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A new approach to hyper dual numbers with tribonacci and tribonacci-Lucas numbers and their generalized summation formulas

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Abstract. Motivated by the definition of Tribonacci quaternions, we define hyper-dual numbers whose components involve Tribonacci and Tribonacci-Lucas numbers. We refer to these new numbers as hyper-dual Tribonacci numbers and hyper-dual Tribonacci-Lucas numbers, respectively.

In this paper, we also establish some properties of these numbers and present useful identities involving them. Furthermore, we investigate formulas for the generalized sum and the sum with alternating signs for Tribonacci and Tribonacci-Lucas numbers using a new method. Based on these results, we derive the corresponding formulas for the generalized and alternating sign sums of hyper-dual Tribonacci and hyper-dual Tribonacci-Lucas numbers.

Keywords: Tribonacci numbers, Tribonacci-Lucas numbers, Hyper dual numbers, Hyper dual tribonacci numbers, Hyper dual tribonacci-Lucas numbers, Generalized sum.

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1 Introduction

In 1873, W. K. Clifford defined dual numbers, an extension of real numbers, as the set

$$\mathbb{D} = \{a + \varepsilon a^* \mid a, a^* \in \mathbb{R},\ \varepsilon^2 = 0\}. \tag{1.1}$$

Let $d_1 = a_1 + \varepsilon a_1^*$ and $d_2 = a_2 + \varepsilon a_2^*$ be two dual numbers. Then, their sum and product are respectively defined as

$$d_1 + d_2 = (a_1 + a_2) + \varepsilon(a_1^* + a_2^*),$$
$$d_1 d_2 = a_1 a_2 + \varepsilon(a_1 a_2^* + a_1^* a_2).$$

The conjugate of $d = a + \varepsilon a^*$ is $\bar{d} = a - \varepsilon a^*$. This number system forms a commutative and associative algebra over the real numbers of dimension two.

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The first application of dual numbers was initiated by Kotelnikov [21]. These numbers and their associated vectors were used by E. Study in kinematics and line geometry. He also proved the bijection between the directed lines in Euclidean 3-space and the points of the unit dual sphere in $\mathbb{D}^3$, which is known as the Study mapping [27].

Dual numbers have been applied in various fields such as displacement analysis of spatial mechanisms, rigid body motion, surface shape analysis, computer graphics, human body motion analysis, kinematic synthesis, and more [2,24,29].

The set of hyper-dual numbers $\mathbb{HD}$ was defined by J. A. Fike [13]. Later, he and J. J. Alonso developed this number system for the calculation of first and second derivatives [14,15]. This system has been applied in many fields, such as open kinematic chain robot manipulators, aerospace systems analysis and design, complex software, and more.

A. Cohen and M. Shoham used hyper-dual numbers in kinematics and dynamics. A hyper-dual number can be formed by two dual numbers. Some kinematic applications of these numbers were studied by S. Aslan [5,8–10].

Hyper-dual numbers are defined by

$$\mathbb{HD} = \left\{ \mathbb{A} = a_1 + a_2\varepsilon_1 + a_3\varepsilon_2 + a_4\varepsilon_1\varepsilon_2 \mid a_i \in \mathbb{R},\ 1 \le i \le 4,\ \varepsilon_1^2 = \varepsilon_2^2 = 0,\ \varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1 \right\}. \quad (1.2)$$

The hyper-dual number $\mathbb{A}$ can also be written as

$$\mathbb{A} = (a_1 + a_2\varepsilon_1) + \varepsilon_2(a_3 + a_4\varepsilon_1) = A + \varepsilon_2 A^*,$$

where A and A^* are dual numbers.

Let $\mathbb{A} = a_0 + a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_1\varepsilon_2$ and $\mathbb{B} = b_0 + b_1\varepsilon_1 + b_2\varepsilon_2 + b_3\varepsilon_1\varepsilon_2$ be two hyper-dual numbers. Then, their addition and multiplication are defined as follows:

$$\mathbb{A} + \mathbb{B} = (a_0 + b_0) + \varepsilon_1(a_1 + b_1) + \varepsilon_2(a_2 + b_2) + \varepsilon_1\varepsilon_2(a_3 + b_3),$$
$$\mathbb{A}\mathbb{B} = a_0b_0 + \varepsilon_1(a_0b_1 + a_1b_0) + \varepsilon_2(a_0b_2 + a_2b_0) + \varepsilon_1\varepsilon_2(a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0).$$

The multiplicative inverse of $\mathbb{A} = A + \varepsilon_2 A^*$ is

$$\mathbb{A}^{-1} = \frac{1}{A} - \varepsilon_2\frac{A^*}{A^2} = \frac{1}{a_0} - \varepsilon_1\frac{a_1}{a_0^2} + \varepsilon_2\frac{a_2}{a_0^2} + \varepsilon_1\varepsilon_2\left(-\frac{a_3}{a_0^2} + \frac{2a_1a_2}{a_0^3}\right), \quad a_0 \neq 0.$$

The square root of a hyper-dual number $\mathbb{A}$ is given by

$$\sqrt{\mathbb{A}} = \sqrt{A} + \varepsilon_2\frac{A^*}{2\sqrt{A}}, \quad a_0 > 0,$$

or equivalently,

$$\sqrt{\mathbb{A}} = \sqrt{a_0} + \varepsilon_1\frac{a_1}{2\sqrt{a_0}} + \varepsilon_2\frac{a_2}{2\sqrt{a_0}} + \varepsilon_1\varepsilon_2\left(\frac{a_3}{2\sqrt{a_0}} - \frac{a_1a_2}{4a_0\sqrt{a_0}}\right), \quad a_0 > 0.$$

The hyper-dual function $f(x_0 + x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_1\varepsilon_2)$ about the point $a_0 + a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_1\varepsilon_2 \in \mathbb{HD}$, [6], is given by

$$f(a_0 + a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_1\varepsilon_2) = f(a_0) + \varepsilon_1 a_1 f'(a_0) + \varepsilon_2 a_2 f'(a_0) + \varepsilon_1\varepsilon_2\left(a_3 f'(a_0) + a_1a_2 f''(a_0)\right),$$

The well-known Fibonacci and Lucas sequences are defined by the second-order recurrence relations:

$$F_n = F_{n-1} + F_{n-2}, \quad n \ge 2,$$
$$L_n = L_{n-1} + L_{n-2}, \quad n \ge 2,$$

for more details, see [17,18,20].

Similarly, the Tribonacci and Tribonacci-Lucas sequences are defined by

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n, \quad T_0 = 0,\ T_1 = 1,\ T_2 = 1, \tag{1.3}$$

$$L_{n+3} = L_{n+2} + L_{n+1} + L_n, \quad L_0 = 3,\ L_1 = 1,\ L_2 = 3. \tag{1.4}$$

The first few elements of these sequences are:

$$0,\ 1,\ 1,\ 2,\ 4,\ 7,\ 13,\ 24,\ 44,\ 81,\ 149,\ 274,\ldots,$$
$$3,\ 1,\ 3,\ 7,\ 11,\ 21,\ 39,\ 71,\ 131,\ 241,\ 443,\ldots$$

The following recurrence relations also hold:

$$T_n = 2T_{n-1} - T_{n-4}, \quad n \geq 4, \quad T_0 = T_1 = 0,\ T_2 = T_3 = 1,$$

$$L_n = 2L_{n-1} - L_{n-4}, \quad n \geq 4, \quad L_0 = 3,\ L_1 = 1,\ L_2 = 3,\ L_3 = 7.$$

Their generating functions are:

$$f(x) = \frac{x}{1 - x - x^2 - x^3}, \quad g(x) = \frac{3 - 2x - x^2}{1 - x - x^2 - x^3}.$$

