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Some spectral problems of a diffusion operator under Paley-Wiener-based high-order approximations

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Abstract. In this study, we acquired spectral results for the diffusion operator under higher-order approximations. We reconstruct the well-known techniques and derive the essential results for the presented problem. The spectral results for the diffusion operator with high-order approximations were evaluated, focusing on solutions in the Paley-Wiener space. Additionally, we consider theorems that involve solutions belonging to the Paley-Wiener space and the applications of Shannon's sampling theorem. We also examine and evaluate the diffusion operator under more general separable boundary conditions.

Keywords: Diffusion operator, Sampling theory, Shannon's sampling theorem.

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1 Introduction

In this study, we consider the following diffusion equation

$$-y'' + [q(x) + 2\lambda p(x)]\, y = \lambda^2 y, \quad x \in [0,\pi], \tag{1.1}$$

where the function $q(x) \in L^2[0,\pi]$, $p(x) \in L^2[0,\pi]$. Note that several spectral problems have been extensively analyzed for the diffusion operator in [1,2,10,11].

We focus on the following problem

$$-y'' + [q(x) + 2\lambda p(x)]\, y = \lambda^2 y, \tag{1.2}$$

$$y(0) = 1, \quad y'(0) = -h, \tag{1.3}$$

where h is a finite number. Let us indicate by $\varphi(x,\lambda)$ the solution of (1.2) satisfying the initial conditions (1.3). Following [10], let

$$\varphi(x,\lambda) = \cos[\lambda x - \tilde{\alpha}(x)] + \int_0^x M(x,\tau)\cos\lambda\tau\, d\tau + \int_0^x N(x,\tau)\sin\lambda\tau\, d\tau, \tag{1.4}$$

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where

$$\tilde{\alpha}(x) = x.p(0) + 2\int_0^x \{M(\zeta,\zeta)\sin\tilde{\alpha}(\zeta) - N(\zeta,\zeta)\cos\tilde{\alpha}(\zeta)\}\,d\zeta,$$
$$q(x) = -p^2(x) + 2\frac{d}{dx}\{M(x,x)\cos\tilde{\alpha}(x) + N(x,x)\sin\tilde{\alpha}(x)\}, \tag{1.5}$$
$$M(0,0) = h,\; N(x,0) = 0,\; \left.\frac{\partial M(x,\tau)}{\partial\tau}\right|_{\tau=0} = 0,\; \tilde{\alpha}(x) = \int_0^x p(\tau)\,d\tau,$$

and the nth eigenvalue is

$$\lambda_n = n + c_0 + \frac{c_1}{n} + \frac{c_{1,n}}{n}, \tag{1.6}$$

where

$$c_0 = \frac{1}{\pi}\int_0^\pi p(x)\,dx, \quad \sum_n |c_{1,n}|^2 < \infty,$$
$$c_1 = \frac{1}{\pi}\left(h + H + \frac{1}{2}\int_0^\pi \left[q(x) + p^2(x)\right]dx\right), \tag{1.7}$$

and H is a finite number.

In this work, we focus on and evaluate the diffusion equation under more general separable boundary conditions

$$-\psi'' + [q(x) + 2\lambda p(x)]\,\psi = \lambda^2\psi, \quad x \in [0,\pi],$$

$$a_{11}\psi(0,\lambda) - a_{12}\psi'(0,\lambda) = 0,$$
$$a_{21}\psi(\pi,\lambda) + a_{22}\psi'(\pi,\lambda) = 0. \tag{1.8}$$

where $a_{11}^2 + a_{12}^2 \neq 0$, $a_{21}^2 + a_{22}^2 \neq 0$. Also, let $\lambda = \mu^2$ and $\psi(x,\mu^2)$ denotes the solution of the following initial value problem

$$-\psi'' + \left[q(x) + 2\mu^2 p(x)\right]\psi = \mu^4\psi,$$
$$\psi(0,\mu^2) = a_{12}, \quad \psi'(0,\mu^2) = a_{11}.$$

Furthermore, the eigenvalues of (1.8) are the squares of the zeroes of the boundary function $B(\mu)$,

$$B(\mu) := a_{21}\psi(\pi,\mu^2) + a_{22}\psi'(\pi,\mu^2).$$

Additionally, the mentioned boundary function is an entire function that belongs to μ in the case of Dirichlet. This function is of type π and order 1. In addition, it belongs to the Paley-Wiener space in the following form

$$PW_\pi = \left\{f \text{ is entire},\; |f(\mu)| \leqslant c\exp[\pi\,|\operatorname{Im}\mu|],\; f \in L^2(\mathbb{R})\right\}.$$

