Journal of Innovative Applied Mathematics and Computational Sciences

J. Innov. Appl. Math. Comput. Sci. **3**(1) (2023), 66–74. DOI: 10.58205/jiamcs.v3i1.204

http://jiamcs.centre-univ-mila.dz/

Existence and asymptotic stability of continuous solutions for a general form of integral equations of product type

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Received 6 May 2023, Accepted 6 June 2023, Published 29 June 2023

Abstract. In this work, we study the existence and the asymptotic stability of the continuous solutions for integral equations of product type in a general form. Our result will be given in more general conditions. Moreover, the integral equation of product type offered in this work contains several specific forms studied recently. The analysis uses the techniques of measures of noncompactness and Darbo's fixed point theorem.

Keywords: Integral equations of product type, Darbo's fixed point theorem, Continuous solution, Asymptotic stable solution.
2020 Mathematics Subject Classification: 45D05, 47H30. MSC2020

1 Introduction

In this paper, we consider the following nonlinear integral equation of product type

$$x(t) = f_1(t, x(t)) + f_2\left(t, \int_0^t u(t, s, x(s))ds\right) \times f_3\left(t, \int_0^t v(t, s, x(s))ds\right), \ t \in \mathbb{R}^+, \tag{1.1}$$

where f_i, $i = 1, 2, 3$ and u, v are continuous functions while $x(t) \in C(\mathbb{R}_+)$ is an unknown function.

Integral equations of product type are an important class of integral equations. They play a fundamental role in modeling, including applied mathematics, physics and biology science, particularly in the study of the spread of an infectious disease that does not induce permanent immunity (see, for example [3, 11, 12, 17, 18]).

Numerous researchers have extensively explored the existence and the qualitative behavior of the solutions for the integral equations of product type. Gripenberg [13] studied the qualitative behavior of solutions of the following integral equation of product type

$$x(t) = k\left[p(t) + \int_0^t A(t-s)x(s)ds\right] \times \left[q(t) + \int_0^t B(t-s)x(s)ds\right]. \tag{1.2}$$

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ISSN (electronic): 2773-4196

Pachpatte [16], Abdeldaim [1] and Li et al. [14] studied the roundedness, the asymptotic behavior and continuous solutions of (1.2).
Bellour et al. [8] studied the existence of an integrable solution of a particular form of (1.1) on the interval $[0,1]$.
Bousselsal and Bellour [10] studied the existence and asymptotic stability of continuous solutions for the following particular form of (1.1)

$$x(t) = f_1(t,x(t)) + \left(p(t) + \int_0^t u(t,s,x(s))ds\right) \times \left(q(t) + \int_0^t v(t,s,x(s))ds\right),\ t \in \mathbb{R}^+,$$

In contrast, Ardjouni and Djoudi [2] investigated the existence and the approximation of solutions to initial value problems of nonlinear hybrid Caputo fractional integro-differential equations, which can be transformed into the following integral equation of product type.

$$x(t) = \left[p(t) + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s,x(s))ds\right] \times \left[\theta + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s))ds\right],$$

on a bounded interval $[0,a]$.
Olaru [15] studied the existence and the uniqueness of the continuous solution of the following integral equation

$$x(t) = \prod_{i=1}^{m}\left(g_i(t) + \int_a^t K_i(t,s,x(s))ds\right), \tag{1.3}$$

on a bounded interval $[a,b]$, where $K_i, i = 1,...,n$ are continuous functions satisfying Lipschitz conditions with respect to the last variable.
Later, Boulfoul et al. [9] studied the existence of an integrable solution of a more generalized version of (1.3) on $\mathbb{R}^+$.
This manuscript is motivated by extending and generalizing the work of [10] and investigating the existence of continuous solutions and their asymptotic stability for (1.1) on $\mathbb{R}^+$ under fairly simple conditions. An example is provided to illustrate the importance and applicability of our results.

