\infty(\sigma_n^0)} + \|\hat{u}_{n,0}^{(v)} - u_n^{0^{(v)}}\|_{L^\infty(\sigma_n^0)}\right) \\
&\leq \sum_{v=0}^{k-1}\|x^{(v)} - \hat{u}_{n,0}^{(v)}\|_{L^\infty(\sigma_n^0)} + \sum_{v=0}^{k-1}\frac{\alpha(m)}{(m+k-v)!}h^{m+k-v} \\
&\leq \frac{1+hk}{1-hk(hk_1+L)}\sum_{v=0}^{k-1}\|e_{n-1}^{0^{(v)}}\|_{L^\infty(\sigma_{n-1}^0)} \\
&\quad + \frac{h^2kk_1}{1-hk(hk_1+L)}\sum_{v=0}^{k-1}\sum_{i=0}^{n-1}\|e_i^{0^{(v)}}\|_{L^\infty(\sigma_i^0)} + Mh^{m+1},
\end{aligned}$$

where $M = \sum_{v=0}^{k-1}\frac{\alpha(m)}{(m+k-v)!}\tau^{k-v-1}$.

Hence by Lemma 3.3, for all $n \in \{0,1,\ldots,N-1\}$ it holds

$$\begin{aligned}
\sum_{v=0}^{k-1}\|e_n^{0^{(v)}}\|_{L^\infty(\sigma_n^0)} &\leq \frac{\sum_{v=0}^{k-1}\|e_0^{0^{(v)}}\|_{L^\infty(\sigma_0^0)}}{R_2-R_1}\left[(R_2-1)R_2^n + (1-R_1)R_1^n\right] + \frac{Mh^{m+1}}{R_2-R_1}\left[R_2^n - R_1^n\right] \\
&\leq Mh^{m+1}\frac{\left[(R_2-1)R_2^n + (1-R_1)R_1^n\right] + \left[R_2^n - R_1^n\right]}{R_2-R_1},
\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}
R_1 &= \left(1 + \frac{1+h(k-\sqrt{\zeta}) + h^2kk_1}{1-hk(hk_1+L)}\right)/2, \\
R_2 &= \left(1 + \frac{1+h(k+\sqrt{\zeta}) + h^2kk_1}{1-hk(hk_1+L)}\right)/2,
\end{aligned} \tag{3.14}$$

with $\zeta = k^2(L+1)^2 + 6kk_1 + 2hk^2k_1(2-L) - 2h^2(kk_1)^2$.
Since $0 < R_1 \leq 1 \leq R_2$, then

$$R_1^n \leq 1 \leq R_2^n \leq R_2^N = R_2^{\frac{\tau}{h}}, n = 0,1,\ldots,N-1,$$

which implies, from (3.13), that

$$\sum_{v=0}^{k-1} \|e_n^{0^{(v)}}\|_{L^\infty(\sigma_n^0)} \leq Mh^m \frac{(1-hk(hk_1+L))\left(\left[(R_2-1)R_2^{\frac{\tau}{h}} + (1-R_1)\right] + R_2^{\frac{\tau}{h}}\right)}{\sqrt{\zeta}}.$$

Therefore, there exist C_0 and h_1 such that, for all $h \in (0, h_1]$,

$$\sum_{v=0}^{k-1} \|e_n^{0^{(v)}}\|_{L^\infty(\sigma_n^0)} \leq C_0 h^m. \tag{3.15}$$

Thus,

$$\|e^0\|_{L^\infty(\sigma^0)} = \max_{n=0,\ldots,N-1} \|e_n^0\|_{L^\infty(\sigma_n^0)} \leq C_0 h^m,$$

and for all $v = 1,\ldots,k-1$, it holds

$$\|e^{0^{(v)}}\|_{L^\infty(\sigma^0)} = \max_{n=0,\ldots,N-1} \|e_n^{0^{(v)}}\|_{L^\infty(\sigma_n^0)} \leq C_0 h^m.$$

