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Quintic and Septic C^2-spline methods for initial fractional differential equations

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Abstract. In this paper, we developed Quintic and Septic C^2-spline methods for solving initial fractional differential equations.
The convergence analysis of the methods is discussed. Illustrative examples are included to demonstrate the validity and applicability of the presented techniques. Our numerical results were compared with those in the recent literature. **Keywords:** Fractional differential equation; Quintic and Septic C^2-splines; Convergence analysis.
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1 Introduction

Fractional calculus has attracted significant interest of many researchers because it has recently gained popularity in the investigation of various areas of science, and engineering, such as nonlinear oscillation of earthquakes [9], fluid-dynamic traffic model [10], quantum and statistical mechanics [16], colored noise [17], solid mechanics [28], economics [3], dynamics of interfaces between nanoparticles and substrates [4].
The existence and uniqueness of solutions to the fractional differential equations have been investigated by the authors [14,24]. During the last decades, several methods have been used to solve fractional differential equations, fractional partial differential equations, fractional integro-differential equations and dynamic systems containing fractional derivatives, such as Adomian's decomposition method [7,18,19,34], variational iteration method [1,12,21,22], spectral methods [6,27,30], homotopy perturbation method [11,23,32], homotopy analysis method [8,13,37].

Consider the following fractional differential equation:

$$y^{''}(x) + D^{\alpha}y(x) = f(x,y), \qquad x \in [0,b]\,, \quad 0 < \alpha < 2, \tag{1.1}$$

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with the initial conditions

$$y(0) = y_0 \ , \ y'(0) = y'_0, \tag{1.2}$$

where $y(x)$ is an unknown function, and D^α is the Caputo fractional differentiation operator and y_0, y'_0 are constants. In [20], Nakhushev investigated the existence and uniqueness for the solutions of (1.1) by considering (1.2).

The Bagley-Torvik equation

$$Ay^{''}(x) + BD^{\frac{3}{2}}y(x) + Cy(x) = f(x) \ ,$$

is a special form of equation(1.1), that arises in the modeling of the motion of a rigid plate immersed in a Newtonian fluid [33].

In this paper we approximate (1.1) subjected to (1.2) by the C^2-spline methods.

The structure of this paper is as follows: In section 2, we introduce some necessary definitions and mathematical preliminaries of fractional calculus theory. In section 3, we use Quintic and Septic C^2-spline methods to solve equations (1.1) and (1.2). In section 4, the convergence of the methods is analyzed. In section 5, the proposed methods are applied to several examples. Comparisons with previously existing methods have been tested.

2 Basic definitions

In this section, basic definitions of fractional derivative and integral along with some properties have been presented. There are different definitions for fractional derivatives, the most commonly used ones are the Riemann-Liouvill and the Caputo derivatives [24].

Definition 2.1. The Riemann-Liouvill fractional derivative is defined by:

$${}^{R}D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_0^x (x-\tau)^{m-\alpha-1} f(\tau)\, d\tau, \qquad m-1 < \alpha < m, m \in \mathbb{N} \tag{2.1}$$

where $\Gamma(.)$ is the Gamma function with the property $\Gamma(x+1) = x\Gamma(x), x \in \mathbb{R}$.

Definition 2.2. The Caputo fractional derivative is defined by :

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\tau)^{m-\alpha-1} f^{(m)}(\tau)\, d\tau, \qquad m-1 < \alpha < m, m \in \mathbb{N} \tag{2.2}$$

Definition 2.3. The Riemann-Liouvill fractional integral is defined by :

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} f(\tau)\, d\tau, \qquad \alpha > 0 \tag{2.3}$$

Suppose that $0 < \alpha < 1$, and f is a continuous function, then

$$D^\alpha (I^\alpha f(t)) = f(t). \tag{2.4}$$

Some important properties of fractional derivative and fractional integral are listed in [35] which are as:

$$\begin{aligned} D^\alpha t^v &= \frac{\Gamma(1+v)}{\Gamma(1+v-\alpha)} t^{v-\alpha}, \\ D^\alpha(f(t).g(t)) &= g(t)D_t^\alpha f(t) + f(t)D^\alpha g(t), \\ D^\alpha f[g(t)] &= f_g^{'}[g(t)]D_t^\alpha g(t) = D_g^\alpha f[g(t)](g^{'}(t)), \\ I^\alpha(D_t^\alpha f(t)) &= f(t) - f(0), \\ D^\alpha(\lambda f(t) + \mu g(t)) &= \lambda D_t^\alpha f(t) + \mu D^\alpha g(t), \\ D^\alpha c &= 0, \end{aligned} \tag{2.5}$$

where λ, μ and c are constants.

