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The extended Exton's quadruple hypergeometric function $K_{15,p}^{(\alpha,\beta)}$ and its properties

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Abstract. In this paper, we introduce the extended Exton's hypergeometric function $K_{15,p}^{(\alpha,\beta)}$ using the extended beta function given by Özergin et al. [10]. For this extended function, we derive various properties, including integral representations, recurrence relations, generating functions, transformation formulas, and summation formulas. Some special cases of the main results are also considered.

Keywords: Extended beta function; Extended Exton's function; Integral representations; Recurrence relations; Generating functions; Transformation formulas; Summation formulas.

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1 Introduction

Special functions play a significant role in many branches of mathematics, physics, and engineering. These functions often arise in solving problems involving differential equations and in representing various physical phenomena through integral expressions in diverse areas such as quantum mechanics and quantum chromodynamics. The generalization of classical special functions has attracted the interest of many researchers. In several recent works, the extended beta function and its generalizations (see [1, 6, 7, 10]) have been used to introduce new extensions of special functions such as Gauss's, Appell's, Lauricella's, Srivastava's, and Exton's hypergeometric functions (see, for example, [2, 4, 5, 12, 15]).

Exton [8] defined the following quadruple hypergeometric function K_{15} as:

$$\begin{aligned} &K_{15}(a,a,a,b_5;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u) \\ &\quad = \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p}(b_5)_q(b_1)_m(b_2)_n(b_3)_p(b_4)_q\ x^m y^n z^p u^q}{(c)_{m+n+p+q}\ m!n!p!q!}, \end{aligned} \tag{1.1}$$

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and obtained the following integral representation:

$$\begin{aligned}&K_{15}(a,a,a,b_5;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u)\\&\quad=\frac{1}{B(a,c-a)B(b_5,c-a-b_5)}\int_0^1\int_0^1 t^{a-1}s^{b_5-1}(1-t)^{c-a-1}(1-s)^{c-a-b_5-1}\\&\quad\times(1-xt)^{-b_1}(1-yt)^{-b_2}(1-zt)^{-b_3}(1-us(1-t))^{-b_4}\,dt\,ds,\end{aligned}\tag{1.2}$$

$$\text{where }\max\{|x|,|y|,|z|,|u|\}<1;\quad \Re(a)>0,\quad \Re(b_5)>0,\quad \Re(c-a-b_5)>0,$$

and $(\lambda)_n$ denotes the Pochhammer symbol, defined by

$$(\lambda)_n=\begin{cases}1, & \text{if } n=0,\\ \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1), & \text{if } n\in\mathbb{N}.\end{cases}\tag{1.3}$$

Özergin et al. [10] defined the extended beta function as follows:

$$B_p^{(\alpha,\beta)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt,\tag{1.4}$$

$$\text{where } \Re(\alpha)>0,\ \Re(\beta)>0,\ \Re(p)\geq 0,\ \Re(x)>0,\ \Re(y)>0.$$

Clearly, we have

$$B_0^{(\alpha,\beta)}(x,y)=B(x,y),$$

where $B(x,y)$ is the well-known Euler beta function, defined as (see [14]):

$$B(x,y)=\begin{cases}\displaystyle\int_0^1 t^{x-1}(1-t)^{y-1}dt, & \text{if } \Re(x)>0,\ \Re(y)>0,\\ \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, & \text{if } x,y\in\mathbb{C}\setminus\mathbb{Z}_0^-.\end{cases}\tag{1.5}$$

Using the extended beta function (1.4), Özergin et al. [10] introduced the extended Gauss hypergeometric function as:

$$F_p^{(\alpha,\beta)}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!},\tag{1.6}$$

$$\text{where } \Re(c)>\Re(b)>0,\quad |z|<1,$$

and presented the following integral representation:

$$\begin{aligned}F_p^{(\alpha,\beta)}(a,b;c;z)&=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\\&\quad\times{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt,\end{aligned}\tag{1.7}$$

$$\text{where } \Re(p)>0,\ p=0,\ \text{and } |\arg(1-z)|<\pi;\ \Re(c)>\Re(b)>0.$$

Clearly, we have

$$F_0^{(\alpha,\beta)}(a,b;c;z)={}_2F_1(a,b;c;z),$$

where ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function, defined by (see [14]):

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},\quad c\neq 0,-1,-2,\ldots.\tag{1.8}$$