Claim 2. There exists a constant C independent of h such that

$$\|e^{p^{(v)}}\|_{L^\infty(I)} \leq Ch^m$$

for all $v = 0,\ldots,k-1$, where the error $e^{p^{(v)}} = e^{(v)}|_{\sigma^p}$ is defined on σ_n^p by $e^{p^{(v)}}(t) = e_n^{p^{(v)}}(t) = x^{(v)}(t) - u_n^{p^{(v)}}(t)$ for all $n \in \{0,1,\ldots,N-1\}$ and $v \in \{1,\ldots,k-1\}$.
First, let $t \in \sigma_0^p$, for all $p \in \{1,\ldots,r-1\}$. For $n = 0$, from (2.6) we have

$$\begin{aligned} x^{(k)}(t) - \hat{u}_{0,p}^{(k)}(t) = &\sum_{v=0}^{k-1} L_v(t)(x^{(v)}(t) - \hat{u}_{0,p}^{(v)}(t)) + \sum_{v=0}^{k-1} M_v(t) e_0^{p-1^{(v)}}(t-\tau) \\ &+ \sum_{v=0}^{k-1}\sum_{i=0}^{p-2}\sum_{d=0}^{N-1} \int_{t_d^i}^{t_{d+1}^i} k_{2,v}(t,s) e_d^{i^{(v)}}(s)ds + \sum_{v=0}^{k-1} \int_{t_0^{p-1}}^{t-\tau} k_{2,v}(t,s) e_0^{p-1^{(v)}}(s)ds \\ &+ \sum_{v=0}^{k-1}\sum_{i=0}^{p-1}\sum_{d=0}^{N-1} \int_{t_d^i}^{t_{d+1}^i} k_{1,v}(t,s) e_d^{i^{(v)}}(s)ds + \sum_{v=0}^{k-1} \int_{t_0^p}^{t} k_{1,v}(t,s)(x^{(v)}(s) - \hat{u}_{0,p}^{(v)}(s))ds, \end{aligned}$$

so that $x^{(v)}(t_0^p) - \hat{u}_{0,p}^{(v)}(t_0^p) = x^{(v)}(t_0^p) - u^{p-1^{(v)}}(t_0^p) = e^{p-1^{(v)}}(t_0^p)$, for all $v = 0,\ldots,k-1$.
This implies that

$$\|x^{(k)} - \hat{u}_{0,p}^{(k)}\|_{L^\infty(\sigma_0^p)} \leq (hk_1 + L)\sum_{v=0}^{k-1} \|x^{(v)} - \hat{u}_{0,p}^{(v)}\|_{L^\infty(\sigma_0^p)} + \widetilde{M} \sum_{v=0}^{k-1}\sum_{i=0}^{p-1} \|e^{i^{(v)}}\|_{L^\infty(\sigma^i)} \tag{3.16}$$

where $\widetilde{M}$ is a positive number.
On the other hand, similar as in (3.12), we have for all $v \in \{1,\ldots,k\}$

$$\|x^{(k-v)} - \hat{u}_{0,p}^{(k-v)}\|_{L^\infty(\sigma_0^p)} \leq \sum_{i=1}^{v} h^{v-i} \|e^{p-1^{(k-i)}}\|_{L^\infty(\sigma^{p-1})} + h^v \|x^{(k)} - \hat{u}_{0,p}^{(k)}\|_{L^\infty(\sigma_0^p)}.$$

Hence, we get

$$\sum_{v=1}^{k} \|x^{(k-v)} - \hat{u}_{0,p}^{(k-v)}\|_{L^\infty(\sigma_0^p)} \leq (1+hk) \sum_{v=0}^{k-1} \|e^{p-1^{(v)}}\|_{L^\infty(\sigma^{p-1})} + hk\|x^{(k)} - \hat{u}_{0,p}^{(k)}\|_{L^\infty(\sigma_0^p)} \tag{3.17}$$

Therefore, by (3.16), we have

$$\begin{aligned}\sum_{v=0}^{k-1} \|x^{(v)} - \hat{u}_{0,p}^{(v)}\|_{L^\infty(\sigma_0^p)} &\leq \frac{1+hk}{1-hk(hk_1+L)} \sum_{v=0}^{k-1} \|e^{p-1^{(v)}}\|_{L^\infty(\sigma^{p-1})} \\ &\quad + \frac{hk\widetilde{M}}{1-hk(hk_1+L)} \sum_{v=0}^{k-1}\sum_{i=0}^{p-1} \|e^{i^{(v)}}\|_{L^\infty(\sigma^i)} \\ &\leq \frac{1+hk(1+\widetilde{M})}{1-hk(hk_1+L)} \sum_{v=0}^{k-1}\sum_{i=0}^{p-1} \|e^{i^{(v)}}\|_{L^\infty(\sigma^i)},\end{aligned}$$