3 Numerical approximation

According to our knowledge Quintic C^2-spline and Septic C^2- spline have been developed by Sallam et al. [31] and Rashidinia et.al [26] respectively to approximate the solution of regular initial value problems of second order. Here we apply these methods for solving of the fractional differential equation (1.1) subjected to initial conditions(1.2).

3.1 Quintic C^2- spline method

Following [31] for a given positive integer n the interval $[0, b]$ is partitioned into n equal subintervals $I_i = [x_{i-1}, x_i]$, $i = 1(1)n$ with the stepsize $h = \frac{b}{n}$. Let Π_5 denotes the collection of all polynomials of degree at most and:

$$S_{n,5}^{(2)} = \{s(x) \ : \ s \in C^2[0, b], \ s \in \Pi_5, \ for \ x \in I_i, \ i = 1(1)n\}.$$

We want to construct a piecewise polynomial $s \in S_{n,5}^{(2)}$ that satisfies 1.1 and 1.2 i.e,

$$s^{''}(x) = -D^\alpha s(x) + f(x, s(x)), \qquad s(0) = y_0, \ s^{'}(0) = y_0^{'}, \tag{3.1}$$

and more ever satisfies the following conditions

(1) $s^{''}(x_i) = -D^\alpha s(x_i) + f(x_i, s(x_i))$

(2) for $x \in [0, b]$, $s(x)$ and its derivatives up to order 2 must be continuous.

Now denoting $s^{''}(x)$ at nodal points $x_{i-1}, x_{i-\frac{2}{3}}, x_{i-\frac{1}{3}}$ and x_i such as , $s_{i-1}^{''}, s_{i-\frac{2}{3}}^{''}, s_{i-\frac{1}{3}}^{''}, s_i^{''}$, $i = 1(1)n$ and using initial conditions in (3.1), then the unique Quintic $s \in S_{n,5}^{(2)}$ in the interval $I_i = [x_{i-1}, x_i]$ defined by

$$s(x) = s_{i-1} + hs_{i-1}^{'}A(t) + h^2 s_{i-1}^{''}B(t) + h^2 s_{i-\frac{2}{3}}^{''}C(t) + h^2 s_{i-\frac{1}{3}}^{''}D(t) + h^2 s_i^{''}E(t), \tag{3.2}$$

where $t=\frac{x-x_{i-1}}{h}$ and A(t), B(t), C(t), D(t) and E(t) are the polynomials of degree at most 5. To determine these coefficients, we differentiate (3.2) twice, we have:

$$s'(x)=s'_{i-1}A'(t)+hs''_{i-1}B'(t)+hs''_{i-\frac{2}{3}}C'(t)+hs''_{i-\frac{1}{3}}D'(t)+hs''_iE'(t),$$
$$s''(x)=\frac{1}{h}s'_{i-1}A''(t)+s''_{i-1}B''(t)+s''_{i-\frac{2}{3}}C''(t)+s''_{i-\frac{1}{3}}D''(t)+s''_iE''(t).$$

At nodal points we have

$$\begin{aligned}
s(x_{i-1})&=s_{i-1}+hs'_{i-1}A(0)+h^2s''_{i-1}B(0)+h^2s''_{i-\frac{2}{3}}C(0)+h^2s''_{i-\frac{1}{3}}D(0)+h^2s''_iE(0),\\
s'(x_{i-1})&=s'_{i-1}A'(0)+hs''_{i-1}B'(0)+hs''_{i-\frac{2}{3}}C'(0)+hs''_{i-\frac{1}{3}}D'(0)+hs''_iE'(0),\\
s''(x_{i-1})&=\frac{1}{h}s'_{i-1}A''(0)+s''_{i-1}B''(0)+s''_{i-\frac{2}{3}}C''(0)+s''_{i-\frac{1}{3}}D''(0)+s''_iE''(0),\\
s''(x_{i-\frac{2}{3}})&=\frac{1}{h}s'_{i-1}A''(\frac{1}{3})+s''_{i-1}B''(\frac{1}{3})+s''_{i-\frac{2}{3}}C''(\frac{1}{3})+s''_{i-\frac{1}{3}}D''(\frac{1}{3})+s''_iE''(\frac{1}{3}),\\
s''(x_{i-\frac{1}{3}})&=\frac{1}{h}s'_{i-1}A''(\frac{2}{3})+s''_{i-1}B''(\frac{2}{3})+s''_{i-\frac{2}{3}}C''(\frac{2}{3})+s''_{i-\frac{1}{3}}D''(\frac{2}{3})+s''_iE''(\frac{2}{3}),\\
s''(x_i)&=\frac{1}{h}s'_{i-1}A''(1)+s''_{i-1}B''(1)+s''_{i-\frac{2}{3}}C''(1)+s''_{i-\frac{1}{3}}D''(1)+s''_iE''(1),
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&A(t)=t,\qquad B(t)=\frac{1}{2}t^2-\frac{11}{12}t^3+\frac{3}{4}t^4-\frac{9}{40}t^5,\qquad C(t)=\frac{3}{2}t^3-\frac{15}{8}t^4+\frac{27}{40}t^5,\\
&D(t)=-\frac{3}{4}t^3+\frac{3}{2}t^4-\frac{27}{40}t^5,\qquad E(t)=\frac{1}{6}t^3-\frac{3}{8}t^4+\frac{9}{40}t^5,
\end{aligned}\tag{3.3}$$