In terms of the extended beta function given in (1.4), Srivastava et al. [15] defined the extended Lauricella function of r variables, $F_{D,p}^{(r,\alpha,\beta)}$, as follows:

$$F_{D,p}^{(r,\alpha,\beta)}(a, b_1, \ldots, b_r; c; x_1, \ldots, x_r) = \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(a + m_1 + \cdots + m_r, c - a)(b_1)_{m_1} \ldots (b_r)_{m_r}\, x_1^{m_1} \ldots x_r^{m_r}}{B(a, c-a)\; m_1! \ldots m_r!}, \tag{1.9}$$

$$\text{where } \max\{|x_1|, \ldots, |x_r|\} < 1, \quad \Re(p) \geq 0,$$

and obtained the following integral representation:

$$F_{D,p}^{(r,\alpha,\beta)}(a, b_1, \ldots, b_r; c; x_1, \ldots, x_r) = \frac{1}{B(a, c-a)} \times \int_0^1 t^{a-1}(1-t)^{c-a-1}(1 - x_1 t)^{-b_1} \ldots (1 - x_r t)^{-b_r} \times {}_1F_1\left(\alpha; \beta; -\frac{p}{t(1-t)}\right) dt. \tag{1.10}$$

Clearly, we have

$$F_{D,0}^{(r,\alpha,\beta)}(a, b_1, \ldots, b_r; c; x_1, \ldots, x_r) = F_D^{(r)}(a, b_1, \ldots, b_r; c; x_1, \ldots, x_r),$$

where $F_D^{(r)}$ is the Lauricella hypergeometric function [14], defined by:

$$F_D^{(r)}(a, b_1, \ldots, b_r; c; x_1, \ldots, x_r) = \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(a)_{m_1+\cdots+m_r}(b_1)_{m_1} \ldots (b_r)_{m_r}}{(c)_{m_1+\cdots+m_r}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_r^{m_r}}{m_r!}. \tag{1.11}$$

2 Extended Exton's quadruple hypergeometric function $K_{15,p}^{(\alpha,\beta)}$

Here, we use the extended beta function (1.4) to define the extended Exton's quadruple hypergeometric function $K_{15,p}^{(\alpha,\beta)}(a, a, a, b_5; b_1, b_2, b_3, b_4; c, c, c, c; x, y, z, u)$ as follows:

$$K_{15,p}^{(\alpha,\beta)}(a, a, a, b_5; b_1, b_2, b_3, b_4; c, c, c, c; x, y, z, u) = \sum_{m,n,r,s=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(a + m + n + r, c - a + s)(b_5)_s (b_1)_m (b_2)_n (b_3)_r (b_4)_s\, x^m y^n z^r u^s}{B(a, c-a)(c-a)_s\; m!n!r!s!}. \tag{2.1}$$

Remark 2.1. The extended Exton's hypergeometric function $K_{15,p}^{(\alpha,\beta)}(a, a, a, b_5; b_1, b_2, b_3, b_4; c, c, c, c; x, y, z, u)$, given in (2.1), can be written as follows:

$$K_{15,p}^{(\alpha,\beta)}(a, a, a, b_5; b_1, b_2, b_3, b_4; c, c, c, c; x, y, z, u) = \sum_{s=0}^{\infty} \frac{(b_4)_s (b_5)_s}{(c)_s} F_{D,p}^{(3,\alpha,\beta)}(a, b_1, b_2, b_3; c + s; x, y, z) \frac{u^s}{s!}, \tag{2.2}$$

where $F_{D,p}^{(3,\alpha,\beta)}$ is the extended Lauricella function of three variables.

Remark 2.2. The special case $b_5 = c - a$ in (2.1) yields the following extended Exton's hypergeometric function $K_{15,p}^{(\alpha,\beta)}$:

$$\begin{aligned} &K_{15,p}^{(\alpha,\beta)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u) \\ &= \sum_{m,n,r,s=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(a+m+n+r,c-a+s)(b_1)_m(b_2)_n(b_3)_r(b_4)_s\, x^m y^n z^r u^s}{B(a,c-a)\; m!n!r!s!}. \end{aligned} \tag{2.3}$$

Remark 2.3. The special case $p = 0$ in (2.1) yields Exton's original hypergeometric function K_{15} given in (1.1):

$$\begin{aligned} &K_{15,0}^{(\alpha,\beta)}(a,a,a,b_5;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u) \\ &\quad = K_{15}(a,a,a,b_5;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u). \end{aligned} \tag{2.4}$$