and therefore, by Lemma 3.4, we deduce that

$$\begin{aligned}\sum_{v=0}^{k-1} \|e_0^{p^{(v)}}\|_{L^\infty(\sigma_0^p)} &\leq \sum_{v=0}^{k-1} \left(\|x^{(v)} - \hat{u}_{0,p}^{(v)}\|_{L^\infty(\sigma_0^p)} + \|\hat{u}_{0,p}^{(v)} - u_0^{p^{(v)}}\|_{L^\infty(\sigma_0^p)} \right) \\ &\leq \sum_{v=0}^{k-1} \|x^{(v)} - \hat{u}_{0,p}^{(v)}\|_{L^\infty(\sigma_0^p)} + \sum_{v=0}^{k-1} \frac{\alpha(m)}{(m+k-v)!} h^{m+k-v} \\ &\leq \frac{1+hk(1+\widetilde{M})}{1-hk(hk_1+L)} \sum_{v=0}^{k-1}\sum_{i=0}^{p-1} \|e^{i^{(v)}}\|_{L^\infty(\sigma^i)} + Mh^{m+1},\end{aligned} \tag{3.18}$$

Next, let $t \in \sigma_n^p$ for $n \in \{1, 2, \ldots, N-1\}$, we have from (2.6),

$$\begin{aligned}x^{(k)}(t) - \hat{u}_{n,p}^{(k)}(t) &= \sum_{v=0}^{k-1} L_v(t)(x^{(v)}(t) - \hat{u}_{n,p}^{(v)}(t)) + \sum_{v=0}^{k-1} M_v(t) e_n^{p-1^{(v)}}(t-\tau) \\ &+ \sum_{v=0}^{k-1}\sum_{i=0}^{p-2}\sum_{d=0}^{N-1} \int_{t_d^i}^{t_{d+1}^i} k_{2,v}(t,s) e_d^{i^{(v)}}(s)ds + \sum_{v=0}^{k-1}\sum_{i=0}^{n-1} \int_{t_i^{p-1}}^{t_{i+1}^{p-1}} k_{2,v}(t,s) e_i^{p-1^{(v)}}(s)ds \\ &+ \sum_{v=0}^{k-1} \int_{t_n^{p-1}}^{t-\tau} k_{2,v}(t,s) e_n^{p-1^{(v)}}(s)ds + \sum_{v=0}^{k-1}\sum_{i=0}^{p-1}\sum_{d=0}^{N-1} \int_{t_d^i}^{t_{d+1}^i} k_{1,v}(t,s) e_d^{i^{(v)}}(s)ds \\ &+ \sum_{v=0}^{k-1}\sum_{d=0}^{n-1} \int_{t_d^p}^{t_{d+1}^p} k_{1,v}(t,s) e_d^{p^{(v)}}(s)ds + \sum_{v=0}^{k-1} \int_{t_n^p}^{t} k_{1,v}(t,s)(x^{(v)}(s) - \hat{u}_{n,p}^{(v)}(s))ds,\end{aligned}$$

so that $x^{(v)}(t_n^p) - \hat{u}_{n,p}^{(v)}(t_n^p) = x^{(v)}(t_n^p) - u^{p-1^{(v)}}(t_n^p) = e^{p-1^{(v)}}(t_n^p)$, for all $v = 0, 1, \ldots, k-1$. This implies that

$$\begin{aligned}\|x^{(k)} - \hat{u}_{n,p}^{(k)}\|_{L^\infty(\sigma_n^p)} &\leq (L+hk_1) \sum_{v=0}^{k-1} \|x^{(v)} - \hat{u}_{n,p}^{(v)}\|_{L^\infty(\sigma_n^p)} + \widehat{M} \sum_{v=0}^{k-1}\sum_{i=0}^{p-1} \|e^{i^{(v)}}\|_{L^\infty(\sigma^i)} \\ &\quad + hk_1 \sum_{v=0}^{k-1}\sum_{d=0}^{n-1} \|e_d^{p^{(v)}}\|_{L^\infty(\sigma_d^p)},\end{aligned} \tag{3.19}$$

being $\widehat{M}$ a positive number. On the other hand, for all $v \in \{1, \ldots, k\}$, we have

$$\|x^{(k-v)} - \hat{u}_{n,p}^{(k-v)}\|_{L^\infty(\sigma_n^p)} \leq \sum_{i=0}^{v} h^{v-i} \|e_{n-1}^{p^{(k-i)}}\|_{L^\infty(\sigma_{n-1}^p)} + h^{v+1}\|x^{(k)} - \hat{u}_{n,p}^{(k)}\|_{L^\infty(\sigma_n^p)}.$$