Now by using Definition (2.2) on equation (3.2), we obtain

$$D^\alpha s(x)=hD^\alpha(A(t))s'_{i-1}+h^2D^\alpha(B(t))s''_{i-1}+h^2D^\alpha(C(t))s''_{i-\frac{2}{3}}+h^2D^\alpha(D(t))s''_{i-\frac{1}{3}}+h^2D^\alpha(E(t))s''_i.\tag{3.4}$$

Setting $x=x_i$ in each subinterval, we have

$$\begin{aligned}
D^\alpha(A(t))|_{t=1}&=\frac{h^{-\alpha}}{\Gamma(2-\alpha)},\\
D^\alpha(B(t))|_{t=1}&=h^{-\alpha}\left(\frac{1}{\Gamma(3-\alpha)}-\frac{33}{\Gamma(4-\alpha)}-\frac{18}{\Gamma(5-\alpha)}-\frac{27}{\Gamma(6-\alpha)}\right),\\
D^\alpha(C(t))|_{t=1}&=h^{-\alpha}\left(\frac{9}{\Gamma(4-\alpha)}-\frac{45}{\Gamma(5-\alpha)}-\frac{81}{\Gamma(6-\alpha)}\right),\\
D^\alpha(D(t))|_{t=1}&=h^{-\alpha}\left(\frac{-18}{4\Gamma(4-\alpha)}-\frac{36}{\Gamma(5-\alpha)}-\frac{81}{\Gamma(6-\alpha)}\right),\\
D^\alpha(E(t))|_{t=1}&=h^{-\alpha}\left(\frac{1}{\Gamma(4-\alpha)}-\frac{9}{\Gamma(5-\alpha)}+\frac{27}{\Gamma(6-\alpha)}\right),
\end{aligned}\tag{3.5}$$

The C^2-spline $s(x)$ for $i = 1(1)n$ has been constructed to approximate the solution $y(x)$ of (1.1) as follows:

$$\begin{aligned}
s_{i-\frac{2}{3}} &= s_{i-1} + \frac{1}{3}hs'_{i-1} + \frac{97}{3240}h^2s''_{i-1} + \frac{19}{540}h^2s''_{i-\frac{2}{3}} - \frac{13}{1080}h^2s''_{i-\frac{1}{3}} + \frac{1}{405}h^2s''_i,\\
s_{i-\frac{1}{3}} &= s_{i-1} + \frac{2}{3}hs'_{i-1} + \frac{28}{405}h^2s''_{i-1} + \frac{22}{135}h^2s''_{i-\frac{2}{3}} - \frac{2}{135}h^2s''_{i-\frac{1}{3}} + \frac{2}{405}h^2s''_i,\\
s_i &= s_{i-1} + hs'_{i-1} + \frac{13}{120}h^2s''_{i-1} + \frac{3}{10}h^2s''_{i-\frac{2}{3}} + \frac{3}{40}h^2s''_{i-\frac{1}{3}} + \frac{1}{60}h^2s''_i,\\
s'_i &= s'_{i-1} + \frac{1}{8}hs''_{i-1} + \frac{3}{8}hs''_{i-\frac{2}{3}} + \frac{3}{8}hs''_{i-\frac{1}{3}} + \frac{1}{8}hs''_i,
\end{aligned} \tag{3.6}$$

where $s''_a = -D^\alpha_x s(x_a) + f(x_a, s_a)$, $a = i-1, i-\frac{2}{3}, i-\frac{1}{3}, i$ with $s_0 = y_0, s'_0 = y'_0$ and finally by solving above system we can obtain $s_{i-\frac{2}{3}}, s_{i-\frac{1}{3}}, s_i$.