The Binet formulas for the Tribonacci and TribonacciLucas sequences are given by

$$T_n = \Lambda_1 \alpha_1^{n+1} + \Lambda_2 \alpha_2^{n+1} + \Lambda_3 \alpha_3^{n+1}, \tag{1.5}$$

$$L_n = \alpha_1^n + \alpha_2^n + \alpha_3^n, \tag{1.6}$$

where α_1, α_2, and α_3 are the roots of the characteristic equation

$$1 - x - x^2 - x^3 = 0,$$

and the constants Λ_i $(i = 1, 2, 3)$ are defined by

$$\Lambda_i = \prod_{\substack{j=1 \\ j \neq i}}^{3} \frac{1}{\alpha_i - \alpha_j}.$$

The sums of the first n terms of these sequences are given by [1,7,11,12,16,19,22,25,26,28,30]:

$$\sum_{k=0}^{n} T_k = \frac{T_n + T_{n+2} - 1}{2},$$
$$\sum_{k=0}^{n} L_k = \frac{L_n + L_{n+2}}{2}.$$

The sums of squares of these sequences are:

$$\sum_{k=0}^{n} T_k^2 = \frac{1 + 4T_n T_{n+1} - (T_{n+1} - T_{n-1})^2}{4},$$
$$\sum_{k=0}^{n} L_k^2 = \frac{-L_{n+1}^2 - L_{n-1}^2 + L_{2n+3} + L_{2n-2}}{2} - 11.$$

The following identity relates Tribonacci numbers with positive and negative indices [4]:

$$T_n^2 - T_{n-1}T_{n+1} = T_{-(n+1)}, \quad \text{for all } n \geq 1. \tag{1.7}$$

Tribonacci numbers for negative indices are:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
T_{-n}	0	1	-1	0	2	-3	1	4	-8	5	7	-20	18	9

In this paper, we define hyper-dual numbers with components involving Tribonacci and Tribonacci-Lucas numbers. We explore their Binet formulas and related identities. Then, we derive generalized summation formulas, including those with alternating signs, for these new types of numbers.

2 Hyper dual tribonacci and hyper dual tribonacci-Lucas numbers

Definition 2.1. Let T_n and L_n be the n^{th} tribonacci and tribonacci-Lucas numbers, then the n^{th} hyper dual tribonacci HT_n and the n^{th} hyper dual tribonacci-Lucas HL_n numbers are defined respectively as follows

$$\mathrm{HT}_n = \mathrm{T}_n + \mathrm{T}_{n+1}\varepsilon_1 + \mathrm{T}_{n+2}\varepsilon_2 + \mathrm{T}_{n+3}\varepsilon_1\varepsilon_2, \tag{2.1}$$

and

$$\mathrm{HL}_n = \mathrm{L}_n + \mathrm{L}_{n+1}\varepsilon_1 + \mathrm{L}_{n+2}\varepsilon_2 + \mathrm{L}_{n+3}\varepsilon_1\varepsilon_2, \tag{2.2}$$

where ε_1 and ε_2 are hyper dual units.

The following identities hold for HT and HL and can be verified easily by using (1.3) and (1.4)

$$\begin{aligned} \mathrm{HT}_{n+3} &= \mathrm{HT}_{n+2} + \mathrm{HT}_{n+1} + \mathrm{HT}_n \ ; n \geqslant 0, \\ \mathrm{HL}_{n+3} &= \mathrm{HL}_{n+2} + \mathrm{HL}_{n+1} + \mathrm{HL}_n; n \geqslant 0, \\ \mathrm{HL}_n &= \mathrm{HT}_n + 2\mathrm{HT}_{n-1} + 3\mathrm{HT}_{n-2}; n \geqslant 2. \end{aligned} \tag{2.3}$$

Definition 2.2. Let A_n and B_n defined as follows

$$\begin{aligned} \mathrm{A}_n &= -\mathrm{A}_{n-1} - \mathrm{A}_{n-2} + \mathrm{A}_{n-3} \quad ; \mathrm{A}_{-1} = 1, \mathrm{A}_0 = \mathrm{A}_1 = 0, \\ \mathrm{B}_n &= -\mathrm{B}_{n-1} - \mathrm{B}_{n-2} + \mathrm{B}_{n-3} \quad ; \mathrm{B}_{-1} = 1, \mathrm{B}_0 = 3, \mathrm{B}_1 = -1, \end{aligned}$$

for $n \geqslant 2$, then

$$\begin{aligned} \mathrm{HT}_{-n} &= \mathrm{A}_n + \mathrm{A}_{n+1}\varepsilon_1 + \mathrm{A}_{n+2}\varepsilon_2 + \mathrm{A}_{n+3}\varepsilon_1\varepsilon_2, \\ \mathrm{HL}_{-n} &= \mathrm{B}_n + \mathrm{B}_{n+1}\varepsilon_1 + \mathrm{B}_{n+2}\varepsilon_2 + \mathrm{B}_{n+3}\varepsilon_1\varepsilon_2. \end{aligned}$$

Theorem 2.3 (Generating functions). *Let $g(x) = \sum_{i=0}^{\infty} \mathrm{HT}_i x^i$ and $h(x) = \sum_{i=0}^{\infty} \mathrm{HL}_i x^i$ be generating functions, then*

$$\begin{aligned} g(x) &= \frac{x + \varepsilon_1 + (1 + x + x^2)\varepsilon_2 + (2 + 2x + x^2)\varepsilon_1\varepsilon_2}{1 - x - x^2 - x^3}, \\ h(x) &= \frac{(3 - 2x - 3x^2) + (1 + 2x + 5x^2)\varepsilon_1 + (3 + 4x + 5x^2)\varepsilon_2 + (7 + 4x + 7x^2)\varepsilon_1\varepsilon_2}{1 - x - x^2 - x^3}, \end{aligned}$$

respectively.

Proof. Suppose $g(x) = \sum_{i=0}^{\infty} \mathrm{HT}_i x^i$ be a generating function for hyper dual tribonacci numbers then

$$\begin{aligned} g(x) &= \mathrm{HT}_0 + \mathrm{HT}_1 x + \mathrm{HT}_2 x^2 + \mathrm{HT}_3 x^3 + ... + \mathrm{HT}_n x^n + ..., \\ xg(x) &= \mathrm{HT}_0 x + \mathrm{HT}_1 x^2 + \mathrm{HT}_2 x^3 + \mathcal{T}_3 x^4 + ... + \mathrm{HT}_{n-1} x^n + ..., \\ x^2 g(x) &= \mathrm{HT}_0 x^2 + \mathrm{HT}_1 x^3 + \mathrm{HT}_2 x^4 + \mathrm{HT}_3 x^5 + ... + \mathrm{HT}_{n-2} x^n + ..., \\ x^3 g(x) &= \mathrm{HT}_0 x^3 + \mathrm{HT}_1 x^4 + \mathrm{HT}_2 x^5 + \mathrm{HT}_3 x^6 + ... + \mathrm{HT}_{n-3} x^n + ..., \end{aligned}$$

and

$$\begin{aligned} (1 - x - x^2 - x^3) g(x) = {} & \mathrm{HT}_0 + (\mathrm{HT}_1 - \mathrm{HT}_0)x + (\mathrm{HT}_2 - \mathrm{HT}_1 - \mathrm{HT}_0)x^2 + (\mathrm{HT}_3 - \mathrm{HT}_2 - \mathrm{HT}_1 - \mathrm{HT}_0)x^3 \\ & + ... + (\mathrm{HT}_n - \mathrm{HT}_{n-1} - \mathrm{HT}_{n-2} - \mathrm{HT}_{n-3})x^n + ..., \end{aligned}$$

therefore

$$g(x) = \frac{\mathrm{HT}_0 + (\mathrm{HT}_1 - \mathrm{HT}_0)x + (\mathcal{T}_2 - \mathrm{HT}_1 - \mathrm{HT}_0)x^2}{1 - x - x^2 - x^3},$$

from (2.1) we get

$$g(x) = \frac{x + \varepsilon_1 + (1 + x + x^2)\varepsilon_2 + (2 + 2x + x^2)\varepsilon_1\varepsilon_2}{1 - x - x^2 - x^3}.$$