3 Integral representations for the extended Exton's hypergeometric function $K_{15,p}^{(\alpha,\beta)}$

Theorem 3.1. *For the extended Exton's hypergeometric function $K_{15,p}^{(\alpha,\beta)}$, the following integral representations hold:*

$$\begin{aligned} &K_{15,p}^{(\alpha,\beta)}(a,a,a,b_5;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u) = \frac{1}{B(a,c-a)} \\ &\quad \times \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b_1}(1-yt)^{-b_2} \\ &\quad \times (1-zt)^{-b_3}\, {}_2F_1\left(b_4,b_5;c-a;u(1-t)\right)\, {}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right) dt, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &K_{15,p}^{(\alpha,\beta)}(a,a,a,b_5;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u) \\ &= \frac{1}{B(a,c-a)B(b_5,c-a-b_5)} \int_0^1\int_0^1 t^{a-1}s^{b_5-1}(1-t)^{c-a-1}(1-s)^{c-a-b_5-1} \\ &\times (1-xt)^{-b_1}(1-yt)^{-b_2}(1-zt)^{-b_3}(1-us(1-t))^{-b_4} \\ &\times {}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right) dtds, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &K_{15,p}^{(\alpha,\beta)}(a,a,a,b_5;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u) \\ &= \frac{1}{B(a,c-a)B(b_5,c-a-b_5)} \int_0^1\int_0^1 t^{a-1}s^{b_5-1}(1-t)^{c-a-1}(1-s)^{c-a-b_5-1} \\ &\times (1-xt)^{-b_1}(1-yt)^{-b_2}(1-zt)^{-b_3}(1-us)^{-b_4}\left(1-\frac{us}{us-1}\right)^{-b_4} \\ &\times {}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right) dtds, \end{aligned} \tag{3.3}$$

$$K_{15,p}^{(\alpha,\beta)}(a,a,a,b_5;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u) = \frac{2}{B(a,c-a)} \times \int_0^{\frac{\pi}{2}} \sin^{2a-1}\theta \cos^{2c-2a-1}\theta (1-x\sin^2\theta)^{-b_1}(1-y\sin^2\theta)^{-b_2} \times (1-z\sin^2\theta)^{-b_3} {}_2F_1\left(b_4,b_5;c-a;u\cos^2\theta\right) {}_1F_1\left(\alpha;\beta;-\frac{p}{\sin^2\theta\cos^2\theta}\right) d\theta, \tag{3.4}$$

$$K_{15,p}^{(\alpha,\beta)}(a,a,a,b_5;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u) = \frac{1}{B(a,c-a)} \times \int_0^{\infty} \xi^{a-1}(1+\xi)^{b_1+b_2+b_3-c}(1+(1-x)\xi)^{-b_1}(1+(1-y)\xi)^{-b_2} \times (1+(1-z)\xi)^{-b_3} {}_2F_1\left(b_4,b_5;c-a;\frac{u}{1+\xi}\right) {}_1F_1\left(\alpha;\beta;-\frac{p(1+\xi)^2}{\xi}\right) d\xi. \tag{3.5}$$

Proof. **of (3.1).** Using (1.4) on the right-hand side of (2.1) and interchanging the order of summation and integration, we have:

$$K_{15,p}^{(\alpha,\beta)}(a,a,a,b_5;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u) = \frac{1}{B(a,c-a)} \times \int_0^1 t^{a-1}(1-t)^{c-a-1}\left(\sum_{m=0}^{\infty}\frac{(b_1)_m(xt)^m}{m!}\right)\left(\sum_{n=0}^{\infty}\frac{(b_2)_n(yt)^n}{n!}\right) \times \left(\sum_{r=0}^{\infty}\frac{(b_3)_r(zt)^r}{r!}\right)\left(\sum_{s=0}^{\infty}\frac{(b_5)_s(b_4)_s(u(1-t))^s}{(c-a)_s s!}\right) {}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right) dt,$$

which, by using the identity [14]

$$\sum_{n=0}^{\infty}(a)_n\frac{x^n}{n!} = (1-x)^{-a}, \tag{3.6}$$

yields the desired result (3.1). The integral representations (3.2) and (3.3) follow from (3.1) and (3.2), respectively, by using the following identities [14]:

$${}_2F_1(a,b;c;x) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt, \tag{3.7}$$

and

$$(1-z(1-t))^{-a} = (1-z)^{-a}\left(1-\frac{zt}{z-1}\right)^{-a}. \tag{3.8}$$

Finally, the integral representations (3.4) and (3.5) can be derived from (3.1) by applying the substitutions $t=\sin^2\theta$ and $t=\frac{\xi}{1+\xi}$, respectively. This completes the proof of Theorem 3.1. □

Remark 3.2. The special case $b_5 = c-a$ in equations (3.1), (3.4), and (3.5) yields the following results:

Corollary 3.3.