3.2 Septic C^2- spline method

Consider equation (1.1) subjected to the initial conditions (1.2). Following [26] for a given positive integer n the interval $[0, b]$ is partitioned into n equal subintervals $I_i = [x_{i-1}, x_i]$, $i = 1(1)n$ with the stepsize $h = \frac{b}{n}$. Let Π_7 denotes the collection of all polynomials of degree at most 7 and

$$S^{(2)}_{n,7} = \{s(x) \ : \ s \in C^2[0, b], \ s \in \Pi_7, \ for x \in I_i, \ \ i = 1(1)n\}.$$

We want to construct a piecewise polynomial $s \in S^{(2)}_{n,7}$ satisfies (1.1) and (1.2) i.e,

$$s''(x) = -D^\alpha_x s(x) + f(x, s(x)), \qquad s(0) = y_0, \ s'(0) = y'_0, \tag{3.7}$$

and more ever satisfies the following conditions:

(1) $s''(x_i) = -D^\alpha_x s(x_i) + f(x_i, s(x_i))$

(2) for $x \in [0, b]$, s(x) and its derivatives up to order 2 must be continuous.

Now denoting $s''(x)$ at nodal points $x_{i-\frac{1}{5}}, x_{i-\frac{2}{5}}, x_{i-\frac{3}{5}}, x_{i-\frac{4}{5}}, x_{i-1}$ and x_i such as $s''_i, s''_{i-\frac{1}{5}}, s''_{i-\frac{2}{5}}, s''_{i-\frac{3}{5}}, s''_{i-\frac{4}{5}}, s''_{i-1}$, $i = 1(1)n$ and s_0, s'_0, then the unique Septic spline $s \in S^{(2)}_{n,7}$ in the interval I_i defined by

$$\begin{aligned}
s(x) = {} & s_{i-1} + hs'_{i-1}A(t) + h^2s''_{i-1}B(t) + h^2s''_{i-\frac{4}{5}}C(t) + h^2s''_{i-\frac{3}{5}}D(t) + h^2s''_{i-\frac{2}{5}}E(t)\\
& + h^2s''_{i-\frac{1}{5}}F(t) + h^2s''_iG(t),
\end{aligned} \tag{3.8}$$

where $t = \frac{x - x_{i-1}}{h}$ and A(t), B(t), C(t), D(t), E(t), F(t) and G(t) are the polynomials of degree at most 7. In the similar manner we did for Quintic C^2-spline, we determine $A(t)$, $B(t)$, $C(t)$, $F(t)$, $G(t)$, $E(t)$

in (3.9) as follows:

$$
\begin{aligned}
A(t) &= t, \\
B(t) &= \frac{-625}{1008}t^7 + \frac{125}{48}t^6 - \frac{425}{96}t^5 + \frac{125}{32}t^4 - \frac{137}{72}t^3 + \frac{1}{2}t^2, \\
C(t) &= \frac{3125}{1008}t^7 - \frac{875}{72}t^6 + \frac{1755}{96}t^5 - \frac{1925}{144}t^4 + \frac{25}{6}t^3, \\
D(t) &= \frac{-3125}{504}t^7 + \frac{1625}{72}t^6 - \frac{1475}{48}t^5 + \frac{2675}{144}t^4 - \frac{25}{6}t^3, \\
E(t) &= \frac{3125}{504}t^7 - \frac{1625}{72}t^6 + \frac{1225}{48}t^5 - \frac{325}{24}t^4 + \frac{25}{9}t^3, \\
F(t) &= \frac{-3125}{1008}t^7 + \frac{1375}{144}t^6 - \frac{1025}{96}t^5 + \frac{1525}{288}t^4 - \frac{25}{24}t^3, \\
G(t) &= \frac{625}{1008}t^7 - \frac{125}{72}t^6 + \frac{175}{96}t^5 - \frac{125}{144}t^4 + \frac{1}{3}t^3.
\end{aligned}
\tag{3.9}
$$

Now by using Definition (2.2) on equation (3.8), so we obtain

$$
\begin{aligned}
D^\alpha s(x) = {} & hs'_{i-1}D^\alpha A(t) + h^2 s''_{i-1}D^\alpha B(t) + h^2 s''_{i-\frac{4}{5}}D^\alpha C(t) + h^2 s''_{i-\frac{3}{5}}D^\alpha D(t) + h^2 s''_{i-\frac{2}{5}}D^\alpha E(t) \\
& + h^2 s''_{i-\frac{1}{5}}D^\alpha F(t) + h^2 s''_{i}D^\alpha G(t).
\end{aligned}
\tag{3.10}
$$