2 Auxiliary facts and results

In this section, we provide some notations, definitions and auxiliary facts which will be needed for stating our results. Denote by $BC(\mathbb{R}^+)$ the Banach space consists of all real functions defined, continuous and bounded on $\mathbb{R}_+$. It is equipped with the standard norm

$$\|x\| = \sup_{t\in\mathbb{R}^+} |x(t)|.$$

For later use, we assume that X is a Banach space. Let $\mathcal{B}(X)$ denote the family of all nonempty bounded subsets of X and $\mathcal{W}(X)$ the subset of $\mathcal{B}(X)$ consisting of all relatively compact subsets of X. Finally, let B_r denote the closed ball centered at 0 with radius r.
Recall the following definition of the concept of the axiomatic measure of noncompactness.

Definition 2.1. [6]. A function $\mu : \mathcal{B}(X) \longrightarrow \mathbb{R}^+$ is said to be a measure of noncompactness if it satisfies the following conditions:

(1) The family $ker(\mu) = \{M \in \mathcal{B}(X) : \mu(M) = 0\}$ is nonempty and $ker(\mu) \subset \mathcal{W}(X)$.

(2) $M_1 \subset M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)$.

(3) $\mu(co(M)) = \mu(M)$, where $co(M)$ is the convex hull of M.

(4) $\mu(\lambda M_1 + (1-\lambda)M_2) \leq \lambda\mu(M_1) + (1-\lambda)\mu(M_2)$ for $\lambda \in [0,1]$.

(5) If $(M_n)_{n\geq 1}$ is a sequence of nonempty, weakly closed subsets of X with M_1 bounded and $M_1 \supseteq M_2 \supseteq ... \supseteq M_n \supseteq ...$ such that $\lim\limits_{n\to\infty} \mu(M_n) = 0$, then $M_\infty := \bigcap_{n=1}^{\infty} M_n$ is nonempty.
A measure μ is said to be sublinear if it satisfies the following two conditions:

(6) $\mu(\lambda M) = |\lambda|\mu(M)$ for $\lambda \in \mathbb{R}$.

(7) $\mu(M_1 + M_2) \leq \mu(M_1) + \mu(M_2)$.

The family $ker(\mu)$ described in (1) is called the kernel of the measure of noncompactness μ. More information about measures of noncompactness and their properties can be found in [5].

In what follows, we will use a measure of noncompactness in the space $BC(\mathbb{R}^+)$, which was introduced in [5]. In order to recall the definition of this measure, let us fix a nonempty bounded subset $X \in BC(\mathbb{R}^+)$ and a positive number $T > 0$. For $x \in X$ and $\varepsilon > 0$, let us define the following quantities (cf. [5]):

$$\omega^T(x,\varepsilon) = \sup\left\{|x(s) - x(t)| : t,s \in [0,T], |t-s| \leq \varepsilon\right\}.$$

Further, let us put

$$\omega^T(X,\varepsilon) = \sup\left\{\omega^T(x,\varepsilon) : x \in X\right\},$$

$$\omega_0^T(X) = \lim_{\varepsilon \longrightarrow 0} \omega^T(X,\varepsilon), \omega_0(X) = \lim_{T \longrightarrow \infty} \omega_0^T(X).$$

For a fixed number $t \geq 0$, we denote

$$d(X(t)) = \sup\left\{|x(t) - y(t)| : x,y \in X\right\},$$

and

$$d(X) = \limsup_{t \longrightarrow \infty} d(X(t)).$$

Finally, the function μ is defined on the family M_C by putting

$$\mu(X) = \omega_0(X) + d(X).$$

It can be shown [5] that the function μ is a measure of noncompactness in the space $BC(\mathbb{R}^+)$ with the kernel $ker(\mu)$ consisting of all nonempty and bounded sets X such that functions from X are equicontinuous and nondecreasing on $\mathbb{R}^+$. For other properties of μ, see [5].

3 Main result

We will use the following fixed point theorem.