2 Main results and discussions under high-order approximations

Following the work in [10], we define the function $y(x,\mu^2)$ as:

$$y(x,\mu^2) = \cos\left[\mu^2 x - \tilde{\alpha}(x)\right] + \int_0^x M(x,t)\cos(\mu^2 t)\,dt + \int_0^x N(x,t)\sin(\mu^2 t)\,dt,$$

We now state the following results:

Theorem 2.1. *Let*

$$v_1^{[0]}(x,\mu) = y(x,\mu),$$

$$v_2^{[0]}(x,\mu) = \int_0^x \left\{ M(x,t)\cos\left(\mu^2 t\right) + N(x,t)\sin\left(\mu^2 t\right) \right\} dt - \int_0^x \left\{ N(x,t)\sin\left(\mu^2 t\right) \right\} dt, \tag{2.1}$$

$$\varphi_0(x,\mu) = \cos\left[\mu^2 x - \tilde{\alpha}(x)\right],$$

and

$$\varphi_n(x,\mu) = \int_0^x \left\{ M(x,t)\cos\left(\mu^2 t\right) + N(x,t)\sin\left(\mu^2 t\right) \right\} \varphi_{n-1}(t,\mu)\, dt,$$

$$v_1^{[n]}(x,\mu) = v_1^{[n-1]}(x,\mu) - \varphi_{n-1}(x,\mu),$$

$$v_2^{[n]}(x,\mu) = v_2^{[n-1]}(x,\mu) - \int_0^x M(x,t)\cos\left(\mu^2 t\right) \varphi_{n-1}(t,\mu) dt, \tag{2.2}$$

$$\tilde{B}^{[n]}(x,\mu) = a_{21} v_1^{[n]}(x,\mu) + a_{22} v_2^{[n]}(x,\mu),$$

for $n \geq 1$. Where upon

$$\varphi_n(x,\mu), v_1^{[n]}(x,\mu), v_2^{[n]}(x,\mu), \tilde{B}^{[n]}(x,\mu) \in PW_x,$$

for $n \geq 1$. Moreover, we have the subsequent estimates,

$$\begin{aligned} |\varphi_n(x,\mu)| &\leq (c_6)^n e^{x\left|\mathrm{Im}\mu^2\right|}, \\ \left|v_1^{[n]}(x,\mu)\right| &\leq c_3 (c_6)^n e^{x\left|\mathrm{Im}\mu^2\right|}, \\ \left|v_2^{[n]}(x,\mu)\right| &\leq c_4 (c_6)^n e^{x\left|\mathrm{Im}\mu^2\right|}, \\ \left|\tilde{B}^{[n]}(x,\mu)\right| &\leq c_5 (c_6)^n e^{x\left|\mathrm{Im}\mu^2\right|}, \end{aligned} \tag{2.3}$$

where

$$c_1 = \int_0^\pi \max_{0\leq x\leq \pi} |M(x,t)|\, dt, c_2 = \int_0^\pi \max_{0\leq x\leq \pi} |N(x,t)|\, dt,$$

$$c_3 = \exp\left(c_1 + c_0 c_2\right), \quad c_4 = c_1 c_3, \quad c_5 = |a_{21}|\, c_3 + |a_{22}|\, c_4, \quad c_6 = c_1 + c_0 c_2.$$

Proof. It is obvious that

$$\begin{aligned} v_1^{[n]}(x,\mu) &= \varphi_n(x,\mu) + \int_0^x \left\{ M(x,t)\cos\left(\mu^2 t\right) + N(x,t)\sin\left(\mu^2 t\right) \right\} v_1^{[n]}(t,\mu)\, dt, \\ v_2^{[n]}(x,\mu) &= \int_0^x M(x,t)\cos\left(\mu^2 t\right) v_1^{[n]}(t,\mu) dt, \end{aligned} \tag{2.4}$$

also, the proof is performed by induction on n.