Hence, similar as in (3.17), we get

$$\sum_{v=1}^{k}\|x^{(k-v)}-\hat{u}_{n,p}^{(k-v)}\|_{L^\infty(\sigma_n^p)} \le (1+hk)\sum_{v=0}^{k-1}\|e_{n-1}^{p^{(v)}}\|_{L^\infty(\sigma_{n-1}^p)} + hk\|x^{(k)}-\hat{u}_{n,p}^{(k)}\|_{L^\infty(\sigma_n^p)}.$$

Therefore, by using (3.19), we have

$$\sum_{v=0}^{k-1}\|x^{(v)}-\hat{u}_{n,p}^{(v)}\|_{L^\infty(\sigma_n^p)} \le \frac{1+hk}{1-hk(hk_1+L)}\sum_{v=0}^{k-1}\|e_{n-1}^{p^{(v)}}\|_{L^\infty(\sigma_{n-1}^p)}$$
$$+\frac{h^2kk_1}{1-hk(hk_1+L)}\sum_{v=0}^{k-1}\sum_{i=0}^{n-1}\|e_i^{p^{(v)}}\|_{L^\infty(\sigma_i^p)} + \frac{hk\widehat{M}}{1-hk(hk_1+L)}\sum_{v=0}^{k-1}\sum_{i=0}^{p-1}\|e^{i^{(v)}}\|_{L^\infty(\sigma^i)},$$

and, by Lemma 3.4, we deduce that

$$\begin{aligned}
\sum_{v=0}^{k-1}\|e_n^{p^{(v)}}\|_{L^\infty(\sigma_n^p)} &\le \sum_{v=0}^{k-1}\left(\|x^{(v)}-\hat{u}_{n,p}^{(v)}\|_{L^\infty(\sigma_n^p)} + \|\hat{u}_{n,p}^{(v)}-u_n^{p^{(v)}}\|_{L^\infty(\sigma_n^p)}\right)\\
&\le \sum_{v=0}^{k-1}\|x^{(v)}-\hat{u}_{n,p}^{(v)}\|_{L^\infty(\sigma_n^p)} + \sum_{v=0}^{k-1}\frac{\alpha(m)}{(m+k-v)!}h^{m+k-v}\\
&\le \frac{1+hk}{1-hk(hk_1+L)}\sum_{v=0}^{k-1}\|e_{n-1}^{p^{(v)}}\|_{L^\infty(\sigma_{n-1}^p)}\\
&\quad+\frac{h^2kk_1}{1-hk(hk_1+L)}\sum_{i=0}^{n-1}\sum_{v=0}^{k-1}\|e_i^{p^{(v)}}\|_{L^\infty(\sigma_i^p)}\\
&\quad+\frac{hk\widehat{M}}{1-hk(hk_1+L)}\sum_{v=0}^{k-1}\sum_{i=0}^{p-1}\|e^{i^{(v)}}\|_{L^\infty(\sigma^i)} + Mh^{m+1},
\end{aligned}$$

from Lemma 3.3, for all $n \in \{0,1,\dots,N-1\}$ it follows that

$$\begin{aligned}
\sum_{v=0}^{k-1}\|e_n^{p^{(v)}}\|_{L^\infty(\sigma_n^p)} &\le \frac{\sum_{v=0}^{k-1}\|e_0^{p^{(v)}}\|_{L^\infty(\sigma_0^p)}}{R_2-R_1}[(R_2-1)R_2^n+(1-R_1)R_1^n]\\
&\quad+\frac{\frac{hk\widehat{M}}{1-hk(hk_1+L)}\sum_{v=0}^{k-1}\sum_{i=0}^{p-1}\|e^{i^{(v)}}\|_{L^\infty(\sigma^i)}+Mh^{m+1}}{R_2-R_1}[R_2^n-R_1^n]\\
&\le \left(\sum_{v=0}^{k-1}\|e_0^{p^{(v)}}\|_{L^\infty(\sigma_0^p)}\right)\frac{(R_2-1)R_2^{\frac{\tau}{h}}+(1-R_1)}{R_2-R_1}\\
&\quad+\frac{k\widehat{M}\sum_{v=0}^{k-1}\sum_{i=0}^{p-1}\|e^{i^{(v)}}\|_{L^\infty(\sigma^i)}+(1-hk(hk_1+L))Mh^m}{\sqrt{\zeta}}R_2^{\frac{\tau}{h}},
\end{aligned}$$