Theorem 3.1. *[4] Let $\mathcal{Q}$ be a nonempty bounded closed convex subset of the space E and let $F : \mathcal{Q} \longrightarrow \mathcal{Q}$ be a continuous operator such that $\mu(FX) \leq k\mu(X)$ for any nonempty subset X of $\mathcal{Q}$, where $k \in [0,1)$ is a constant. Then F has a fixed point in the set $\mathcal{Q}$.*

Equation (1.1) will be studied under the following assumptions:

(i) The functions $u, v : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ are continuous and there exist functions $a_i \in BC(\mathbb{R}^+)$ $(i = 1, 2)$ such that $|u(t,s,x)| \leq k_1(t,s)a_1(s)$ and $|v(t,s,x)| \leq k_2(t,s)a_2(s)$ for $(t,s,x) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$, where $k_i : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ $(i = 1, 2)$ are measurable functions and the linear Volterra operators K_i generated by k_i,

$$(K_i x)(t) = \int_0^t k_i(t,s)x(s)ds, \tag{3.1}$$

transform the space $BC(\mathbb{R}^+)$ into itself. Let $\|K_i\|$ be the norm of the operator K_i, $i = 1, 2$.

(ii) $\lim\limits_{t \longrightarrow +\infty} (K_i 1)(t) = \lim\limits_{t \longrightarrow +\infty} \int_0^t k_i(t,s)ds = 0$, for $i = 1, 2$.

(iii) The functions $f_i : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2, 3)$ are continuous such that $f_i(t, 0) \in BC(\mathbb{R}^+)$. Let $\overline{f_i}$ be the norm of $f_i(t, 0)$ in $BC(\mathbb{R}^+)$ for $i = 1, 2, 3$.

(iv) There exist constants $p_i > 0$ and functions $\lambda_i \in BC(\mathbb{R}^+)$ for $i = 1, 2, 3$ such that $|f_i(t,x) - f_i(t,y)| \leq \lambda_i(t) |x - y|^{p_i}$ with $p_1 = 1$ and $\|\lambda_1\| < 1$.

To prove our main result, we need the following lemma.

Lemma 3.2. *Under the assumptions $(i) - (iv)$, the operators*

$$(F_2 x)(t) = f_2\left(t, \int_0^t u(t,s,x(s))ds\right),$$
$$(F_3 x)(t) = f_3\left(t, \int_0^t v(t,s,x(s))ds\right),$$

map $BC(\mathbb{R}^+)$ continuously into itself.

Proof. We only prove that F_2 maps $BC(\mathbb{R}^+)$ continuously into itself and the proof of F_3 is similar.
The operator F_2 maps $BC(\mathbb{R}^+)$ into $C(\mathbb{R}^+)$. Moreover, let $x \in BC(\mathbb{R}^+)$, since

$$|(F_2 x)(t)| \leq \|\lambda_2\| \|a_1\|^{p_2} \|K_1\|^{p_2} + \overline{f_2}.$$

Hence, F_2 maps $BC(\mathbb{R}^+)$ into itself.
Now, to prove that F_2 is continuous, let $\{x_n\}$ be an arbitrary sequence in $BC(\mathbb{R}^+)$ which converges to $x \in BC(\mathbb{R}^+)$.
Then, from the assumption (ii), for $\varepsilon > 0$, there exist $n_1 \in \mathbb{N}$ and $T > 0$, such that for all $n \geq n_1$ and $t \geq T$, we have

$$\|x_n\| \leq 1 + \|x\|, (K_1 1)^{p_2}(t) \leq \frac{\varepsilon}{2^{p_2}\|a_1\|^{p_2} + 1}.$$

It follows that, for $n \geq n_1$ and $t \geq T$, we have

$$|(F_2 x_n - F_2 x)(t)| \leq 2^{p_2} \|\lambda_2\| \|a_1\|^{p_2} (K_1 1)(t) \leq \varepsilon. \tag{3.2}$$

On the other hand, since u is uniformly bounded on the compact set $[0, T] \times [0, T] \times [-1 - \|x\|, 1 + \|x\|]$, hence there exists $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$, we have

$$\sup\{|u(t,s,x_n(s)) - u(t,s,x(s))|, (t,s) \in [0,T] \times [0,T], n \geq n_2\} \leq \left(\frac{\varepsilon}{T\|\lambda_2\| + 1}\right)^{p_2},$$