$$K_{15,p}^{(\alpha,\beta)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u) = \frac{1}{B(a,c-a)} \times \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b_1}(1-yt)^{-b_2}(1-zt)^{-b_3}(1-u(1-t))^{-b_4} \times {}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right) dt, \tag{3.9}$$

$$K_{15,p}^{(\alpha,\beta)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u)=\frac{2}{B(a,c-a)}$$
$$\times\int_0^{\frac{\pi}{2}}\sin^{2a-1}\theta\cos^{2c-2a-1}\theta(1-x\sin^2\theta)^{-b_1}(1-y\sin^2\theta)^{-b_2}$$
$$\times(1-z\sin^2\theta)^{-b_3}(1-u\cos^2\theta)^{-b_4}{}_1F_1\left(\alpha;\beta;-\frac{p}{\sin^2\theta\cos^2\theta}\right)d\theta, \tag{3.10}$$

and

$$K_{15,p}^{(\alpha,\beta)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u)=\frac{1}{B(a,c-a)}$$
$$\times\int_0^{\infty}\xi^{a-1}(1+\xi)^{b_1+b_2+b_3+b_4-c}(1+(1-x)\xi)^{-b_1}(1+(1-y)\xi)^{-b_2}$$
$$\times(1+(1-z)\xi)^{-b_3}(1+\xi-u)^{-b_4}{}_1F_1\left(\alpha;\beta;-\frac{p(1+\xi)^2}{\xi}\right)d\xi. \tag{3.11}$$

4 Recurrence relations

Theorem 4.1. *For the extended Extons hypergeometric function $K_{15,p}^{(\alpha,\beta)}$, the following recurrence relations hold:*

(i)

$$(\beta-\alpha)K_{15,p}^{(\alpha-1,\beta)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u)$$
$$-\alpha K_{15,p}^{(\alpha+1,\beta)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u)$$
$$+(2\alpha-\beta)K_{15,p}^{(\alpha,\beta)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u)$$
$$-\frac{pB(a-1,c-a-1)}{B(a,c-a)}K_{15,p}^{(\alpha,\beta)}(a-1,a-1,a-1,c-a-1;b_1,b_2,b_3,b_4;c-2,c-2,c-2,c-2;$$
$$x,y,z,u)=0, \tag{4.1}$$

(ii)

$$\beta(\beta-1)K_{15,p}^{(\alpha,\beta-1)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u)$$
$$-\beta(\beta-1)K_{15,p}^{(\alpha,\beta)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u)$$
$$+\frac{p\beta B(a-1,c-a-1)}{B(a,c-a)}K_{15,p}^{(\alpha,\beta)}(a-1,a-1,a-1,c-a-1;b_1,b_2,b_3,b_4;c-2,c-2,c-2,c-2;x,y,z,u)$$
$$+\frac{p(\alpha-\beta)B(a-1,c-a-1)}{B(a,c-a)}K_{15,p}^{(\alpha,\beta+1)}(a-1,a-1,a-1,c-a-1;b_1,b_2,b_3,b_4;c-2,c-2,c-2,c-2;$$
$$x,y,z,u)=0, \tag{4.2}$$

(iii)

$$\alpha\beta K_{15,p}^{(\alpha,\beta)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u)$$
$$-\alpha\beta K_{15,p}^{(\alpha+1,\beta)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u)$$
$$-\frac{p\beta B(a-1,c-a-1)}{B(a,c-a)}K_{15,p}^{(\alpha,\beta)}(a-1,a-1,a-1,c-a-1;b_1,b_2,b_3,b_4;c-2,c-2,c-2,c-2;x,y,z,u)$$
$$+\frac{p(\beta-\alpha)B(a-1,c-a-1)}{B(a,c-a)}K_{15,p}^{(\alpha,\beta+1)}(a-1,a-1,a-1,c-a-1;b_1,b_2,b_3,b_4;c-2,c-2,c-2,c-2;$$
$$x,y,z,u)=0, \tag{4.3}$$

(iv)

$$\beta K_{15,p}^{(\alpha,\beta)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u)$$
$$-\beta K_{15,p}^{(\alpha-1,\beta)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u)$$
$$+\frac{pB(a-1,c-a-1)}{B(a,c-a)}K_{15,p}^{(\alpha,\beta+1)}(a-1,a-1,a-1,c-a-1;b_1,b_2,b_3,b_4;c-2,c-2,c-2,c-2;$$
$$x,y,z,u)=0, \tag{4.4}$$

(v)

$$(\beta-\alpha-1)K_{15,p}^{(\alpha,\beta)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u)$$
$$+\alpha K_{15,p}^{(\alpha+1,\beta)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u)$$
$$-(\beta-1)K_{15,p}^{(\alpha,\beta-1)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u)=0, \tag{4.5}$$