Similarly by using the same method we get

$$h(x) = \frac{\mathrm{HL}_0 + (\mathrm{HL}_1 - \mathrm{HL}_0)x + (\mathrm{HL}_2 - \mathrm{HL}_1 - \mathrm{HL}_0)x^2}{1 - x - x^2 - x^3},$$

From equation (2.2), it follows that

$$h(x) = \frac{(3 - 2x - 3x^2) + (1 + 2x + 5x^2)\varepsilon_1 + (3 + 4x + 5x^2)\varepsilon_2 + (7 + 4x + 7x^2)\varepsilon_1\varepsilon_2}{1 - x - x^2 - x^3}.$$

□

Theorem 2.4 (Binet's formulas). *The explicit formulas for* HT_n *and* HL_n *are*

$$\mathrm{HT}_n = \Lambda_1 \alpha_1^{n+1} \breve{\alpha}_1 + \Lambda_2 \alpha_2^{n+1} \breve{\alpha}_2 + \Lambda_3 \alpha_3^{n+1} \breve{\alpha}_3, \tag{2.4}$$

and

$$\mathrm{HL}_n = \alpha_1^n \breve{\alpha}_1 + \alpha_2^n \breve{\alpha}_2 + \alpha_3^n \breve{\alpha}_3, \tag{2.5}$$

respectively, where $\Lambda_i = \prod_{\substack{i,j=1 \\ j \neq i}}^{3} \frac{1}{\alpha_i - \alpha_j}$ *and*

$$\begin{cases} \breve{\alpha}_1 = 1 + \alpha_1 \varepsilon_1 + \alpha_1^2 \varepsilon_2 + \alpha_1^3 \varepsilon_1 \varepsilon_2, \\ \breve{\alpha}_2 = 1 + \alpha_2 \varepsilon_1 + \alpha_2^2 \varepsilon_2 + \alpha_2^3 \varepsilon_1 \varepsilon_2, \\ \breve{\alpha}_3 = 1 + \alpha_3 \varepsilon_1 + \alpha_3^2 \varepsilon_2 + \alpha_3^3 \varepsilon_1 \varepsilon_2. \end{cases} \tag{2.6}$$

Proof. By using (1.5) and (2.1) we get

$$\begin{aligned} \mathrm{HT}_n = {} & (\Lambda_1 \alpha_1^{n+1} + \Lambda_2 \alpha_2^{n+1} + \Lambda_3 \alpha_3^{n+1}) + (\Lambda_1 \alpha_1^{n+2} + \Lambda_2 \alpha_2^{n+2} + \Lambda_3 \alpha_3^{n+2})\varepsilon_1 \\ & + (\Lambda_1 \alpha_1^{n+3} + \Lambda_2 \alpha_2^{n+3} + \Lambda_3 \alpha_3^{n+3})\varepsilon_2 + (\Lambda_1 \alpha_1^{n+4} + \Lambda_2 \alpha_2^{n+4} + \Lambda_3 \alpha_3^{n+4})\varepsilon_1\varepsilon_2 \\ = {} & \Lambda_1 \alpha_1^{n+1}(1 + \alpha_1 \varepsilon_1 + \alpha_1^2 \varepsilon_2 + \alpha_1^3 \varepsilon_1 \varepsilon_2) + \Lambda_2 \alpha_2^{n+1}(1 + \alpha_2 \varepsilon_1 + \alpha_2^2 \varepsilon_2 + \alpha_2^3 \varepsilon_1 \varepsilon_2) \\ & + \Lambda_3 \alpha_3^{n+1}(1 + \alpha_3 \varepsilon_1 + \alpha_3^2 \varepsilon_2 + \alpha_3^3 \varepsilon_1 \varepsilon_2) \\ = {} & \Lambda_1 \alpha_1^{n+1} \breve{\alpha}_1 + \Lambda_2 \alpha_2^{n+1} \breve{\alpha}_2 + \Lambda_3 \alpha_3^{n+1} \breve{\alpha}_3. \end{aligned}$$

Similarly, by using (1.6) and (2.2) we obtain

$$
\begin{aligned}
HL_n &= (\alpha_1^n + \alpha_2^n + \alpha_3^n) + (\alpha_1^{n+1} + \alpha_2^{n+1} + \alpha_3^{n+1})\varepsilon_1 + (\alpha_1^{n+2} + \alpha_2^{n+2} + \alpha_3^{n+2})\varepsilon_2 + (\alpha_1^{n+3} + \alpha_2^{n+3} + \alpha_3^{n+3})\varepsilon_1\varepsilon_2 \\
&= \alpha_1^n(1 + \alpha_1\varepsilon_1 + \alpha_1^2\varepsilon_2 + \alpha_1^3\varepsilon_1\varepsilon_2) + \alpha_2^n(1 + \alpha_2\varepsilon_1 + \alpha_2^2\varepsilon_2 + \alpha_2^3\varepsilon_1\varepsilon_2) + \alpha_3^n(1 + \alpha_3\varepsilon_1 + \alpha_3^2\varepsilon_2 + \alpha_3^3\varepsilon_1\varepsilon_2) \\
&= \alpha_1^n\breve{\alpha}_1 + \alpha_2^n\breve{\alpha}_2 + \alpha_3^n\breve{\alpha}_3.
\end{aligned}
$$

□

Lemma 2.5. *The following properties hold*

$$T_{k+n} = T_k L_n - T_{k-n} C_n + T_{k-2n}, \tag{2.7}$$

and

$$L_{k+n} = L_k L_n - L_{k-n} C_n + L_{k-2n}, \tag{2.8}$$

where C_n *is a sequence of the following form*

$$C_n = \alpha^n \beta^n + \alpha^n \gamma^n + \beta^n \gamma^n,$$

or equivalently

$$C_n = -C_{n-1} - C_{n-2} + C_{n-3}\,; C_{-1} = 1, C_0 = 3, C_1 = -1;\ n \geqslant 2. \tag{2.9}$$

Theorem 2.6. *Any hyper dual tribonacci* HT_n *and hyper dual tribonacci-Lucas* HL_n *numbers satisfy*

$$HT_{m+n} = HT_m L_n - HT_{m-n} C_n + HT_{m-2n}, \tag{2.10}$$
$$HL_{m+n} = HL_m L_n - HL_{m-n} C_n + HL_{m-2n}. \tag{2.11}$$

Proof. By using (2.1) and (2.7) we get

$$
\begin{aligned}
HT_{m+n} &= T_{m+n} + T_{m+n+1}\varepsilon_1 + T_{m+n+2}\varepsilon_2 + T_{m+n+3}\varepsilon_1\varepsilon_2 \\
&= (T_m L_n - T_{m-n} C_n + T_{m-2n}) + (T_{m+1} L_n - T_{m-n+1} C_n + T_{m+1-2n})\varepsilon_1 \\
&+ (T_{m+2} L_n - T_{m+2-n} C_n + T_{m+2-2n})\varepsilon_2 + (T_{m+3} L_n - T_{m+3-n} C_n + T_{m+3-2n})\varepsilon_1\varepsilon_2 \\
&= HT_m L_n - HT_{m-n} C_n + HT_{m-2n}.
\end{aligned}
$$

Similarly, by using (2.3) and (2.10) we obtain

$$
\begin{aligned}
HL_{m+n} &= HT_{m+n} + 2HT_{m+n-1} + 3HT_{m+n-2} \\
&= HT_m L_n - HT_{m-n} C_n + HT_{m-2n} + 2(HT_{m-1} L_n - HT_{m-1-n} C_n + HT_{m-1-2n}) \\
&+ 3(HT_{m-2} L_n - HT_{m-2-n} C_n + HT_{m-2-2n}) \\
&= HL_m L_n - HL_{m-n} C_n + HL_{m-2n}.
\end{aligned}
$$