where R_1 and R_2 are defined by (3.14), and

$$\zeta = k^2(L+1)^2+6kk_1+2hk^2k_1(2-L)-2h^2(kk_1)^2$$

So, there exist C_1 and h_2 such that, for all $h \in (0,h_2]$,

$$\sum_{v=0}^{k-1}\|e_n^{p^{(v)}}\|_{L^\infty(\sigma_n^p)} \le \left(\sum_{v=0}^{k-1}\|e_0^{p^{(v)}}\|_{L^\infty(\sigma_0^p)}+\sum_{v=0}^{k-1}\sum_{i=0}^{p-1}\|e^{i^{(v)}}\|_{L^\infty(\sigma^i)}+h^m\right)C_1,$$

which implies, by (3.18), that for all $h \leq h_2$,

$$\sum_{v=0}^{k-1} \|e_n^{p^{(v)}}\|_{L^\infty(\sigma_n^p)} \leq \left(\frac{1+hk(1+\widetilde{M})}{1-hk(hk_1+L)}+1\right) C_1 \sum_{v=0}^{k-1}\sum_{i=0}^{p-1} \|e^{i^{(v)}}\|_{L^\infty(\sigma^i)} + (Mh+1)C_1 h^m$$

$$\leq \left(\frac{1+h_2k(1+\widetilde{M})}{1-h_2k(h_2k_1+L)}+1\right) C_1 \sum_{v=0}^{k-1}\sum_{i=0}^{p-1} \|e^{i^{(v)}}\|_{L^\infty(\sigma^i)} + (Mh_2+1)C_1 h^m.$$

Hence, for $C_2 = \max\{\left(\frac{1+h_2k(1+\widetilde{M})}{1-h_2k(h_2k_1+L)}+1\right) C_1, (Mh_2+1)C_1\}$, we obtain for all $n = 0, \ldots, N-1$,

$$\sum_{v=0}^{k-1} \|e_n^{p^{(v)}}\|_{L^\infty(\sigma_n^p)} \leq C_2 \sum_{v=0}^{k-1}\sum_{i=0}^{p-1} \|e_n^{i^{(v)}}\|_{L^\infty(\sigma_n^i)} + C_2 h^m.$$

We deduce that

$$\sum_{v=0}^{k-1} \|e_n^{p^{(v)}}\|_{L^\infty(\sigma_n^p)} \leq C_2 \sum_{i=0}^{p-1}\sum_{v=0}^{k-1} \|e_n^{i^{(v)}}\|_{L^\infty(\sigma_n^i)} + C_3 h^m, \tag{3.20}$$

where $C_3 = \max\{C_0, C_2\}$.
Then, from (3.15), (3.20) and by Lemma 3.1, we get

$$\sum_{v=0}^{k-1} \|e_n^{p^{(v)}}\|_{L^\infty(\sigma_n^p)} \leq \underbrace{C_3 \exp(rC_2)}_{C} h^m.$$

This implies that for all $p = 0, 1, \ldots, r-1$ and $v = 0, \ldots, k-1$

$$\|e^{p^{(v)}}\|_{L^\infty(\sigma^p)} \leq Ch^m.$$

Thus, the proof of theorem 3.5 is completed. □

4 Numerical Examples

We will give some numerical examples to illustrate the theoretical results obtained in the previous section. In each example, we calculate the error between x and the Taylor collocation solution u. We also compute the resulting error $e^{(v)}$ between exact derivatives $x^{(v)}$ and approximate derivatives $u^{(v)}$ for $v \in \{1, \ldots, k-1\}$ in the example 4.3 and 4.4. We compare our results with other well-known methods: the classical method [6], the Hermite method [13], the multistep method given in [12]. The results in these examples confirm the theoretical ones and suggest that the experimental order of convergence (EOC) is m (see EOC of example 4.1 and example 4.2 in Table 4.3).