This implies that, for all $n \geq n_2$ and $t \in [0, T]$

$$|(F_2x_n - F_2x)(t)| \leq \varepsilon. \tag{3.3}$$

Then, from (3.2) and (3.3), we deduce that, for all $n \geq n_0 = \max(n_1, n_2)$

$$\|F_2x_n - F_2x\| \leq \varepsilon.$$

Thus, F_2 maps $BC(\mathbb{R}^+)$ continuously into itself. $\square$

Remark 3.3. [7] The concept of the asymptotic stability of a solution $x = x(t)$ of Eq. (1.1) is understood in the following sense.
For any $\varepsilon > 0$ there exist $T > 0$ and $r > 0$ such that if $x, y \in B_r$ and $x = x(t), y = y(t)$ are solutions of (1.1) then $|x(t) - y(t)| \leq \varepsilon$ for $t \geq T$.

Now we are in a position to state our main result.

Theorem 3.4. *Under the assumptions above the nonlinear integral equation (1.1) has at least an asymptotically stable solution $x \in BC(\mathbb{R}^+)$.*

Proof. Solving Eq. (1.1) is equivalent to finding a fixed point of the operator A, where $Ax(t) = f_1(t, x(t)) + (F_2x)(t) \times (F_3x)(t)$. We will show that A satisfies the conditions of Theorem 3.1. The proof is split into four steps.
Step 1. We first show that there exists B_{r_0} from $BC(\mathbb{R}^+)$ such that $A(B_{r_0}) \subset B_{r_0}$. To see this, let $x \in B_r$. Then

$$\begin{aligned}
\|Ax\| \leq & \|f_1(t, x(t))\| + \|(F_2x)(t) \times (F_3x)(t)\| \\
\leq & \|\lambda_1\|\|x\| + \overline{f_1} + (\overline{f_2} + \|\lambda_2\|\|K_1\|^{p_2}\|a_1\|^{p_2}) \times (\overline{f_3} + \|\lambda_3\|\|K_2\|^{p_3}\|a_2\|^{p_3}) \\
\leq & \|\lambda_1\|r + \overline{f_1} + (\overline{f_2} + \|\lambda_2\|\|K_1\|^{p_2}\|a_1\|^{p_2}) \times (\overline{f_3} + \|\lambda_3\|\|K_2\|^{p_3}\|a_2\|^{p_3}).
\end{aligned}$$

Since $\|\lambda_1\| < 1$, we deduce that the operator A transforms the ball B_{r_0} into itself for $r_0 = \frac{\overline{f_1} + (\overline{f_2} + \|\lambda_2\|\|K_1\|^{p_2}\|a_1\|^{p_2}) \times (\overline{f_3} + \|\lambda_3\|\|K_2\|^{p_3}\|a_2\|^{p_3})}{1 - \|\lambda_1\|}$.
Step 2. The operator A maps B_{r_0} continuously into itself. To see this, take an arbitrary number $\epsilon > 0$ and a convergent sequence (x_n) to (x) in B_{r_0}.
Hence, by Lemma 3.2, there exists n_0 such that for all $n \geq n_0$, we have

$$\begin{aligned}
\|x_n - x\| \leq & \frac{\epsilon}{3\|\lambda_1\|}, \|F_2x_n - F_2x\| \leq \frac{\epsilon}{3(\|\lambda_3\|\|a_2\|^{p_3}\|K_2\|^{p_3} + \overline{f_3})}, \\
\|F_3x_n - F_3x\| \leq & \frac{\epsilon}{3(\|\lambda_2\|\|a_1\|^{p_2}\|K_1\|^{p_2} + \overline{f_2})}.
\end{aligned}$$

Which implies, for all $n \geq n_0$,

$$\begin{aligned}
\|Ax_n - Ax\| \leq & \|\lambda_1\|\|x_n - x\| + \|(F_2x_n) \times (F_3x_n) - (F_2x) \times (F_3x)\| \\
\leq & \|\lambda_1\|\|x_n - x\| + \|F_2x_n\|\|F_3x_n - F_3x\| + \|F_3x\|\|F_2x_n - F_2x\| \\
\leq & \|\lambda_1\|\|x_n - x\| + (\|\lambda_2\|\|a_1\|^{p_2}\|K_1\|^{p_2} + \overline{f_2})\|F_3x_n - F_3x\| \\
& + (\|\lambda_3\|\|a_2\|^{p_3}\|K_2\|^{p_3} + \overline{f_3})\|F_2x_n - F_2x\| \\
\leq & \epsilon.
\end{aligned}$$