Setting $x = x_i$ in each subinterval, so we have

$$
\begin{aligned}
D^\alpha(A(t))|_{t=1} &= \frac{h^{-\alpha}}{\Gamma(2-\alpha)}, \\
D^\alpha(B(t))|_{t=1} &= h^{-\alpha}\left(\frac{-3125}{\Gamma(8-\alpha)} + \frac{1875}{\Gamma(7-\alpha)} - \frac{2125}{\Gamma(6-\alpha)} + \frac{375}{4\Gamma(5-\alpha)}\right), \\
D^\alpha(C(t))|_{t=1} &= h^{-\alpha}\left(\frac{15625}{\Gamma(8-\alpha)} - \frac{8750}{\Gamma(7-\alpha)} + \frac{8875}{4\Gamma(6-\alpha)} - \frac{1975}{6\Gamma(5-\alpha)} + \frac{25}{\Gamma(4-\alpha)}\right), \\
D^\alpha(D(t))|_{t=1} &= h^{-\alpha}\left(\frac{-31250}{\Gamma(8-\alpha)} - \frac{16250}{\Gamma(7-\alpha)} + \frac{7375}{2\Gamma(6-\alpha)} + \frac{2675}{6\Gamma(5-\alpha)} - \frac{25}{\Gamma(4-\alpha)}\right), \\
D^\alpha(E(t))|_{t=1} &= h^{-\alpha}\left(\frac{-31250}{\Gamma(8-\alpha)} - \frac{1500}{\Gamma(7-\alpha)} + \frac{6125}{2\Gamma(6-\alpha)} - \frac{325}{\Gamma(5-\alpha)} - \frac{50}{3\Gamma(4-\alpha)}\right), \\
D^\alpha(E(t))|_{t=1} &= h^{-\alpha}\left(\frac{-31250}{\Gamma(8-\alpha)} - \frac{1500}{\Gamma(7-\alpha)} + \frac{6125}{2\Gamma(6-\alpha)} - \frac{325}{\Gamma(5-\alpha)} - \frac{50}{3\Gamma(4-\alpha)}\right).
\end{aligned}
\tag{3.11}
$$

Finally the C^2-spline $s(x)$ *for* $i = 1(1)n$ has been constructed to approximate solution $y(x)$

of (1.1) as follows:

$$
\begin{aligned}
s_{i-\frac{4}{5}} &= s_{i-1} + \frac{1}{5}hs'_{i-1} + \frac{1}{125}\Big(\frac{1231}{1008}h^2s''_{i-1} + \frac{4315}{2016}h^2s''_{i-\frac{4}{5}} - \frac{761}{504}h^2s''_{i-\frac{3}{5}} + \frac{941}{1008}h^2s''_{i-\frac{2}{5}}\\
&-\frac{341}{1008}h^2s''_{i-\frac{1}{5}} + \frac{107}{2016}h^2s''_{i}\Big),\\
s_{i-\frac{3}{5}} &= s_{i-1} + \frac{2}{5}hs'_{i-1} + \frac{1}{125}\Big(\frac{355}{126}h^2s''_{i-1} + \frac{544}{63}h^2s''_{i-\frac{4}{5}} - \frac{185}{63}h^2s''_{i-\frac{3}{5}} + \frac{136}{63}h^2s''_{i-\frac{2}{5}}\\
&-\frac{101}{126}h^2s''_{i-\frac{1}{5}} + \frac{8}{63}h^2s''_{i}\Big),\\
s_{i-\frac{2}{5}} &= s_{i-1} + \frac{3}{5}hs'_{i-1} + \frac{1}{125}\Big(\frac{4428}{1008}h^2s''_{i-1} + \frac{31509}{2016}h^2s''_{i-\frac{4}{5}} - \frac{9}{8}h^2s''_{i-\frac{3}{5}} + \frac{435}{112}h^2s''_{i-\frac{2}{5}}\\
&-\frac{9}{7}h^2s''_{i-\frac{1}{5}} + \frac{45}{224}h^2s''_{i}\Big),\\
s_{i-\frac{1}{5}} &= s_{i-1} + \frac{4}{5}hs'_{i-1} + \frac{1}{125}\Big(\frac{376}{63}h^2s''_{i-1} + \frac{1424}{63}h^2s''_{i-\frac{4}{5}} - \frac{176}{63}h^2s''_{i-\frac{3}{5}} + \frac{608}{63}h^2s''_{i-\frac{2}{5}}\\
&-\frac{80}{63}h^2s''_{i-\frac{1}{5}} + \frac{16}{63}h^2s''_{i}\Big),\\
s_i &= s_{i-1} + hs'_{i-1} + \frac{61}{1008}h^2s''_{i-1} + \frac{475}{2016}h^2s''_{i-\frac{4}{5}} + \frac{25}{504}h^2s''_{i-\frac{3}{5}} + \frac{125}{1008}h^2s''_{i\frac{2}{5}} + \frac{25}{1008}h^2s''_{i-\frac{1}{5}}\\
&+\frac{11}{2016}h^2s''_{i},\\
s'_i &= s'_{i-1} + \frac{19}{288}hs''_{i-1} + \frac{25}{96}hs''_{i-\frac{4}{5}} + \frac{25}{144}hs''_{i-\frac{3}{5}} + \frac{25}{144}hs''_{i-\frac{2}{5}} + \frac{25}{96}hs''_{i-\frac{1}{5}} + \frac{19}{288}hs''_{i},
\end{aligned}
\tag{3.12}
$$