We would like to remind that we will use the following estimates [5] to prove the estimates (2.3) for $n=0$,

$$\left|\cos u\right| \leq e^{\left|\operatorname{Im} u\right|}, \quad \left|\sin u\right| \leq c_0 e^{\left|\operatorname{Im} u\right|}, \tag{2.5}$$

where c_0 is an arbitrary constant (we could get $c_0 = 1.72$). By means of this approach we obtain,

$$\left|\varphi_0\left(x,\mu\right)\right| \leq e^{x\left|\operatorname{Im}\mu^2\right|},$$

and from (2.4), we have

$$\begin{aligned}
\left|v_1^{[0]}\left(x,\mu\right)\right| &\leq \left|\varphi_0\left(x,\mu\right)\right| + \left|\int_0^x \left\{M\left(x,t\right)\cos\left(\mu^2 t\right) + N\left(x,t\right)\sin\left(\mu^2 t\right)\right\} v_1^{[0]}\left(t,\mu\right)dt\right| \\
&\leq \left|\varphi_0\left(x,\mu\right)\right| + \int_0^x \left|M\left(x,t\right)\right|\left|\cos\left(\mu^2 t\right)\right|\left|v_1^{[0]}\left(t,\mu\right)\right|dt \\
&\quad + \int_0^x \left|N\left(x,t\right)\right|\left|\sin\left(\mu^2 t\right)\right|\left|v_1^{[0]}\left(t,\mu\right)\right|dt \\
&\leq e^{x\left|\operatorname{Im}\mu^2\right|} + e^{x\left|\operatorname{Im}\mu^2\right|}\left(\int_0^x \left|M\left(x,t\right)\right| e^{-t\left|\operatorname{Im}\mu^2\right|}\left|v_1^{[0]}\left(t,\mu\right)\right|dt\right. \\
&\quad \left. + c_0\int_0^x \left|N\left(x,t\right)\right| e^{-t\left|\operatorname{Im}\mu^2\right|}\left|v_1^{[0]}\left(t,\mu\right)\right|dt\right),
\end{aligned}$$

from which we get

$$\left|v_1^{[0]}\left(x,\mu\right)\right| e^{-x\left|\operatorname{Im}\mu^2\right|} \leq 1 + \int_0^x \left[\left|M\left(x,t\right)\right| + c_0\left|N\left(x,t\right)\right|\right] e^{-t\left|\operatorname{Im}\mu^2\right|}\left|v_1^{[0]}\left(t,\mu\right)\right|dt,$$

and using Gronwall's Lemma, yields

$$\begin{aligned}
\left|v_1^{[0]}\left(x,\mu\right)\right| &\leq e^{\int_0^\pi \max\limits_{0\leq x\leq\pi}\left(\left|M(x,t)\right|\right)dt + c_0\int_0^\pi \max\limits_{0\leq x\leq\pi}\left(\left|N(x,t)\right|\right)dt} e^{x\left|\operatorname{Im}\mu^2\right|}, \\
&\leq c_3 e^{x\left|\operatorname{Im}\mu^2\right|},
\end{aligned}$$

and

$$\begin{aligned}
\left|v_2^{[0]}\left(x,\mu\right)\right| &= \int_0^x \left|M\left(x,t\right)\right|\left|\cos\left(\mu^2 t\right)\right|\left|v_1^{[0]}\left(t,\mu\right)\right|dt, \\
&\leq e^{x\left|\operatorname{Im}\mu^2\right|} c_3 \int_0^\pi \max_{0\leq x\leq\pi}\left(\left|M\left(x,t\right)\right|\right)dt, \\
&\leq c_4 e^{x\left|\operatorname{Im}\mu^2\right|},
\end{aligned}$$

Furthermore,

$$\left|\tilde{B}^{[0]}\left(x,\mu\right)\right| \leq \left|a_{21}\right|\left|v_1^{[0]}\left(x,\mu\right)\right| + \left|a_{22}\right|\left|v_2^{[0]}\left(x,\mu\right)\right|,$$

$$\leq c_5 e^{x\left|\operatorname{Im}\mu^2\right|}.$$

Therefor, the estimates of (2.3) are true for $n = 0$.