We deduce that the operator A maps B_{r_0} continuously into itself.
Step 3. We illustrate that there exists $\gamma \in [0,1)$ such that $\mu(AX) \leq \gamma\mu(X)$ for all subset X of B_{r_0}. To see this, take an arbitrary number $t \geq 0$. Then for any $x, y \in X$, we have

$$\begin{aligned} |Ax(t) - Ay(t)| \leq & \|\lambda_1\||x(t) - y(t)| + |F_2x(t)||F_3x(t) - F_3y(t)| + |F_3y(t)||F_2x(t) - F_2y(t)| \\ \leq & \|\lambda_1\|\|x - y\| + (\overline{f_2} + \|\lambda_2\|\|K_1\|^{p_2}\|a_1\|^{p_2})|F_3x(t) - F_3y(t)| \\ & + (\overline{f_3} + \|\lambda_3\|\|K_2\|^{p_3}\|a_2\|^{p_3})|F_2x(t) - F_2y(t)| \\ \leq & \|\lambda_1\|\|x - y\| + (\overline{f_2} + \|\lambda_2\|\|K_1\|^{p_2}\|a_1\|^{p_2})2^{p_3}\|\lambda_3\|\|a_2\|^{p_3}\,(K_2 1)\,(t) \\ & + (\overline{f_3} + \|\lambda_3\|\|K_2\|^{p_3}\|a_2\|^{p_3})2^{p_2}\|\lambda_2\|\|a_1\|^{p_2}\,(K_1 1)\,(t). \end{aligned} \tag{3.4}$$

Which implies that

$$\begin{aligned} d(AX(t)) \leq & \|\lambda_1\|d(X(t)) + (\overline{f_2} + \|\lambda_2\|\|K_1\|^{p_2}\|a_1\|^{p_2})2^{p_3}\|\lambda_3\|\|a_2\|^{p_3}\,(K_2 1)\,(t) \\ & + (\overline{f_3} + \|\lambda_3\|\|K_2\|^{p_3}\|a_2\|^{p_3})2^{p_2}\|\lambda_2\|\|a_1\|^{p_2}\,(K_1 1)\,(t). \end{aligned}$$

Now, taking into account the assumption (ii), we obtain the following estimate:

$$d(AX) \leq \|\lambda_1\|d(X). \tag{3.5}$$

Further, let us fix numbers $T > 0$, $\varepsilon > 0$ arbitrarily, let $x \in X$ and take $t_1, t_2 \in [0, T]$ such that $|t_2 - t_1| \leq \varepsilon$. Without loss of generality, we assume that $t_1 < t_2$.
Then, in view of our assumptions, we have

$$\begin{aligned} |Ax(t_2) - Ax(t_1)| \leq & \|\lambda_1\||x(t_2) - x(t_1)| + |f_1(t_2, x(t_2)) - f_1(t_1, x(t_2))| + \\ & + |F_2x(t_2)||F_3x(t_2) - F_3x(t_1)| + |F_3x(t_1)||F_2x(t_2) - F_2x(t_1)| \\ \leq & \|\lambda_1\||x(t_2) - x(t_1)| + \overline{\omega}^T(f_1, \varepsilon) \\ & + (\overline{f_2} + \|\lambda_2\|\|K_1\|^{p_2}\|a_1\|^{p_2})|F_3x(t_2) - F_3x(t_1)| \\ & + (\overline{f_3} + \|\lambda_3\|\|K_2\|^{p_3}\|a_2\|^{p_3})|F_2x(t_2) - F_2x(t_1)|, \end{aligned} \tag{3.6}$$

where $\overline{\omega}^T(f_1, \varepsilon) = \sup\{|f_1(t_2, x) - f_1(t_1, x)|, t_1, t_2 \in [0, T], |t_2 - t_1| \leq \varepsilon, |x| \leq r_0\}$.
Now, from the assumption (i), we have