(vi)

$$(\alpha-1)K_{15,p}^{(\alpha,\beta)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u)$$
$$+(\beta-\alpha)K_{15,p}^{(\alpha-1,\beta)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u)$$
$$-(\beta-1)K_{15,p}^{(\alpha,\beta-1)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u)$$
$$-\frac{pB(a-1,c-a-1)}{B(a,c-a)}K_{15,p}^{(\alpha,\beta)}(a-1,a-1,a-1,c-a-1;b_1,b_2,b_3,b_4;c-2,c-2,c-2,c-2;$$
$$x,y,z,u)=0. \tag{4.6}$$

Proof. To establish these results, we employ the known recurrence identities of the confluent hypergeometric function ${}_1F_1$ [9]:

$$(\beta-\alpha){}_1F_1(\alpha-1;\beta;z)-\alpha\,{}_1F_1(\alpha+1;\beta;z)+(2\alpha-\beta+z){}_1F_1(\alpha;\beta;z)=0, \tag{4.7}$$

$$\beta(\beta-1){}_1F_1(\alpha;\beta-1;z)-\beta(\beta-1+z){}_1F_1(\alpha;\beta;z)+(\beta-\alpha)z\,{}_1F_1(\alpha;\beta+1;z)=0, \tag{4.8}$$

$$\beta(\alpha+z){}_1F_1(\alpha;\beta;z)-\alpha\beta\,{}_1F_1(\alpha+1;\beta;z)-(\beta-\alpha)z\,{}_1F_1(\alpha;\beta+1;z)=0, \tag{4.9}$$

$$\beta\,{}_1F_1(\alpha;\beta;z)-\beta\,{}_1F_1(\alpha-1;\beta;z)-z\,{}_1F_1(\alpha;\beta+1;z)=0, \tag{4.10}$$

$$(\beta-\alpha-1){}_1F_1(\alpha;\beta;z)+\alpha\,{}_1F_1(\alpha+1;\beta;z)-(\beta-1){}_1F_1(\alpha;\beta-1;z)=0, \tag{4.11}$$

$$(\alpha+z-1){}_1F_1(\alpha;\beta;z)+(\beta-\alpha){}_1F_1(\alpha-1;\beta;z)-(\beta-1){}_1F_1(\alpha;\beta-1;z)=0. \tag{4.12}$$

Proof of (4.1). Replace z with $-\frac{p}{t(1-t)}$ in (4.7), multiply both sides by

$$\frac{t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b_1}(1-yt)^{-b_2}(1-zt)^{-b_3}(1-u(1-t))^{-b_4}}{B(a,c-a)},$$

and integrate over $t\in[0,1]$. Using the integral representation (3.9) of $K_{15,p}^{(\alpha,\beta)}$, we obtain the identity (4.1).

The remaining relations (4.2)-(4.6) follow analogously by applying the corresponding identities (4.8)-(4.12) and proceeding similarly. This concludes the proof of Theorem 4.1. □

5 Generating functions

Theorem 5.1. *For the extended Exton's hypergeometric function $K_{15,p}^{(\alpha,\beta)}$, the following generating functions hold:*

$$\sum_{k=0}^{\infty}\frac{(b_5)_k\,t^k}{k!}K_{15,p}^{(\alpha,\beta)}(a,a,a,b_5+k;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u)$$
$$=(1-t)^{-b_5}K_{15,p}^{(\alpha,\beta)}\left(a,a,a,b_5;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,\frac{u}{1-t}\right), \tag{5.1}$$

$$\sum_{k=0}^{\infty}\frac{(b_1)_k\,t^k}{k!}K_{15,p}^{(\alpha,\beta)}(a,a,a,b_5;b_1+k,b_2,b_3,b_4;c,c,c,c;x,y,z,u)$$
$$=(1-t)^{-b_1}K_{15,p}^{(\alpha,\beta)}\left(a,a,a,b_5;b_1,b_2,b_3,b_4;c,c,c,c;\frac{x}{1-t},y,z,u\right), \tag{5.2}$$

$$\sum_{k=0}^{\infty}\frac{(b_2)_k\,t^k}{k!}K_{15,p}^{(\alpha,\beta)}(a,a,a,b_5;b_1,b_2+k,b_3,b_4;c,c,c,c;x,y,z,u)$$
$$=(1-t)^{-b_2}K_{15,p}^{(\alpha,\beta)}\left(a,a,a,b_5;b_1,b_2,b_3,b_4;c,c,c,c;x,\frac{y}{1-t},z,u\right), \tag{5.3}$$

$$\sum_{k=0}^{\infty}\frac{(b_3)_k\,t^k}{k!}K_{15,p}^{(\alpha,\beta)}(a,a,a,b_5;b_1,b_2,b_3+k,b_4;c,c,c,c;x,y,z,u)$$
$$=(1-t)^{-b_3}K_{15,p}^{(\alpha,\beta)}\left(a,a,a,b_5;b_1,b_2,b_3,b_4;c,c,c,c;x,y,\frac{z}{1-t},u\right), \tag{5.4}$$

$$\sum_{k=0}^{\infty}\frac{(b_4)_k\,t^k}{k!}K_{15,p}^{(\alpha,\beta)}(a,a,a,b_5;b_1,b_2,b_3,b_4+k;c,c,c,c;x,y,z,u)$$
$$=(1-t)^{-b_4}K_{15,p}^{(\alpha,\beta)}\left(a,a,a,b_5;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,\frac{u}{1-t}\right). \tag{5.5}$$