□

Theorem 2.7. *For integers* $n \geqslant 0$ *and* $m \geqslant 3$*, the following property holds*

$$HT_{m+n} = T_{m-2} HT_n + (T_{m-3} + T_{m-2}) HT_{n+1} + T_{m-1} HT_{n+2}, \tag{2.12}$$

where T_n *is the* n^{th} *tribonacci number.*

Proof. We use induction on m, for $m = 3$,

$$HT_{n+3} = T_1HT_n + (T_0 + T_1)HT_{n+1} + T_2HT_{n+2} = HT_n + HT_{n+1} + HT_{n+2}.$$

Now suppose the equality is true for $m \leqslant k$, then for $m = k+1$ we have,

$$\begin{aligned}HT_{n+k+1} &= HT_{n+k} + HT_{n+k-1} + HT_{n+k-2}\\ &= (T_{k-2}HT_n + (T_{k-3} + T_{k-2})HT_{n+1} + T_{k-1}HT_{n+2}) + (T_{k-3}HT_n + (T_{k-4} + T_{k-3})HT_{n+1}\\ &+ T_{k-2}HT_{n+2}) + (T_{k-4}HT_n + (T_{k-5} + T_{k-4})HT_{n+1} + T_{k-3}HT_{n+2})\\ &= (T_{k-2} + T_{k-3} + T_{k-4})HT_n + ((T_{k-3} + T_{k-2}) + (T_{k-4} + T_{k-3}) + (T_{k-5} + T_{k-4}))HT_{n+1}\\ &+ (T_{k-1} + T_{k-2} + T_{k-3})HT_{n+2}\\ &= T_{k-1}HT_n + (T_{k-2} + T_{k-1})HT_{n+1} + T_kHT_{n+2},\end{aligned}$$

then we get the result for all $m \geqslant 3$. □

Definition 2.8. Let R_n and U_n be sequences which are defined for $n \geqslant 3$, by

$$R_n = R_{n-1} + R_{n-2} + R_{n-3};\ R_0 = 3, R_1 = 2, R_2 = 5, \tag{2.13}$$

and

$$U_n = U_{n-1} + U_{n-2} + U_{n-3};\ U_0 = 0, U_1 = 1, U_2 = 0, \tag{2.14}$$

respectively. These sequences are listed in OEIS (The On-Line Encyclopedia of Integer Sequences) in (A274761) and (A001590) [23], then we define the following hyper dual sequences

$$HR_n = R_n + R_{n+1}\varepsilon_1 + R_{n+2}\varepsilon_2 + R_{n+3}\varepsilon_1\varepsilon_2, \tag{2.15}$$

and

$$HU_n = U_n + U_{n+1}\varepsilon_1 + U_{n+2}\varepsilon_2 + U_{n+3}\varepsilon_1\varepsilon_2. \tag{2.16}$$

The following relations can be proved easily

$$R_n = 3T_{n+1} - T_n;\ n \geqslant 0, \tag{2.17}$$

$$U_n = T_{n-1} + T_{n-2};\ n \geqslant 2, \tag{2.18}$$

$$HR_n = HR_{n-1} + HR_{n-2} + HR_{n-2};\ n \geqslant 2, \tag{2.19}$$

$$HU_n = HU_{n-1} + HU_{n-2} + HU_{n-2};\ n \geqslant 2. \tag{2.20}$$

Theorem 2.9. *Let HT_n be hyper dual tribonacci and HU_n be the hyper dual number sequence defined in (2.16), then the following identities hold for $n \geqslant 2$*

$$HT_n^2 - HT_{n-1}^2 = HU_{n+1}HU_{n-1},$$
$$HU_{n+1}^2 + HU_{n-1}^2 = 2(HT_n^2 + HT_{n-1}^2).$$

Proof. By using (2.1) we get

$$\begin{aligned}HT_n^2 - HT_{n-1}^2 &= (HT_n - HT_{n-1})(HT_n + HT_{n-1})\\ &= [(T_n - T_{n-1}) + (T_{n+1} - T_n)\varepsilon_1 + (T_{n+2} - T_{n+1})\varepsilon_2 + (T_{n+3} - T_{n+2})\varepsilon_1\varepsilon_2]\\ &\times [(T_n + T_{n-1}) + (T_{n+1} + T_n)\varepsilon_1 + (T_{n+2} + T_{n+1})\varepsilon_2 + (T_{n+3} + T_{n+2})\varepsilon_1\varepsilon_2]\\ &= [(T_{n-2} + T_{n-3}) + (T_{n-1} + T_{n-2})\varepsilon_1 + (T_n + T_{n-1})\varepsilon_2 + (T_{n+1} + T_n)\varepsilon_1\varepsilon_2]\\ &\times [(T_n + T_{n-1}) + (T_{n+1} + T_n)\varepsilon_1 + (T_{n+2} + T_{n+1})\varepsilon_2 + (T_{n+3} + T_{n+2})\varepsilon_1\varepsilon_2]\\ &= (U_{n-1} + U_n\varepsilon_1 + U_{n+1}\varepsilon_2 + U_{n+2}\varepsilon_1\varepsilon_2) \times (U_{n+1} + U_{n+2}\varepsilon_1 + U_{n+3}\varepsilon_2 + U_{n+4}\varepsilon_1\varepsilon_2)\\ &= HU_{n-1}HU_{n+1}.\end{aligned}$$

Similarly, by using (2.16) and (2.18) we get

$$\begin{aligned}
\mathrm{HU}_{n+1}-\mathrm{HU}_{n-1} &= (\mathrm{U}_{n+1}-\mathrm{U}_{n-1})+(\mathrm{U}_{n+2}-\mathrm{U}_{n})\varepsilon_1+(\mathrm{U}_{n+3}-\mathrm{U}_{n+1})\varepsilon_2+(\mathrm{U}_{n+4}-\mathrm{U}_{n+2})\varepsilon_1\varepsilon_2\\
&= (\mathrm{T}_{n}+\mathrm{T}_{n-1}-\mathrm{T}_{n-2}-\mathrm{T}_{n-3})+(\mathrm{T}_{n+1}+\mathrm{T}_{n}-\mathrm{T}_{n-1}-\mathrm{T}_{n-2})\varepsilon_1\\
&+(\mathrm{T}_{n+2}+\mathrm{T}_{n+1}-\mathrm{T}_{n}-\mathrm{T}_{n-1})\varepsilon_2+(\mathrm{T}_{n+3}+\mathrm{T}_{n+2}-\mathrm{T}_{n+1}-\mathrm{T}_{n})\varepsilon_1\varepsilon_2\\
&= (\mathrm{T}_{n}+\mathrm{T}_{n-4})+(\mathrm{T}_{n+1}+\mathrm{T}_{n-3})\varepsilon_1+(\mathrm{T}_{n+2}+\mathrm{T}_{n-2})\varepsilon_2+(\mathrm{T}_{n+3}+\mathrm{T}_{n-1})\varepsilon_1\varepsilon_2\\
&= \mathrm{HT}_{n}+\mathrm{HT}_{n-4}\\
&= 2\mathrm{HT}_{n-1}.
\end{aligned}$$

Then,

$$(\mathrm{HU}_{n+1}-\mathrm{HU}_{n-1})^2 = 4\mathrm{HT}_{n-1}^2,$$

$$\mathrm{HU}_{n+1}^2-2\mathrm{HU}_{n+1}\mathrm{HU}_{n-1}+\mathrm{HU}_{n-1}^2 = 4\mathrm{HT}_{n-1}^2,$$

therefore

$$\begin{aligned}
\mathrm{HU}_{n+1}^2+\mathrm{HU}_{n-1}^2 &= 4\mathrm{HT}_{n-1}^2+2(\mathcal{T}_n^2-\mathrm{HT}_{n-1}^2)\\
&= 2(\mathrm{HT}_n^2+\mathrm{HT}_{n-1}^2).
\end{aligned}$$