$$\begin{aligned} \left|\int_0^{t_2} u(t_2, s, x(s))ds - \int_0^{t_1} u(t_1, s, x(s))ds\right| \leq & \int_0^{t_2} |u(t_2, s, x(s)) - u(t_1, s, x(s))|ds \\ & + \int_{t_1}^{t_2} |u(t_1, s, x(s))|ds \\ \leq & T\overline{\omega}^T(u, \varepsilon) + |t_2 - t_1|\overline{u} \\ \leq & T\overline{\omega}^T(u, \varepsilon) + \varepsilon\overline{u}, \end{aligned} \tag{3.7}$$

where,

$$\begin{aligned} \overline{\omega}^T(u, \varepsilon) &= \sup\{|u(t_2, s, x) - u(t_1, s, x)|, t_1, t_2, s \in [0, T], |t_2 - t_1| \leq \varepsilon, |x| \leq r_0\}, \\ \overline{u} &= \sup\{|u(t, s, x)|, t, s \in [0, T], |x| \leq r_0\}. \end{aligned}$$

Similarly, from the assumption (i), we obtain

$$\left|\int_0^{t_2} v(t_2, s, x(s))ds - \int_0^{t_1} v(t_1, s, x(s))ds\right| \leq T\overline{\omega}^T(v, \varepsilon) + \varepsilon\overline{v}, \tag{3.8}$$

where,

$$\begin{aligned}\overline{\omega}^T(v,\varepsilon) &= \sup\{|v(t_2,s,x)-v(t_1,s,x)|, t_1,t_2,s\in[0,T], |t_2-t_1|\le\varepsilon, |x|\le r_0\},\\ \overline{v}^T &= \sup\{|v(t,s,x)|, t,s\in[0,T], |x|\le r_0\}.\end{aligned}$$

Hence, from (3.7) and (3.8), we obtain

$$|F_2x(t_2)-F_2x(t_1)| \le \overline{\omega}^T(f_2,\varepsilon)+\|\lambda_2\|(T\overline{\omega}^T(u,\varepsilon)+\varepsilon\overline{u})^{p_2} \tag{3.9}$$

and

$$|F_3x(t_2)-F_3x(t_1)| \le \overline{\omega}^T(f_3,\varepsilon)+\|\lambda_3\|(T\overline{\omega}^T(v,\varepsilon)+\varepsilon\overline{v})^{p_3}, \tag{3.10}$$

where,

$$\begin{aligned}\overline{\omega}^T(f_2,\varepsilon) &= \sup\{|f_2(t_2,x)-f_2(t_1,x)|, t_1,t_2\in[0,T], |t_2-t_1|\le\varepsilon, |x|\le\|K_1\|\|a_1\|\},\\ \overline{\omega}^T(f_3,\varepsilon) &= \sup\{|f_3(t_2,x)-f_2(t_1,x)|, t_1,t_2\in[0,T], |t_2-t_1|\le\varepsilon, |x|\le\|K_2\|\|a_2\|\}.\end{aligned}$$

We deduce, from (3.6), (3.9) and (3.10), that

$$\begin{aligned}\omega^T(Ax,\varepsilon) \le& \|\lambda_1\|\omega^T(x,\varepsilon)+\overline{\omega}^T(f_1,\varepsilon)\\ &+\left(\overline{f_2}+\|\lambda_2\|\|K_1\|^{p_2}\|a_1\|^{p_2}\right)\left(\overline{\omega}^T(f_3,\varepsilon)+\|\lambda_3\|(T\overline{\omega}^T(v,\varepsilon)+\varepsilon\overline{v})^{p_3}\right)\\ &+\left(\overline{f_3}+\|\lambda_3\|\|K_2\|^{p_3}\|a_2\|^{p_3}\right)\left(\overline{\omega}^T(f_2,\varepsilon)+\|\lambda_2\|(T\overline{\omega}^T(u,\varepsilon)+\varepsilon\overline{u})^{p_2}\right).\end{aligned}$$