Example 4.1. ([13]) We consider the linear delay Volterra integral equation

$$x(t) = \begin{cases} g(t) + \int_0^t (t+s+1)x(s)ds + \int_0^{t-\frac{1}{2}} sin(s-t)x(s)ds, & t \in [0,1], \\ \Phi(t) = tsin(t), & t \in [-\frac{1}{2}, 0). \end{cases}$$

g is chosen so that the exact solution is $x(t) = tsin(t)$. The maximum errors $\|x-u\|_{L^\infty(I)}$ obtained by the present method for $m = 8$ and $N = 4, 8, 16, 32$ are compared in Table 4.1 and Fig 4.1 with the errors of the classical method [6], the multistep method [12] and the Hermite method given in [13].

Table 4.1: Comparison of the maximum errors of Example 4.1

N	Classical method	Multistep method	Hermite method	Present method
4	9.82×10^{-1}	3.18×10^{-3}	6.85×10^{-11}	1.63×10^{-11}
8	7.85×10^{-1}	3.66×10^{-4}	3.49×10^{-13}	6.37×10^{-14}
16	3.72×10^{-1}	2.65×10^{-5}	1.66×10^{-15}	2.48×10^{-16}
32	1.05×10^{-1}	1.79×10^{-6}	5.60×10^{-16}	9.67×10^{-19}

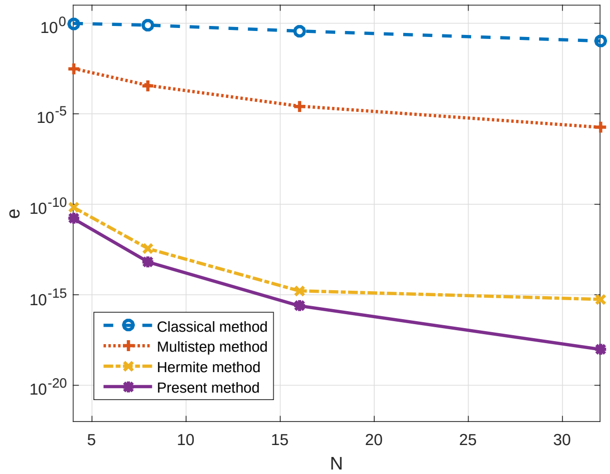


Figure 4.1: Comparison of the errors $\|x-u\|_{L^\infty(I)}$ of Example 4.1.

Example 4.2. We consider the neutral delay second order linear VIDE

$$x''(t) = \begin{cases} 1+t-(t+\frac{1}{\sqrt{e}})e^t + x(t) + x(t-\dfrac{1}{2}) + \int_0^t tx(s)ds + \int_0^t (t+s)x'(s)ds \\ \quad + \int_0^{t-\tau} x(s)ds + \int_0^{t-\tau} tsx'(s)ds, \ t \in [0,3], \\ \\ \Phi(t) = e^t, \qquad t \in [-\frac{1}{2}, 0). \end{cases}$$

The exact solution is given by $x(t) = e^t$ and the errors e between x and u for $(m,N) = \{(4,4),(5,5),(6,6),(8,8)\}$ at $t = 0, 0.5, \ldots, 3$ are presented In Table 4.2.

Table 4.2: Absolute errors of Example 4.2

t	$m=4, N=4$	$m=5, N=5$	$m=6, N=6$	$m=8, N=8$
0.0	0.0	0.0	0.0	0.0
0.5	2.44×10^{-7}	1.36×10^{-9}	1.96×10^{-10}	6.43×10^{-11}
1.0	1.45×10^{-6}	8.56×10^{-9}	1.13×10^{-9}	9.31×10^{-11}
1.5	5.36×10^{-6}	3.17×10^{-8}	2.81×10^{-8}	5.56×10^{-10}
2.0	1.88×10^{-5}	1.16×10^{-7}	1.14×10^{-8}	5.45×10^{-9}
2.5	7.05×10^{-5}	4.58×10^{-7}	5.73×10^{-9}	1.78×10^{-8}
3.0	2.91×10^{-4}	1.96×10^{-6}	1.83×10^{-8}	8.24×10^{-8}

Table 4.3: Experimental orders of convergence (EOC)