□

Theorem 2.10. *For* HT_n, HL_n, HR_n *defined in* (2.15) *and* HU_n *defined in* (2.16), *then the following summation formulas hold for any positive integer* n

$$\begin{aligned}
\sum_{k=0}^{n}\mathrm{HU}_k &= \mathrm{HT}_{n+1}-(1+\varepsilon_1+\varepsilon_2+2\varepsilon_1\varepsilon_2),\\
\sum_{k=1}^{n}\mathrm{HL}_k &= 2\mathrm{HU}_{n+2}+\mathrm{HU}_n-(3+4\varepsilon_1+7\varepsilon_2+14\varepsilon_1\varepsilon_2),\\
\sum_{k=0}^{n}\mathrm{HT}_k &= \frac{1}{2}(\mathrm{HU}_{n+2}+\mathrm{HU}_{n+1}-(1+\varepsilon_1+3\varepsilon_2+5\varepsilon_1\varepsilon_2)),\\
\sum_{k=0}^{n}\mathrm{HR}_k &= \frac{1}{2}(3\mathrm{HU}_{n+3}+2\mathrm{HU}_{n+2}-\mathrm{HU}_{n+1}-(2+8\varepsilon_1+12\varepsilon_2+22\varepsilon_1\varepsilon_2)).
\end{aligned}$$

Proof. This identities can be proved by induction. □

3 Generalized sum formulas for hyper dual tribonacci and hyper dual tribonacci-Lucas numbers

Lemma 3.1 ([16]). *Let* $f(k)$ *be any real sequence then for any integer* m *and* j, *we have*

$$\sum_{k=1}^{n}[f(m(k+j))-f(m(k-j))] = \sum_{k=n-1+j}^{n+j} f(mk)-\sum_{k=1-j}^{j} f(mk), \tag{3.1}$$

and

$$\sum_{k=1}^{n}(-1)^{k-1}[f(m(k+j))-f(m(k-j))] = \sum_{k=n+1-j}^{n+j}(-1)^{k+j-1}f(mk)-\sum_{k=1-j}^{j}(-1)^{k+j-1}f(mk). \tag{3.2}$$

Theorem 3.2. *For any integers m and r, the generalized sum of first n terms of tribonacci numbers T_n is*

$$\sum_{k=1}^{n}\mathrm{T}_{mk+r}=\frac{1}{\mathrm{L}_m-\mathrm{C}_m}[\mathrm{T}_{m(n+1)+r}+\mathrm{T}_{m(n-1)+r}+(\mathrm{C}_m-1)(\mathrm{T}_r-\mathrm{T}_{mn+r})-\mathrm{T}_{m+r}-\mathrm{T}_{-m+r}]. \quad (3.3)$$

Proof. From using (3.1) for $f(k)=\mathrm{T}_{k+r}$ and $j=1$ we get

$$\sum_{k=1}^{n}[\mathrm{T}_{m(k+1)+r}-\mathrm{T}_{m(k-1)+r}]=\mathrm{T}_{mn+r}+\mathrm{T}_{m(n+1)+r}-\mathrm{T}_r-\mathrm{T}_{m+r}.$$

If we replace n by m and k by $mk+r$ in (2.7) then we get

$$\mathrm{T}_{m(k+1)+r}-\mathrm{T}_{m(k-1)+r}=\mathrm{T}_{mk+r}\mathrm{L}_m-(1+\mathrm{C}_m)\mathrm{T}_{m(k-1)+r}+\mathrm{T}_{m(k-2)+r},$$

therefore

$$\mathrm{L}_m\sum_{k=1}^{n}\mathrm{T}_{mk+r}-(1+\mathrm{C}_m)\sum_{k=1}^{n}\mathrm{T}_{m(k-1)+r}+\sum_{k=1}^{n}\mathrm{T}_{m(k-2)+r}=\mathrm{T}_{mn+r}+\mathrm{T}_{m(n+1)+r}-\mathrm{T}_r-\mathrm{T}_{m+r}.$$

From using

$$\begin{aligned}\sum_{k=1}^{n}\mathrm{T}_{m(k-1)+r}&=\mathrm{T}_r-\mathrm{T}_{mn+r}+\sum_{k=1}^{n}\mathrm{T}_{mk+r},\\ \sum_{k=1}^{n}\mathrm{T}_{m(k-2)+r}&=\mathrm{T}_{-m+r}+\mathrm{T}_r-\mathrm{T}_{m(n-1)+r}-\mathrm{T}_{mn+r}+\sum_{k=1}^{n}\mathrm{T}_{mk+r},\end{aligned}$$

we get the result as

$$\sum_{k=1}^{n}\mathrm{T}_{mk+r}=\frac{1}{\mathrm{L}_m-\mathrm{C}_m}[\mathrm{T}_{m(n+1)+r}+\mathrm{T}_{m(n-1)+r}+(\mathrm{C}_m-1)(\mathrm{T}_r-\mathrm{T}_{mn+r})-\mathrm{T}_{m+r}-\mathrm{T}_{-m+r}].$$

□

Corollary 3.3. *For k^{th} tribonacci number T_k and positive integer n, we have*

$$\begin{aligned}\sum_{k=1}^{n}\mathrm{T}_k&=\frac{1}{2}(\mathrm{T}_{n+2}+\mathrm{T}_n-1)\\ \sum_{k=1}^{n}\mathrm{T}_{2k}&=\frac{1}{2}(\mathrm{T}_{2n+1}+\mathrm{T}_{2n}-1)\\ \sum_{k=1}^{n}\mathrm{T}_{2k+1}&=\frac{1}{2}(\mathrm{T}_{2n+2}+\mathrm{T}_{2n+1}-2)\\ \sum_{k=1}^{n}\mathrm{T}_{3k}&=\frac{1}{2}(\mathrm{T}_{3n+1}+\mathrm{T}_{3n-1}-1)\end{aligned}$$

Theorem 3.4. *For any integers m and r, the generalized sum of tribonacci numbers T_n with alternative signs is*

$$\begin{aligned}\sum_{k=1}^{n}(-1)^{k-1}\mathrm{T}_{mk+r}=\frac{1}{\mathrm{L}_m+\mathrm{C}_m+2}&[(-1)^{n+1}(\mathrm{T}_{m(n+1)+r}-\mathrm{T}_{m(n-1)+r}+(1+\mathrm{C}_m)\mathrm{T}_{mn+r})\\ &+(1+\mathrm{C}_m)\mathrm{T}_r+\mathrm{T}_{m+r}-\mathrm{T}_{-m+r}]. \quad (3.4)\end{aligned}$$

Proof. Set $f(k) = T_{k+r}$ and $j = 1$ in (3.2), then we get

$$\sum_{k=1}^{n}(-1)^{k-1}(T_{m(k+1)+r} - T_{m(k-1)+r}) = (-1)^n T_{mn+r} + (-1)^{n+1} T_{m(n+1)+r} - T_r + T_{m+r}.$$

Let replace n by m and k by $mk + r$ in (2.7), then we have

$$T_{m(k+1)+r} - T_{m(k-1)+r} = T_{mk+r} L_m - (1 + C_m) T_{m(k-1)+r} + T_{m(k-2)+r},$$

therefore

$$\begin{aligned} L_m \textstyle\sum_{k=1}^{n}(-1)^{k-1}T_{mk+r} - (1+C_m)\sum_{k=1}^{n}(-1)^{k-1}T_{m(k-1)+r} + \sum_{k=1}^{n}(-1)^{k-1}T_{m(k-2)+r} \\ = (-1)^n T_{mn+r} + (-1)^{n+1} T_{m(n+1)+r} - T_r + T_{m+r}, \end{aligned}$$

by using of

$$\begin{aligned} \sum_{k=1}^{n}(-1)^{k-1}T_{m(k-1)+r} &= T_r - (-1)^n T_{mn+r} - \sum_{k=1}^{n}(-1)^{k-1}T_{mk+r}, \\ \sum_{k=1}^{n}(-1)^{k-1}T_{m(k-2)+r} &= T_{-m+r} - T_r - (-1)^n T_{m(n-1)+r} - (-1)^{n+1}T_{mn+r} + \sum_{k=1}^{n}(-1)^{k-1}T_{mk+r}, \end{aligned}$$

we get the result as

$$\begin{aligned} \sum_{k=1}^{n}(-1)^{k-1}T_{mk+r} = \frac{1}{L_m + C_m + 2}[(-1)^{n+1}(T_{m(n+1)+r} - T_{m(n-1)+r} + (1+C_m)T_{mn+r}) \\ + (1+C_m)T_r + T_{m+r} - T_{-m+r}]. \end{aligned}$$