where $s''_a = -D_x^{\alpha}\, s(x_a) + f(x_a, s_a),\ a = i-1, i-\frac{4}{5}, i-\frac{3}{5}, i-\frac{2}{5}, i-\frac{1}{5}, i$ with $s_0 = y_0, s'_0 = y'_0$ and coefficients $s_{i-\frac{4}{5}}, s_{i-\frac{3}{5}}, s_{i-\frac{2}{5}}, s_{i-\frac{1}{5}}, s_i$ can be determined by solving system (3.12).

4 Convergence analysis

In this section, without loss of generality we will consider problem (1.1) with homogenous conditions.

Considering

$$y''(x) = z(x), \tag{4.1}$$

with the initial conditions

$$y(0) = 0, y'(0) = 0, \tag{4.2}$$

has a unique solution, then there is a Green's function $G(x,s)$ for the problem, where

$$y(x) = \int_0^x G(x,s)z(s)\, ds = G_z(x), \tag{4.3}$$

and

$$G(x,s) = (x - s). \tag{4.4}$$

Since the operator $Gz(x)$ satisfies the following conditions :

(1) $lim_{h\to 0}(max_{t,s\in[0,b]} \max_{|t-s|\le h} \int_0^b |G(t,x) - G(s,x)|\, dx) = 0.$

(2) $\max_{t\in[0,b]} \int_0^b |G(t,s)|\, ds < \infty.$

Therefore $Gz(x)$ is a compact and bounded operator [2].
Now we will prove the following theorem.

Theorem 4.1. *Let $y(x)$ satisfies (4.3), then*

$$D^{\alpha}y(x) = D^{\alpha} \int_0^x G(x,s)\, z(s)\, ds = \int_0^x (D^{\alpha}\, G(x,s))\, z(s)\, ds = D^{\alpha}\, Gz(x). \tag{4.5}$$

Proof. From the Caputo fractional derivative $D^{\alpha}y(x)$, we get

$$D^{\alpha}y(x) = D^{\alpha} \int_{s=0}^{s=x} G(x,s)\, z(s)\, ds = \frac{1}{\Gamma(m-\alpha)} \int_{t=0}^{t=x} (x-t)^{m-\alpha-1} \Big(\frac{d^m}{dt^m} \Big[\int_{s=0}^{s=t} G(t,s)\, z(s)\, ds \Big] \Big)\, dt, \tag{4.6}$$

where

$$\frac{d^m}{dt^m} \Big[\int_{s=0}^{s=t} G(t,s)\, z(s)\, ds \Big] = \int_{s=0}^{s=t} \frac{\partial^m}{\partial t^m} G(t,s)\, z(s)\, ds, \tag{4.7}$$

so we have:

$$D^{\alpha}y(x) = \frac{1}{\Gamma(m-\alpha)} \int_{t=0}^{t=x} (x-t)^{m-\alpha-1} \Big[\int_{s=0}^{s=x} \frac{\partial^m}{\partial t^m} G(t,s)\, z(s)\, ds \Big]\, dt. \tag{4.8}$$