Since $\lim\limits_{\varepsilon\longrightarrow0}\overline{\omega}^T(f_2,\varepsilon)=\lim\limits_{\varepsilon\longrightarrow0}\overline{\omega}^T(f_3,\varepsilon)=\lim\limits_{\varepsilon\longrightarrow0}\overline{\omega}^T(u,\varepsilon)=\lim\limits_{\varepsilon\longrightarrow0}\overline{\omega}^T(v,\varepsilon)=0$, then

$$\omega_0(AX)\le\|\lambda_1\|\omega_0(X). \tag{3.11}$$

We deduce, from (3.5) and (3.11), that

$$\mu(AX)\le\|\lambda_1\|\mu(X).$$

Hence the third step is completed by taking $\gamma=\|\lambda_1\|<1$.
Finally, applying Theorem 3.1, Equation (1.1) has at least one solution $x\in BC(\mathbb{R}^+)$.
Step 4. The solution x is asymptotically stable on $\mathbb{R}^+$.
Let $\varepsilon>0$, and taking $r=r_0$, then, for any other solution $y\in B_{r_0}(\mathbb{R}^+)$, we have from (3.4)

$$\begin{aligned}|x(t)-y(t)| \le& \|\lambda_1\|\|x-y\|+(\overline{f_2}+\|\lambda_2\|\|K_1\|^{p_2}\|a_1\|^{p_2})2^{p_3}\|\lambda_3\|\|a_2\|^{p_3}(K_21)(t)\\ &+(\overline{f_3}+\|\lambda_3\|\|K_2\|^{p_3}\|a_2\|^{p_3})2^{p_2}\|\lambda_2\|\|a_1\|^{p_2}(K_11)(t).\end{aligned}$$

Since $\|\lambda_1\|<1$, we obtain

$$\begin{aligned}|x(t)-y(t)| \le& \frac{(\overline{f_2}+\|\lambda_2\|\|K_1\|^{p_2}\|a_1\|^{p_2})2^{p_3}\|\lambda_3\|\|a_2\|^{p_3}}{1-\|\lambda_1\|}(K_21)(t)\\ &+\frac{(\overline{f_3}+\|\lambda_3\|\|K_2\|^{p_3}\|a_2\|^{p_3})2^{p_2}\|\lambda_2\|\|a_1\|^{p_2}}{1-\|\lambda_1\|}(K_11)(t).\end{aligned}$$

By using Assumption (ii), we deduce that there exists $T>0$ such that for all $t\ge T$

$$|x(t)-y(t)|\le\varepsilon.$$

Which implies that the solution is asymptotically stable on $\mathbb{R}^+$. □

4 Example

Consider the following integral equation

$$\begin{aligned} x(t) = t\exp(-t) + 1 + \frac{1}{2}x(t) + f_1\left(t, \int_0^t \frac{5}{(2+s+t+(x(s))^2)^2}\right) \times \\ f_2\left(t, \int_0^t e^{s-2t}\cos(t+x(s))ds\right), \end{aligned} \tag{4.1}$$

where $t \in \mathbb{R}^+$. Set

$$f_1(t,x) = t^2\exp(-t) + \frac{1}{3}x, f_2(t,x) = \frac{1}{2+t} + 5x, f_3(t,x) = 2\sin(t) + 3x, k_1(t,s) = \frac{5}{(2+s+t)^2}$$

and

$$k_2(t,s) = e^{s-2t}, a_1(s) = a_2 = 1, a_2(s) = \sin(t).$$

Using the notations of Theorem 3.4, we can easily show that

$$\lambda_1 = \frac{1}{3}, \lambda_2 = 5, \lambda_3 = 3.$$

Hence all the assumptions of Theorem 3.4 are satisfied.
Then by Theorem 3.4, we conclude that the integral equation (4.1) has an asymptotically stable solution $x \in BC(\mathbb{R}^+)$.

5 Conclusions

In this paper, we have considered a general form of the integral equations of product type and studied the existence of continuous solutions on the real half-line. Moreover, we have studied the asymptotic stability of the solutions. The existence of solutions has been investigated, under fairly simple conditions, by using techniques of measures of non-compactness and Darbo's fixed point theorem. Finally, an example is provided to illustrate our main result.

Conflict of interest

The authors have no conflicts of interest to disclose.

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