N 2	$m=3$	$m=4$	$m=5$
4	2.68	3.92	4.72
8	2.85	3.98	4.87
16	2.93	3.99	4.94
32	2.97	4.01	4.97

EOC of Example 4.1

N 2	$m=2$	$m=3$	$m=4$
4	1.66	2.70	3.69
8	1.83	2.84	3.84
16	1.92	2.91	4.45
32	1.96	2.95	3.42

EOC of Example 4.2

Example 4.3. Let us consider the following fourth order linear NDVIDE

$$\begin{aligned} x^{(4)}(t) &= g(t) + \sum_{v=0}^{3} L_v(t)x^{(v)}(t) + \sum_{v=0}^{3} M_v(t)x^{(v)}(t-0.5) \\ &+ \int_0^t \sin(s-t)x(s)ds + \int_0^{t-0.5} \frac{3s}{1+t}x(s)ds, \ t \in [0,2], \end{aligned} \tag{4.1}$$

and $\qquad x(t) = \frac{1+t}{e^t}, \qquad t \in [-\frac{1}{2}, 0).$

Here, the functions characterizing equation (4.1) are given by $L_0 = 1$, $L_1 = t^2$, $L_2 = L_3 = 0$, $M_0 = \dfrac{1}{3}$, $M_1 = M_3 = 1$, $M_2 = t$ and g is chosen so that the exact solution is $x(t) = \frac{1+t}{e^t}$.

Table 4.4 shows the errors e between x and u obtained by using the Taylor collocation method. We also compute the errors $e^{(v)} := x^{(v)} - u^{(v)}$ for $v = 1,2,3$ and $(m, N) = \{(4,4),(4,8),(8,4),(7,8)\}$ at $t = 0, 0.5, \ldots, 2$, the results are shown in tables 4.5-4.7. Moreover, Fig 4.2 shows some numerical results.

Table 4.4: Absolute errors of Example 4.3

t	$m=4, N=4$	$m=4, N=8$	$m=8, N=4$	$m=7, N=8$
0.0	0.0	0.0	0.0	0.0
0.5	2.38×10^{-8}	1.81×10^{-9}	3.88×10^{-11}	5.29×10^{-11}
1.0	4.25×10^{-7}	2.85×10^{-8}	3.69×10^{-10}	3.35×10^{-10}
1.5	2.18×10^{-6}	1.42×10^{-7}	1.24×10^{-9}	8.80×10^{-10}
2.0	7.35×10^{-6}	4.71×10^{-7}	3.27×10^{-9}	9.41×10^{-10}

Table 4.5: The resulting error e' of Example 4.3

t	$m=4, N=4$	$m=4, N=8$	$m=8, N=4$	$m=7, N=8$
0.0	0.0	0.0	0.0	0.0
0.5	2.06×10^{-7}	1.46×10^{-8}	4.64×10^{-10}	4.92×10^{-10}
1.0	1.71×10^{-6}	1.12×10^{-7}	1.26×10^{-9}	6.67×10^{-10}
1.5	5.95×10^{-6}	3.81×10^{-7}	2.38×10^{-9}	1.00×10^{-9}
2.0	1.62×10^{-5}	1.03×10^{-6}	6.07×10^{-9}	1.07×10^{-9}

Table 4.6: The resulting error e'' of Example 4.3

t	$m=4, N=4$	$m=4, N=8$	$m=8, N=4$	$m=7, N=8$
0.0	0.0	0.0	0.0	0.0
0.5	1.31×10^{-6}	8.73×10^{-8}	1.83×10^{-10}	4.92×10^{-10}
1.0	5.14×10^{-6}	3.28×10^{-7}	1.42×10^{-9}	1.98×10^{-9}
1.5	1.27×10^{-5}	8.11×10^{-7}	4.96×10^{-9}	2.06×10^{-9}
2.0	3.11×10^{-5}	1.98×10^{-6}	1.30×10^{-8}	2.08×10^{-9}

Table 4.7: The resulting error $e^{(3)}$ of Example 4.3

t	$m=4, N=4$	$m=4, N=8$	$m=8, N=4$	$m=7, N=8$
0.0	0.0	0.0	0.0	0.0
0.5	5.68×10^{-6}	3.47×10^{-7}	3.45×10^{-11}	1.07×10^{-9}
1.0	1.06×10^{-5}	6.66×10^{-7}	8.13×10^{-10}	7.53×10^{-10}
1.5	2.21×10^{-5}	1.40×10^{-6}	1.31×10^{-9}	1.28×10^{-9}
2.0	5.90×10^{-5}	3.75×10^{-6}	1.64×10^{-8}	4.85×10^{-9}