Proof. **Proof of (5.1).** Using the series definition of $K_{15,p}^{(\alpha,\beta)}$ from (2.1) in the left-hand side of (5.1), we have:

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{(b_5)_k\,t^k}{k!}K_{15,p}^{(\alpha,\beta)}(a,a,a,b_5+k;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u)\\
&=\sum_{m,n,r,s,k=0}^{\infty}\frac{B_p^{(\alpha,\beta)}(a+m+n+r,c-a+s)(b_5)_{s+k}(b_1)_m(b_2)_n(b_3)_r(b_4)_s}{B(a,c-a)(c-a)_s\,m!n!r!s!k!}\\
&\quad\times x^m y^n z^r u^s t^k\\
&=\sum_{m,n,r,s=0}^{\infty}\frac{B_p^{(\alpha,\beta)}(a+m+n+r,c-a+s)(b_5)_s(b_1)_m(b_2)_n(b_3)_r(b_4)_s}{B(a,c-a)(c-a)_s\,m!n!r!s!}\\
&\quad\times x^m y^n z^r u^s\sum_{k=0}^{\infty}\frac{(b_5+s)_k\,t^k}{k!}.
\end{aligned}$$

Using the identity

$$\sum_{k=0}^{\infty}\frac{(b+s)_k\,t^k}{k!}=(1-t)^{-b-s},$$

and simplifying, we get the right-hand side of (5.1).

The generating functions (5.2)–(5.5) can be proved similarly. This completes the proof of Theorem 5.1. □

Remark 5.2. Setting $p=0$ in equations (5.1)-(5.5) recovers classical generating functions (see [13]).

Theorem 5.3. *The following generating functions for the extended Extons function* $K_{15,p}^{(\alpha,\beta)}$ *also hold:*

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{(\lambda)_k\,t^k}{k!}K_{15,p}^{(\alpha,\beta)}(a,a,a,-k;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u)\\
&=(1-t)^{-\lambda}K_{15,p}^{(\alpha,\beta)}\left(a,a,a,\lambda;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,\frac{-ut}{1-t}\right),
\end{aligned}\tag{5.6}$$

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{(\lambda)_k\,t^k}{k!}K_{15,p}^{(\alpha,\beta)}(a,a,a,b_5;-k,b_2,b_3,b_4;c,c,c,c;x,y,z,u)\\
&=(1-t)^{-\lambda}K_{15,p}^{(\alpha,\beta)}\left(a,a,a,b_5;\lambda,b_2,b_3,b_4;c,c,c,c;\frac{-xt}{1-t},y,z,u\right),
\end{aligned}\tag{5.7}$$

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{(\lambda)_k\,t^k}{k!}K_{15,p}^{(\alpha,\beta)}(a,a,a,b_5;b_1,-k,b_3,b_4;c,c,c,c;x,y,z,u)\\
&=(1-t)^{-\lambda}K_{15,p}^{(\alpha,\beta)}\left(a,a,a,b_5;b_1,\lambda,b_3,b_4;c,c,c,c;x,\frac{-yt}{1-t},z,u\right),
\end{aligned}\tag{5.8}$$

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{(\lambda)_k\,t^k}{k!}K_{15,p}^{(\alpha,\beta)}(a,a,a,b_5;b_1,b_2,-k,b_4;c,c,c,c;x,y,z,u)\\
&=(1-t)^{-\lambda}K_{15,p}^{(\alpha,\beta)}\left(a,a,a,b_5;b_1,b_2,\lambda,b_4;c,c,c,c;x,y,\frac{-zt}{1-t},u\right),
\end{aligned}\tag{5.9}$$

$$\sum_{k=0}^{\infty} \frac{(\lambda)_k\, t^k}{k!} K_{15,p}^{(\alpha,\beta)}(a,a,a,b_5;b_1,b_2,b_3,-k;c,c,c,c;x,y,z,u)$$
$$= (1-t)^{-\lambda} K_{15,p}^{(\alpha,\beta)}\left(a,a,a,b_5;b_1,b_2,b_3,\lambda;c,c,c,c;x,y,z,\frac{-ut}{1-t}\right). \quad (5.10)$$