□

Corollary 3.5. *Let T_k be the k^{th} Tribonacci number, then for positive integer n we have*

$$\begin{aligned} \sum_{k=1}^{n}(-1)^{k-1}T_k &= \frac{1}{2}((-1)^{n+1}(T_n + T_{n-2}) + 1), \\ \sum_{k=1}^{n}(-1)^{k-1}T_{2k} &= \frac{(-1)^{n+1}}{2}(T_{2n} + T_{2n-1}), \\ \sum_{k=1}^{n}(-1)^{k-1}T_{2k+1} &= \frac{1}{2}((-1)^{n+1}(T_{2n+1} + T_{2n}) + 1), \\ \sum_{k=1}^{n}(-1)^{k-1}T_{3k} &= \frac{1}{14}\left((-1)^{n+1}(T_{3n+3} + T_{3n+1} + 4T_{3n}) + 3\right). \end{aligned}$$

Theorem 3.6. *For any integer m and r, the generalized sum of tribonacci-Lucas numbers L_n is*

$$\sum_{k=1}^{n} L_{mk+r} = \frac{1}{L_m - C_m}[L_{m(n+1)+r} + L_{m(n-1)+r} + (C_m - 1)(L_r - L_{mn+r}) - L_{m+r} - L_{-m+r}]. \quad (3.5)$$

Proof. This proof can be done by the same method as Theorem 3.2 and using (2.8). □

Corollary 3.7. *Let L_k be the k^{th} tribonacci-Lucas number, then for positive integer n we have*

$$\sum_{k=1}^{n} L_k = \frac{1}{2}(L_{n+2} + L_n - 6),$$
$$\sum_{k=1}^{n} L_{2k} = \frac{1}{2}(L_{2n+1} + L_{2n} - 4),$$
$$\sum_{k=1}^{n} L_{2k+1} = \frac{1}{2}(L_{2n+2} + L_{2n+1} - 4),$$
$$\sum_{k=1}^{n} L_{3k} = \frac{1}{2}(L_{3n+1} + L_{3n-1}).$$

Theorem 3.8. *For any integers m and r, the generalized sum of tribonacci-Lucas numbers L_n with alternative signs is*

$$\sum_{k=1}^{n}(-1)^{k-1}L_{mk+r} = \frac{1}{L_m + C_m + 2}[(-1)^{n+1}(L_{m(n+1)+r} - L_{m(n-1)+r} + (1 + C_m)L_{mn+r}) + (1 + C_m)L_r + L_{m+r} - L_{-m+r}]. \tag{3.6}$$

Proof. The proof can be done as the proof of Theorem 3.4 and using (2.8). □

Corollary 3.9. *For the k^{th} tribonacci-Lucas number L_k and positive integer n we have*

$$\sum_{k=1}^{n}(-1)^{k-1}L_k = \frac{1}{2}((-1)^{n+1}(L_{n+1} - L_{n-1}) + 2),$$
$$\sum_{k=1}^{n}(-1)^{k-1}L_{2k} = \frac{1}{2}((-1)^{n+1}(L_{2n} + L_{2n-1}) + 2),$$
$$\sum_{k=1}^{n}(-1)^{k-1}L_{2k+1} = \frac{1}{2}[(-1)^{n+1}(L_{2n+1} + L_{2n}) + 4],$$
$$\sum_{k=1}^{n}(-1)^{k-1}L_{3k} = \frac{1}{14}((-1)^{n+1}(L_{3n+2} + 9L_{3n}) + 20).$$

Theorem 3.10. *For any integers m and p, the generalized sum of hyper dual triboancci numbers HT_n is*

$$\sum_{k=1}^{n} HT_{mk+p} = \frac{1}{L_m - C_m}[HT_{m(n+1)+p} + HT_{m(n-1)+p} + (C_m - 1)(HT_p - HT_{mn+p}) - HT_{m+p} - HT_{-m+p}]. \tag{3.7}$$

Proof. From using (2.1) and (3.3) we have

$$\sum_{k=1}^{n} HT_{mk+p} = \sum_{k=1}^{n} T_{mk+p} + \sum_{k=1}^{n} T_{mk+p+1}\varepsilon_1 + \sum_{k=1}^{n} T_{mk+p+2}\varepsilon_2 + \sum_{k=1}^{n} T_{mk+p+3}\varepsilon_1\varepsilon_2$$
$$= \frac{1}{L_m - C_m}[T_{m(n+1)+p} + T_{m(n-1)+p} + (C_m - 1)(T_p - T_{mn+p}) - T_{m+p} - T_{-m+p}]$$
$$+ \frac{1}{L_m - C_m}[T_{m(n+1)+p+1} + T_{m(n-1)+p+1} + (C_m - 1)(T_{p+1} - T_{mn+p+1}) - T_{m+p+1} - T_{-m+p+1}]\varepsilon_1$$
$$+ \frac{1}{L_m - C_m}[T_{m(n+1)+p+2} + T_{m(n-1)+p+2} + (C_m - 1)(T_{p+2} - T_{mn+p+2}) - T_{m+p+2} - T_{-m+p+2}]\varepsilon_2$$
$$+ \frac{1}{L_m - C_m}[T_{m(n+1)+p+3} + T_{m(n-1)+p+3} + (C_m - 1)(T_{p+3} - T_{mn+p+3}) - T_{m+p+3} - T_{-m+p+3}]\varepsilon_1\varepsilon_2,$$

therefore

$$\sum_{k=1}^{n} \mathrm{HT}_{mk+p} = \frac{1}{\mathrm{L}_m - \mathrm{C}_m}[\mathrm{HT}_{m(n+1)+p} + \mathrm{HT}_{m(n-1)+p} + (\mathrm{C}_m - 1)(\mathrm{HT}_p - \mathrm{HT}_{mn+p}) - \mathrm{HT}_{m+p} - \mathrm{HT}_{-m+p}].$$

□

Corollary 3.11. *Let* HT_k *be* k^{th} *hyper dual tribonacci number and n be any positive integer, then we have*

$$\sum_{k=1}^{n} \mathrm{HT}_k = \frac{1}{2}(\mathrm{HT}_{n+2} + \mathrm{HT}_n - (1 + 3\varepsilon_1 + 5\varepsilon_2 + 9\varepsilon_1\varepsilon_2)),$$
$$\sum_{k=1}^{n} \mathrm{HT}_{2k} = \frac{1}{2}(\mathrm{HT}_{2n+1} + \mathrm{HT}_{2n} - (1 + 2\varepsilon_1 + 3\varepsilon_2 + 6\varepsilon_1\varepsilon_2)),$$
$$\sum_{k=1}^{n} \mathrm{HT}_{2k+1} = \frac{1}{2}(\mathrm{HT}_{2n+2} + \mathrm{HT}_{2n+1} - (2 + 3\varepsilon_1 + 6\varepsilon_2 + 11\varepsilon_1\varepsilon_2)),$$
$$\sum_{k=1}^{n} \mathrm{HT}_{3k} = \frac{1}{2}(\mathrm{HT}_{3n+1} + \mathrm{HT}_{3n-1} - (1 + \varepsilon_1 + 3\varepsilon_2 + 5\varepsilon_1\varepsilon_2)).$$