Example 4.4. Consider the following 19th NDVIDE.

$$\begin{aligned} x^{(19)}(t) &= g(t) + \sum_{v=0}^{18} L_v(t)x^{(v)}(t) + \sum_{v=0}^{18} M_v(t)x^{(v)}(t-0.5) \\ &+ \int_0^t e^{s-t}x(s)ds + \int_0^{t-0.5} \frac{3s}{1+2t}x(s)ds, \ t \in [0,3], \end{aligned} \tag{4.2}$$

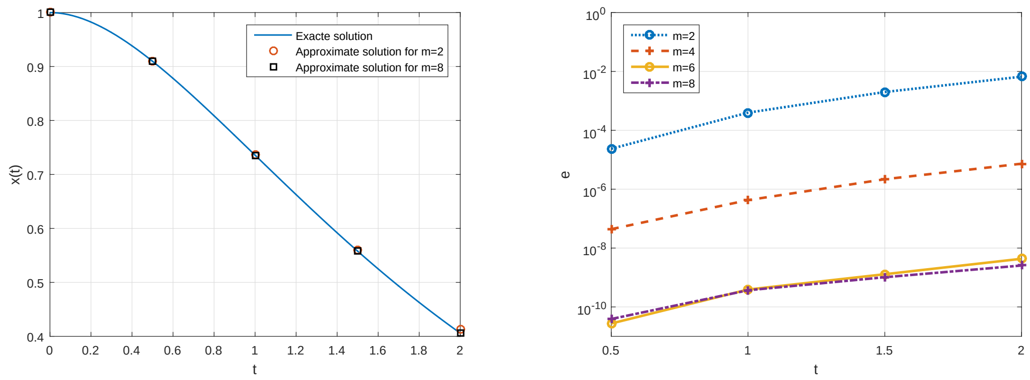
and $\quad x(t) = 1 - cos(t), \qquad t \in [-\frac{1}{2}, 0).$

Here, g is chosen so that the exact solution is $x(t) = 1 - cos(t)$, the functions characterizing equation (4.2) are given by $L_v = 1$ and $M_v = \dfrac{1}{3}$ for $v = 0, 2, 8$, $L_v = 2t + 1$ and $M_v = t^2 + 1$ for $v = 3, 6, 9, 12, 17, 18$, $L_v = M_v = 0$ for $v = 1, 4, 5, 7, 10, 11, 13, 14, 15, 16$.

The resulting errors e, $e^{(2)}$, $e^{(11)}$ and $e^{(17)}$ for $(m, N) = (7, 2)$ at $t = 0, 0.5, \ldots, 2$ are presented in Table 4.8.

Table 4.8: Absolute errors of Example 4.4

t	$e^{(17)}$	$e^{(11)}$	$e^{(2)}$	e
0.0	0.0	0.0	0.0	0.0
0.5	2.29×10^{-10}	1.51×10^{-10}	1.53×10^{-10}	3.40×10^{-13}
1.0	7.68×10^{-10}	4.79×10^{-10}	3.52×10^{-12}	3.01×10^{-11}
1.5	1.36×10^{-8}	9.94×10^{-10}	2.68×10^{-9}	4.31×10^{-11}
2.0	2.46×10^{-7}	2.24×10^{-11}	2.19×10^{-9}	1.80×10^{-10}



(a) The exact solution and approximate solutions.

(b) Comparison of the absolute error functions.

Figure 4.2: Numerical results of Example 4.3.

5 Conclusion

In this paper, we have proposed a collocation method for the numerical solution of the high-order neutral delay linear Volterra integro-differential equations (1.1), which has been derived by using Taylor polynomials. The main novelty of this method is the study of the convergence of the approximate solution and approximate derivatives up to order $k-1$ of the solution for kth-order neutral delay VIDE. Moreover, this method is easy to implement. Finally, numerical examples were introduced, showing that the method is convergent with good accuracy, and the numerical results confirmed the theoretical estimates. Further research on this kind of problem will be conducted by generalizing the work done to a system of kth-order neutral delay VIDEs.

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Conflict of Interest

The authors have no conflicts of interest to declare.

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