Proof. **Proof of (5.6).** From the series form of $K_{15,p}^{(\alpha,\beta)}$ and the identities

$$(-n)_k = \frac{(-1)^k n!}{(n-k)!}, \quad 0 \le k \le n, \qquad \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+k),$$

we proceed by transforming the summation indices and apply the identity

$$\sum_{k=0}^{\infty} \frac{(\lambda+s)_k t^k}{k!} = (1-t)^{-\lambda-s}.$$

Following the same approach, we obtain (5.6). The remaining generating functions (5.7)-(5.10) follow analogously. □

6 Transformation and summation formulas

Theorem 6.1. *For the extended Exton's hypergeometric function $K_{15,p}^{(\alpha,\beta)}$, the following transformation formula holds:*

$$K_{15,p}^{(\alpha,\beta)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u)$$
$$= (1-u)^{-b_4} F_{D,p}^{(4,\alpha,\beta)}\left(a,b_1,b_2,b_3,b_4;c;x,y,z,\frac{u}{u-1}\right), \quad (6.1)$$

where $F_{D,p}^{(4,\alpha,\beta)}$ is the extended Lauricella function defined in (1.9).

Proof. Using (3.8) in (3.9), we obtain

$$K_{15,p}^{(\alpha,\beta)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u)$$
$$= \frac{(1-u)^{-b_4}}{B(a,c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b_1}(1-yt)^{-b_2}(1-zt)^{-b_3}$$
$$\times \left(1-\frac{ut}{u-1}\right)^{-b_4} {}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right) dt,$$

which corresponds to the integral representation of the function $F_{D,p}^{(4,\alpha,\beta)}$ in (1.10) (with $r=4$). This proves (6.1). □

Remark 6.2. Setting $p=0$ in (6.1), we recover the classical transformation formula [8]:

$$K_{15}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,u) = (1-u)^{-b_4} F_D^{(4)}\left(a,b_1,b_2,b_3,b_4;c;x,y,z,\frac{u}{1-u}\right), \quad (6.2)$$

where $F_D^{(4)}$ is the classical Lauricella function as defined in (1.11).

Theorem 6.3. *For the extended Exton's hypergeometric function $K_{15,p}^{(\alpha,\beta)}$, the following summation formula holds:*

$$K_{15,p}^{(\alpha,\beta)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;1,1,1,1) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} B_p^{(\alpha,\beta)}(a-b_4,c-a-b_1-b_2-b_3). \tag{6.3}$$

Proof. Setting $x=y=z=u=1$ in (3.9), we obtain

$$K_{15,p}^{(\alpha,\beta)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;1,1,1,1) = \frac{1}{B(a,c-a)} \int_0^1 t^{a-b_4-1}(1-t)^{c-a-b_1-b_2-b_3-1} \times {}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right) dt. \tag{6.4}$$

Using the definition of the extended beta function $B_p^{(\alpha,\beta)}$ in (1.4), we obtain the desired result (6.3). □

Remark 6.4. Setting $p=0$ in (6.3) and applying the identity (1.5), we recover the classical Exton's summation formula [3]:

$$K_{15}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;1,1,1,1) = \frac{\Gamma(c)\Gamma(c-a-b_1-b_2-b_3)\Gamma(a-b_4)}{\Gamma(a)\Gamma(c-a)\Gamma(c-b_1-b_2-b_3-b_4)}. \tag{6.5}$$

Theorem 6.5. *For the extended Exton's hypergeometric function $K_{15,p}^{(\alpha,\beta)}$, the following summation formula holds:*

$$K_{15,p}^{(\alpha,\beta)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;1,1,1,-1) = \frac{\Gamma(c)\Gamma(c-a-b_1-b_2-b_3)}{\Gamma(c-a)\Gamma(c-b_1-b_2-b_3)} F_p^{(\alpha,\beta)}(b_4,c-a-b_1-b_2-b_3;c-b_1-b_2-b_3;-1). \tag{6.6}$$

Proof. Substitute $x=y=z=1$, $u=-1$ into (3.9):

$$K_{15,p}^{(\alpha,\beta)}(a,a,a,c-a;b_1,b_2,b_3,b_4;c,c,c,c;1,1,1,-1) = \frac{1}{B(a,c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-b_1-b_2-b_3-1}(1+(1-t))^{-b_4} \times {}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right) dt.$$

Changing variables via $t \to 1-t$ and using identity (1.7) yields the result (6.6). □