By changing the order of integration we have:

$$D^{\alpha}y(x) = \int_{s=0}^{s=x} \Big[\frac{1}{\Gamma(m-\alpha)} \int_{t=a}^{t=x} (x-t)^{m-\alpha-1} \frac{\partial^m}{\partial t^m} G(t,s)\, dt \Big] z(s)\, ds.$$
$$\int_{s=0}^{s=x} (D^{\alpha} G(x,s))\, z(s)\, ds = D^{\alpha}\, Gz(x). \tag{4.9}$$

So that the proof is complete. □

Theorem 4.2. *Assuming that $s(x) \in S^{(2)}_{n,i}$, $i = 5,7$ be the solution of (3.1) and $y(x)$ be the solution of (1.1)-(1.2). If $n \ge N_0$, then for constants c_k and c_0 independent of h, we have:*

$$\begin{aligned} &\|y - s(x)\| \le c_k \|y^{(k+2)}\| h^k, \qquad for\ \ y \in C^{k+2}[0,b], \qquad 1 \le k \le 2, \\ &\|y - s(x)\| \le c_0 \psi(y'', h), \qquad for\ \ y \in C^2[0,b], \end{aligned} \tag{4.10}$$

where

$$\psi(\phi, h) = sup\{|\phi(x+h) - \phi(x)| : x, x+h \in [0,b]\}. \tag{4.11}$$

Proof. Following [15] by using equation (4.1) and theorem 4.1, equation (1.1) can be written in the following form

$$z(x) + \int_0^x D^\alpha\, G(x,s)\, z(s)\, ds = f(x, Gz(x)), \tag{4.12}$$

where the operator $D^\alpha G$ is compact. Therefore, the solution of equations (1.1)-(1.2) is equivalent to the solution of equation (4.12).

Equation (4.12) can be written in operator form as:

$$(I + D^\alpha G)z = f. \tag{4.13}$$

Since $s(x) \in S^{(2)}_{n,i}$, $i = 5, 7$, therefore, $s \in C^2[0,b]$ and so $s''(x) \in C[0,b]$.
Setting

$$s''(x) = z_n(x), \tag{4.14}$$

so $z_n(x)$ is a continuous piecewise polynomial that satisfies homogeneous initial conditions.
Now define a linear projection Pc which maps each continuous function into

$$S_j = \{s(x) \;:\; s \in C^2[0,b],\ s \in \Pi_j\},\ j = 3, 5,$$

where S_j is a spline function of degree j, and by following [25] for continuous function z, $\lim\limits_{h\to 0} \|P_c z - z\|_\infty \to 0$ and this implies that $\lim\limits_{h\to 0} \|P_c D^\alpha G - D^\alpha G\|_\infty \to 0$.

By using theorem (4.1), we obtain

$$s''(x) = -D^\alpha G s(x) + f(x, s(x)). \tag{4.15}$$

Substitute (4.14) in (4.15) and operating P_c on both sides of (4.15) and since $P_c z_n = z_n$, then after simplification we obtain:

$$z_n + P_c D^\alpha G z_n = P_c f. \tag{4.16}$$

Operating the linear projection operator P_c on both sides of (4.13) we have

$$P_c z + P_c D^\alpha G z = P_c f. \tag{4.17}$$

By using (4.16) and (4.17), we easily obtain that

$$(I + P_c D^\alpha G)(z - z_n) = z - P_c z. \tag{4.18}$$

Following [29], $(I + P_c D^\alpha G)^{-1}$ exists and it is bounded. then we have

$$z - z_n = (I + P_c D^\alpha G)^{-1}(z - P_c z). \tag{4.19}$$

By operating G on both sides of (4.19) and using (4.1) and (4.3), we obtain

$$y - s(x) = G(I + P_c D^\alpha G)^{-1}(y'' - P_c y''). \tag{4.20}$$

Since operator G is bounded we have

$$\|y - s(x)\| \le \|G\| \|(I + P_c D^\alpha G)^{-1}\| \|y'' - P_c y''\|. \tag{4.21}$$

According to the theory of interpolation [25], we have

$$\begin{aligned} &\|y'' - P_c y''\| \le \eta_k \|y^{(k+2)}\| h^k, && for\;\; y \in C^{k+2}[0,b],\ 1 \le k \le 2, \\ &\|y'' - P_c y''\| \le \eta_0 \psi(y'', h), && for\;\; y \in C^2[0,b]. \end{aligned} \tag{4.22}$$

Following [29], $\|(I + P_c D^\alpha G)^{-1}\| \le \delta$, for $n \ge N_0$. Finally $c_0 = \delta\eta_0\|G\|$ and $c_k = \delta\eta_k\|G\|$, $k = 1, 2$. Then the proof is complete. □

5 Numerical results

In this section, we test our presented methods to solve the following examples. Numerical computations reported here have been carried out in a Mathematica environment. We verify that our approaches are efficient and applicable to fractional differential equations (1.1). The computed errors in the solutions are given:

$$RMS = \left[\sum_{j=0}^{n} \frac{e_n^2(x_j)}{n}\right]^{\frac{1}{2}},$$

where.$e_n = y(x_n) - s(x_n)$.
We compare our results with the results given in [5], [15] and [36].