In this part of our work, we suppose the estimates (2.3) are true for $n - 1$, and then we prove for the value of n. From equalities (2.2), we have

$$\begin{aligned}
|\varphi_n(x,\mu)| &\leq \int_0^x |M(x,t)|\left|\cos\left(\mu^2 t\right)\right| |\varphi_{n-1}(t,\mu)|\,dt \\
&\quad + \int_0^x |N(x,t)|\left|\sin\left(\mu^2 t\right)\right|\, |\varphi_{n-1}(t,\mu)|\,dt, \\
&\leq \int_0^x \{|M(x,t)| + c_0 |N(x,t)|\}\, e^{(x-t)\left|\operatorname{Im}\mu^2\right|}(c_6)^{n-1} e^{t\left|\operatorname{Im}\mu^2\right|} dt, \\
&\leq (c_6)^n e^{x\left|\operatorname{Im}\mu^2\right|},
\end{aligned}$$

and from (2.4), we acquire

$$\begin{aligned}
\left|v_1^{[n]}(x,\mu)\right| &\leq |\varphi_n(x,\mu)| + \int_0^x |M(x,t)|\left|\cos\left(\mu^2 t\right)\right|\left|v_1^{[n]}(t,\mu)\right| dt, \\
&\quad + \int_0^x |N(x,t)|\left|\sin\left(\mu^2 t\right)\right|\left|v_1^{[n]}(t,\mu)\right| dt, \\
&\leq (c_6)^n e^{x\left|\operatorname{Im}\mu^2\right|} + e^{x\left|\operatorname{Im}\mu^2\right|}\left(\int_0^x \{|M(x,t)| + c_0 |N(x,t)|\}\, e^{-t\left|\operatorname{Im}\mu^2\right|}\left|v_1^{[n]}(t,\mu)\right| dt\right),
\end{aligned}$$

so that

$$\left|v_1^{[n]}(x,\mu)\right| e^{-x\left|\operatorname{Im}\mu^2\right|} \leq (c_6)^n + \int_0^x \{|M(x,t)| + c_0 |N(x,t)|\}\, e^{-t\left|\operatorname{Im}\mu^2\right|}\left|v_1^{[n]}(t,\mu)\right| dt,$$

from which we get

$$\begin{aligned}
\left|v_1^{[n]}(x,\mu)\right| &\leq (c_6)^n e^{\int_0^\pi \max\limits_{0\leq x\leq\pi}(|M(x,t)|)dt + c_0\int_0^\pi \max\limits_{0\leq x\leq\pi}(|N(x,t)|)dt}\, e^{x\left|\operatorname{Im}\mu^2\right|}, \\
&\leq c_3 (c_6)^n e^{x\left|\operatorname{Im}\mu^2\right|},
\end{aligned}$$

and so,

$$\begin{aligned}
\left|v_2^{[n]}(x,\mu)\right| &\leq \int_0^x |M(x,t)|\left|\cos\left(\mu^2 t\right)\right|\left|v_1^{[n]}(t,\mu)\right| dt, \\
&\leq e^{x\left|\operatorname{Im}\mu^2\right|} c_3 (c_6)^n \int_0^\pi \max_{0\leq x\leq\pi}(|M(x,t)|)\,dt, \\
&\leq c_4 (c_6)^n e^{x\left|\operatorname{Im}\mu^2\right|}.
\end{aligned}$$

Moreover,

$$\left|\tilde{B}^{[n]}(x,\mu)\right| \leq |a_{21}| \left|v_1^{[n]}(x,\mu)\right| + |a_{22}| \left|v_2^{[n]}(x,\mu)\right|,$$
$$\leq c_5 (c_6)^n e^{x\left|\operatorname{Im}\mu^2\right|}.$$

Therefor, the proof is complete. The estimates (2.3) are true for all values of n. □

Theorem 2.2. *(Whittaker-Shannon-Kotel'nikov) Let $f \in PW_\pi$, then*

$$f(\mu) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin \pi(\mu - k)}{\pi(\mu - k)},$$

where the series converges uniformly on the compact subsets of $\mathbb{R}$ and also in $L^2_{d\mu}$ [15].

3 Conclusion

In the present research, we investigated the diffusion operator in detail and derived essential spectral results for the diffusion equation under high-order approximations. We considered a diffusion operator under more general separable boundary conditions. We then obtained the necessary results by modifying existing techniques for the presented problem. The applied approach is based on Shannons sampling theorem, a well-established technique in the literature. The significant results obtained are evaluated using the Paley-Wiener spaces. The mathematical framework is firmly based on spectral theory, and the proofs follow well-established approaches to Sturm-Liouville problems. These assessments demonstrate the validity and strength of the obtained results.

Declarations

Availability of data and materials

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Conflict of interest

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