Remark 6.6. Setting $p=0$, $b_4=1-a$ in (6.6) and applying the classical identity [11]:

$${}_2F_1(a,b;1+a-b;-1) = \frac{\Gamma(1+a-b)\Gamma(1+\frac{1}{2}a)}{\Gamma(1+\frac{1}{2}a-b)\Gamma(1+a)}, \tag{6.7}$$

we obtain the known Extons summation formula [3]:

$$K_{15}(a,a,a,c-a;b_1,b_2,b_3,1-a;c,c,c,c;1,1,1,-1) = \frac{\Gamma(c)\Gamma(c-a-b_1-b_2-b_3)\Gamma\left(1+\frac{1}{2}c-\frac{1}{2}a-\frac{1}{2}b_1-\frac{1}{2}b_2-\frac{1}{2}b_3\right)}{\Gamma(c-a)\Gamma(1+c-a-b_1-b_2-b_3)\Gamma\left(\frac{1}{2}c+\frac{1}{2}a-\frac{1}{2}b_1-\frac{1}{2}b_2-\frac{1}{2}b_3\right)}. \tag{6.8}$$

7 Conclusion

In this paper, we introduced an extension of Exton's hypergeometric function, denoted by $K_{15,p}^{(\alpha,\beta)}$, defined in (2.1), by employing the extended beta function $B_p^{(\alpha,\beta)}(x,y)$ given in (1.4). Several fundamental properties of this extended function were studied, including its integral representations, recurrence relations, generating functions, transformation formulas, and summation formulas.

Each of the results derived in this work reduces either to known or novel identities for the classical Exton's hypergeometric function K_{15} when the extension parameter $p = 0$ (see, e.g., [3,8,13]). This illustrates the consistency and natural generalization of our approach.

Given the broad applicability of hypergeometric-type functions in mathematical physics, engineering, and applied sciences, the extended function $K_{15,p}^{(\alpha,\beta)}$ and the associated results presented herein are expected to be of interest for further theoretical developments and potential applications in these domains.

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Conflict of interest

The authors declare that they have no conflicts of interest related to this work.

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References

[1] A. A. Al-Gonah and W. K. Mohammed, *A new extension of extended Gamma and Beta functions and their properties*, *Journal of Scientific and Engineering Research* **5**(9) (2018), 257–270. URL

[2] A. A. Atash, S. S. Barahmah and M. A. Kulib, *On Extensions of Extended Gauss Hypergeometric Function*, *Communications in Advanced Mathematical Sciences* **2**(3) (2019), 199–205. DOI

[3] A. A. Atash and H. S. Bellehaj, *Transformation and summation formulas for Exton's functions* K_{11}, K_{14} *and* K_{15}, *Electronic Journal of University of Aden for Basic and Applied Sciences* **1**(1) (2020), 49–53. URL

[4] A. A. Atash and M. A. Kulib, *Extension of Extons hypergeometric function* K_{16}, *CUBO, A Mathematical Journal* **23**(3) (2021), 489–501. DOI

[5] A. Çetinkaya, M. B. Yabasan and I. O. Kymaz, *The Extended Srivastava's Triple Hypergeometric Functions and Their Integral Representations*, *Journal of Nonlinear Sciences and Applications* **10** (2016), 1–7. DOI

[6] M. A. Chaudhry, A. Qadir, M. Rafique and S. M. Zubair, *Extension of Euler's beta function*, *Journal of Computational and Applied Mathematics* **78** (1997), 19–32. DOI

[7] J. Choi, A. K. Rathie and R. K. Parmar, *Extension of extended Beta, Hypergeometric and confluent hypergeometric functions*, *Honam Mathematical Journal* **36** (2014), 357–385. DOI

[8] H. Exton, *Multiple Hypergeometric Functions and Applications*, Halsted Press, New York, 1976.

[9] Y. L. Luke, *The Special Functions and Their Approximations*, Vol. 1, Academic Press, New York, 1969.

[10] E. Özergin, M. A. Özarslan and A. Altin, *Extension of gamma, beta and hypergeometric functions*, *Journal of Computational and Applied Mathematics* **235** (2011), 4601–4610. DOI

[11] E. D. Rainville, *Special Functions*, Macmillan, New York, 1960.

[12] M. Shadab and J. Choi, *Extensions of Appell and Lauricella's hypergeometric functions*, *Far East Journal of Mathematical Sciences* **102** (2017), 1301–1317. DOI

[13] R. C. Singh Chandel and A. Tiwari, *Generating relations involving hypergeometric functions of four variables*, *Pure and Applied Mathematical Sciences* **36** (1991), 15–25. URL

[14] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press, New York, 1984.

[15] H. M. Srivastava, R. K. Parmar and P. Chopra, *A class of extended fractional derivative operators and associated generating relations involving hypergeometric functions*, *Axioms* **1** (2012), 238–258. DOI