Example 5.1. Consider the Bagley-Torvik equation

$$y^{''}(x) + BD^{\frac{3}{2}}y(x) + Cy(x) = f(x)\ ,$$

with

$$y(0) = 1,\ y^{'}(0) = 1.$$

In order to make a comparison with the numerical solution in [5] we have solved this problem on interval [0, b]. The numerical results at $x = 5$, are listed in Table 5.1. the exact solution to this problem is $y(x) = x + 1$.

Table 5.1: Absolute errors.

h	Quintic spline method	Septic spline method	[5]
$\frac{1}{2}$	$5.9\ E-2$	$4.24\ E-2$	$1.51\ E-1$
$\frac{1}{4}$	$1.9\ E-2$	$1.62\ E-2$	$4.68\ E-2$
$\frac{1}{8}$	$6.66\ E-3$	$6.071\ E-3$	$1.602\ E-2$
$\frac{1}{16}$	$2.25\ E-3$	$2.23\ E-3$	$5.62\ E-3$

Example 5.2. Consider the following fractional differential equation

$$y^{''}(x) = -D^{\alpha}y(x) + 30x^4 - 56x^6 + \frac{1024}{231\sqrt{\pi}}x^{5.5} - \frac{32768}{6435\sqrt{\pi}}x^{7.5},$$

with

$$y(0) = 0,\ y^{'}(0) = 0.$$

The exact solution to this problem is $y(x) = x^6 - x^8$. RMS errors with solutions are presented in Tables 5.2,5.3, 5.4.

Example 5.3. Consider the following fractional differential equation

$$y^{''}(x) = -D^{0.5}y(x) + \frac{256}{64\sqrt{\pi}}x^{4.5} - \frac{128}{35\sqrt{\pi}}x^{3.5} + 20x^3 - 12x^2,$$

with

$$y(0) = 0,\ y^{'}(0) = 0.$$

the exact solution of this problem is $y(x) = x^5 - x^4$. RMS errors with solutions is presented in Table 5.5.

Table 5.2: RMS errors for $\alpha = 0$

h	Quintic spline method	Septic spline method
$\frac{1}{8}$	$0.488524\ E-4$	$7.44652\ E-8$
$\frac{1}{16}$	$0.358964\ E-4$	$1.01118\ E-7$
$\frac{1}{32}$	$0.383262\ E-4$	$1.24191\ E-7$
$\frac{1}{64}$	$0.414685\ E-4$	$1.39324\ E-7$

Table 5.3: RMS errors for $\alpha = 0.2$

h	Quintic spline method	Septic spline method
$\frac{1}{8}$	$0.119991\ E-3$	$1.34581\ E-7$
$\frac{1}{16}$	$0.675742\ E-4$	$1.85004\ E-7$
$\frac{1}{32}$	$0.691067\ E-4$	$2.38609\ E-7$
$\frac{1}{64}$	$0.768007\ E-4$	$2.80765\ E-7$

Table 5.4: RMS errors for $\alpha = 0.4$

h	Quintic spline method	Septic spline method
$\frac{1}{8}$	$0.335703\ E-3$	$2.24091\ E-7$
$\frac{1}{16}$	$0.124821\ E-3$	$3.57712\ E-7$
$\frac{1}{32}$	$0.111639\ E-3$	$4.05279\ E-7$
$\frac{1}{64}$	$0.126435\ E-3$	$5.05839\ E-7$

Table 5.5: RMS errors.

h	Quintic spline method	Septic spline method	[15]
$\frac{1}{8}$	$0.44981\ E-3$	$0.6988\ E-2$	$6.9291\ E-3$
$\frac{1}{16}$	$0.27243\ E-3$	$1.7222\ E-4$	$1.7368\ E-3$
$\frac{1}{32}$	$0.28483\ E-3$	$2.2623\ E-4$	$4.3646\ E-4$
$\frac{1}{64}$	$0.31768\ E-3$	$2.7225\ E-4$	$1.0914\ E-4$

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Conflict of interest

The authors have no conflicts of interest to declare.

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