Theorem 3.12. *For any integers m and p, the generalized sum of hyper dual tribonacci numbers* HT_n *with alternative signs is*

$$\sum_{k=1}^{n}(-1)^{k-1}\mathrm{HT}_{mk+p} = \frac{1}{\mathrm{L}_m + \mathrm{C}_m + 2}\left[(-1)^{n+1}(\mathrm{HT}_{m(n+1)+p} - \mathrm{HT}_{m(n-1)+p} + (1 + \mathrm{C}_m)\mathrm{HT}_{mn+p}) + (1 + \mathrm{C}_m)\mathrm{HT}_p + \mathrm{HT}_{m+p} - \mathrm{HT}_{-m+p}\right]. \tag{3.8}$$

Proof. By using (2.1) and (3.4) we get

$$\begin{aligned}
&\sum_{k=1}^{n}(-1)^{k-1}\mathrm{HT}_{mk+p}\\
&= \frac{1}{\mathrm{L}_m + \mathrm{C}_m + 2}[(-1)^{n+1}(\mathrm{T}_{m(n+1)+p} - \mathrm{T}_{m(n-1)+p} + (1 + \mathrm{C}_m)\mathrm{T}_{mn+p}) + (1 + \mathrm{C}_m)\mathrm{T}_p + \mathrm{T}_{m+p} - \mathrm{T}_{-m+p}]\\
&+ \varepsilon_1\frac{1}{\mathrm{L}_m + \mathrm{C}_m + 2}[(-1)^{n+1}(\mathrm{T}_{m(n+1)+p+1} - \mathrm{T}_{m(n-1)+p+1} + (1 + \mathrm{C}_m)\mathrm{T}_{mn+p+1})\\
&+ (1 + \mathrm{C}_m)\mathrm{T}_{p+1} + \mathrm{T}_{m+p+1} - \mathrm{T}_{-m+p+1}]\\
&+ \varepsilon_2\frac{1}{\mathrm{L}_m + \mathrm{C}_m + 2}[(-1)^{n+1}(\mathrm{T}_{m(n+1)+p+2} - \mathrm{T}_{m(n-1)+p+2} + (1 + \mathrm{C}_m)\mathrm{T}_{mn+p+2})\\
&+ (1 + \mathrm{C}_m)\mathrm{T}_{p+2} + \mathrm{T}_{m+p+2} - \mathrm{T}_{-m+p+2}]\\
&+ \varepsilon_1\varepsilon_2\frac{1}{\mathrm{L}_m + \mathrm{C}_m + 2}[(-1)^{n+1}(\mathrm{T}_{m(n+1)+p+3} - \mathrm{T}_{m(n-1)+p+3} + (1 + \mathrm{C}_m)\mathrm{T}_{mn+p+3})\\
&+ (1 + \mathrm{C}_m)\mathrm{T}_{p+3} + \mathrm{T}_{m+p+3} - \mathrm{T}_{-m+p+3}],
\end{aligned}$$

therefore

$$\sum_{k=1}^{n}(-1)^{k-1}\mathrm{HT}_{mk+p} = \frac{1}{\mathrm{L}_m + \mathrm{C}_m + 2}[(-1)^{n+1}(\mathrm{HT}_{m(n+1)+p} - \mathrm{HT}_{m(n-1)+p} + (1 + \mathrm{C}_m)\mathrm{HT}_{mn+p}) + (1 + \mathrm{C}_m)\mathrm{HT}_p + \mathrm{HT}_{m+p} - \mathrm{HT}_{-m+p}].$$

□

Corollary 3.13. *Let* HT_k *be* k^{th} *hyper dual tribonacci number and* n *be any positive integer, then we have*

$$\sum_{k=1}^{n}(-1)^{k-1}\mathrm{HT}_k = \frac{1}{2}((-1)^{n+1}(\mathrm{HT}_{n+1}-\mathrm{HT}_{n-1})+(1+\varepsilon_1+\varepsilon_2+3\varepsilon_1\varepsilon_2)),$$
$$\sum_{k=1}^{n}(-1)^{k-1}\mathrm{HT}_{2k} = \frac{1}{2}((-1)^{n+1}(\mathrm{HT}_{2n}+\mathrm{HT}_{2n-1})+(\varepsilon_1+2\varepsilon_2+3\varepsilon_1\varepsilon_2)),$$
$$\sum_{k=1}^{n}(-1)^{k-1}\mathrm{HT}_{2k+1} = \frac{1}{2}((-1)^{n+1}(\mathrm{HT}_{2n+1}+\mathrm{HT}_{2n})+(1+2\varepsilon_1+3\varepsilon_2+6\varepsilon_1\varepsilon_2)),$$
$$\sum_{k=1}^{n}(-1)^{k-1}\mathrm{HT}_{3k} = \frac{1}{14}((-1)^{n+1}(\mathrm{HT}_{3n+2}+2\mathrm{HT}_{3n+1}+5\mathrm{HT}_{3n})+(3+9\varepsilon_1+13\varepsilon_2+25\varepsilon_1\varepsilon_2)).$$

Theorem 3.14. *For any integers* m *and* p*, the generalized sum of hyper dual tribonacci-Lucas numbers* HL_n *is*

$$\sum_{k=1}^{n}\mathrm{HL}_{mk+p} = \frac{1}{\mathrm{L}_m-\mathrm{C}_m}[\mathrm{HL}_{m(n+1)+p}+\mathrm{HL}_{m(n-1)+p}+(\mathrm{C}_m-1)(\mathrm{HL}_p-\mathrm{HL}_{mn+p})-\mathrm{HL}_{m+p}-\mathrm{HL}_{-m+p}]. \tag{3.9}$$

Proof. The proof can be done similar to Theorem 3.10 by using (2.2) and (3.5). □

Corollary 3.15. *Let* HL_k *be* k^{th} *hyper dual tribonacci-Lucas number and* n *be any positive integer, then we have*

$$\sum_{k=1}^{n}\mathrm{HL}_k = \frac{1}{2}(\mathrm{HL}_{n+2}+\mathrm{HL}_n)-(3+4\varepsilon_1+7\varepsilon_2+14\varepsilon_1\varepsilon_2),$$
$$\sum_{k=1}^{n}\mathrm{HL}_{2k} = \frac{1}{2}(\mathrm{HL}_{2n+1}+\mathrm{HL}_{2n})-(2+2\varepsilon_1+5\varepsilon_2+9\varepsilon_1\varepsilon_2),$$
$$\sum_{k=1}^{n}\mathrm{HL}_{2k+1} = \frac{1}{2}(\mathrm{HL}_{2n+2}+\mathrm{HL}_{2n+1})-(2+5\varepsilon_1+9\varepsilon_2+16\varepsilon_1\varepsilon_2),$$
$$\sum_{k=1}^{n}\mathrm{HL}_{3k} = \frac{1}{2}(\mathrm{HL}_{3n+1}+\mathrm{HL}_{3n-1})-(3\varepsilon_1+4\varepsilon_2+7\varepsilon_1\varepsilon_2).$$

Theorem 3.16. *For any integers* m *and* p *the generalized sum of hyper dual tribonacci-Lucas numbers* HL_n *with alternative signs is*

$$\sum_{k=1}^{n}(-1)^{k-1}\mathrm{HL}_{mk+p} = \frac{1}{\mathrm{L}_m+\mathrm{C}_m+2}[(-1)^{n+1}(\mathrm{HL}_{m(n+1)+p}-\mathrm{HL}_{m(n-1)+p}+(1+\mathrm{C}_m)\mathrm{HL}_{mn+p})$$
$$+(1+\mathrm{C}_m)\mathrm{HL}_p+\mathrm{HL}_{m+p}-\mathrm{HL}_{-m+p}]. \tag{3.10}$$

Proof. The proof can be done by using (2.2) and (3.6) similar to Theorem 3.12. □

Corollary 3.17. *Let* HL_k *be* k^{th} *hyper dual tribonacci-Lucas number and* n *be any positive integer,* *then we have*

$$\sum_{k=1}^{n}(-1)^{k-1}\mathrm{HL}_k = \frac{(-1)^{n+1}}{2}(\mathrm{HL}_{n+1} - \mathrm{HL}_{n-1}) + (1 + 3\varepsilon_2 + 4\varepsilon_1\varepsilon_2),$$

$$\sum_{k=1}^{n}(-1)^{k-1}\mathrm{HL}_{2k} = \frac{(-1)^{n+1}}{2}(\mathrm{HL}_{2n} + \mathrm{HL}_{2n-1}) + (1 + 2\varepsilon_1 + 2\varepsilon_2 + 5\varepsilon_1\varepsilon_2),$$

$$\sum_{k=1}^{n}(-1)^{k-1}\mathrm{HL}_{2k+1} = \frac{(-1)^{n+1}}{2}(\mathrm{HL}_{2n+1} + \mathrm{HL}_{2n}) + (2 + 2\varepsilon_1 + 5\varepsilon_2 + 9\varepsilon_1\varepsilon_2),$$

$$\sum_{k=1}^{n}(-1)^{k-1}\mathrm{HL}_{3k} = \frac{(-1)^{n+1}}{14}(\mathrm{HL}_{3n+2} + 2\mathrm{HL}_{3n+1} + 5\mathrm{HL}_{3n}) + \frac{1}{7}(10 + 9\varepsilon_1 + 20\varepsilon_2 + 39\varepsilon_1\varepsilon_2).$$

4 Conclusions

Hyper-dual Tribonacci and hyper-dual Tribonacci-Lucas numbers are hyper-dual numbers whose components are Tribonacci and Tribonacci-Lucas numbers, respectively. We derive several important formulas and identities for these new types of numbers. Another main focus of this paper is the investigation of formulas for generalized summation and generalized summation with alternating signs for these number sequences using new methods.

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Conflict of interest

The authors have no conflicts of interest to declare.

References

[1] K. Adegoke, A. Olatinwo, W. Oyekanmi, *New Tribonacci recurrence relations and addition formulas*, Notes on Number Theory and Discrete Mathematics, **26**(4) (2020), 164–172. https://doi.org/10.7546/nntdm.2020.26.4.164-172

[2] O.P. Agrawal, *Hamilton operators and dual-number-quaternions in spatial kinematics*, Mechanism and Machine Theory, **22**(6) (1987), 569–575. https://doi.org/10.1016/0094-114X(87)90019-6

[3] I. Akkus, G. Kizilaslan, *On some properties of Tribonacci quaternions*, Analele tiinifice ale Universitii Ovidius Constana - Seria Matematic, **26**(3) (2018), 5–20. https://doi.org/10.2478/auom-2018-0044

[4] P. Anantakitpaisal, K. Kuhapatanakul, *Reciprocal sums of the Tribonacci numbers*, Journal of Integer Sequences, **19** (2016), Article 16.2.1. https://cs.uwaterloo.ca/journals/JIS/VOL19/Kuhapatanakul/kuhapat2.pdf

[5] S. Aslan, *Kinematic applications of hyper-dual numbers*, International Electronic Journal of Geometry, **14**(2) (2021), 292–304. https://doi.org/10.36890/iejg.884998

[6] S. Aslan, M. Bekar, Y. Yayli, *Ruled surfaces corresponding to hyper-dual curves*, Hacettepe Journal of Mathematics and Statistics, **51**(1) (2022), 187–198. https://doi.org/10.15672/hujms.739628

[7] M. Catalani, *Identities for Tribonacci-related sequences*, arXiv preprint (2002). https://arxiv.org/pdf/math/0209179.pdf

[8] A. Cohen, M. Shoham, *Application of hyper-dual numbers to multi-body kinematics*, Journal of Mechanisms and Robotics, **8** (2015). https://doi.org/10.1115/1.4030588

[9] A. Cohen, M. Shoham, *Application of hyper-dual numbers to rigid bodies equations of motion*, Mechanism and Machine Theory, **111** (2017), 76–84. https://doi.org/10.1016/j.mechmachtheory.2017.01.012

[10] A. Cohen, M. Shoham, *Principle of transferenceAn extension to hyper-dual numbers*, Mechanism and Machine Theory, **125** (2018), 101–110. https://doi.org/10.1016/j.mechmachtheory.2018.03.014

[11] M. Feinberg, *Fibonacci-Tribonacci*, The Fibonacci Quarterly, **1** (1963), 71–74. https://www.mathstat.dal.ca/FQ/Scanned/1-3/feinberg.pdf

[12] S. Feng, *More identities on the Tribonacci numbers*, Ars Combinatoria (2011). https://www.researchgate.net/publication/285584356

[13] J. A. Fike, *Numerically exact derivative calculations using hyper-dual numbers*, 3rd Annual Student Joint Workshop in Simulation-Based Engineering and Design (2009). https://www.researchgate.net/publication/228961292

[14] J.A. Fike, J.J. Alonso, *The development of hyper-dual numbers for exact second-derivative calculations*, Proceedings of the 49th AIAA Aerospace Sciences Meeting including the New Horizons Forum and Aerospace Exposition (2011), 4–7. https://doi.org/10.2514/6.2011-886

[15] J.A. Fike, J.J. Alonso, *Automatic differentiation through the use of hyper-dual numbers for second derivatives*, Lecture Notes in Computational Science and Engineering, **87** (2011), 163–173. https://doi.org/10.1007/978-3-642-15391-7-8

[16] R. Frontczak, *Sums of Tribonacci and Tribonacci-Lucas numbers*, International Journal of Mathematical Analysis, **12**(1) (2018), 19–24.

[17] A.F. Horadam, *Complex Fibonacci numbers and Fibonacci quaternions*, The American Mathematical Monthly, **70** (1963), 289–291.

[18] A.F. HORADAM, *A generalized Fibonacci sequence*, The American Mathematical Monthly, **68**(5) (1961), 455–459.

[19] E. KILIÇ, *Tribonacci sequences with certain indices and their sum*, Ars Combinatoria, **86** (2008), 13–22.

[20] T. KOSHY, *Fibonacci and Lucas Numbers with Applications*, Wiley-Interscience, New York, (2001).

[21] A.P. KOTELNIKOV, *Screw calculus and some applications to geometry and mechanics*, Annales de lUniversité Impériale de Kazan, Russia, (1895).

[22] J. LI, Z. JIANG, F. LIU, *Determinants, norms, and the spread of circulant matrices with Tribonacci and generalized Lucas numbers*, Abstract and Applied Analysis, **2014**, Article ID 381829, 9 pp.

[23] P.Y. LIN, *De Moivre-type identities for the Tribonacci numbers*, The Fibonacci Quarterly, **26** (1988), 131–134.

[24] B. ONEILL, *Elementary Differential Geometry*, Revised 2nd ed., Academic Press, USA, (2006).

[25] S. PETHE, *Some identities for Tribonacci sequences*, The Fibonacci Quarterly, **26**(2) (1988), 144–151.

[26] A. SCOTT, T. DELANEY, V. HOGGATT JR., *The Tribonacci sequence*, The Fibonacci Quarterly, **15**(3) (1977), 193–200.

[27] E. STUDY, *Geometry der Dynamen*, Leipzig, (1901).

[28] Y. TAYURDU, *On the sums of Tribonacci and Tribonacci-Lucas numbers*, Applied Mathematical Sciences, **13**(24) (2019), 1201–1208.

[29] G.R. VELDKAMP, *On the use of dual numbers, vectors and matrices in instantaneous, spatial kinematics*, Mechanism and Machine Theory, **11**(2) (1976), 141–156.

[30] C.C. YALAVIGI, *Properties of Tribonacci numbers*, The Fibonacci Quarterly, **10**(3) (1972), 231–246.