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# Results in semi-E-convex functions

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**Abstract.** The concept of convexity and its various generalizations is important for quantitative and qualitative studies in operations research or applied mathematics. Recently, E-convex sets and functions were introduced with important implications across numerous branches of mathematics. By relaxing the definition of convex sets and functions, a new concept of semi-$E$-convex functions was introduced, and its properties are discussed. It has been demonstrated that if a function $f : M \to \mathbb{R}$ is semi-$E$-convex on an $E$-convex set $M \subset \mathbb{R}^n$ then, $f(E(x)) \leq f(x)$ for each $x \in M$. This article discusses the inverse of this proposition and presents some results for convex functions.

**Keywords:** Semi-E-convex functions, convex functions, Lower semi-continuous functions.

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## 1 Introduction

Youness in [5] introduced a class of sets and functions called $E$-convex sets and $E$-convex functions by relaxing the definition of convex sets and convex functions. Following this, Xiusu Chen [1] introduced a new class of semi-$E$-convex functions and applied these functions to nonlinear programming problems see for instance [3,4]. In this paper, we give weak conditions for a lower semi-continuous function on $\mathbb{R}^n$ to be a semi-$E$-convex function, we also present some results for convex functions.

## 2 Preliminaries

Let $M$ be a nonempty subset of $\mathbb{R}^n$ and let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a map. We recall:

**Definition 2.1.** [5] A set $M \subseteq \mathbb{R}^n$ is said to be $E$-convex in $\mathbb{R}^n$ if

$$tE(x) + (1-t)E(y) \in M,$$

for each $x, y \in M$ and all $t \in [0, 1]$.

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**Definition 2.2.** [5] A function $f : M \to \mathbb{R}$ is said to be $E$-convex on $M$ if $M$ is $E$-convex and

$$f\left(tE(x)+(1-t)E(y)\right) \leq tf\left(E\left(x\right)\right)+(1-t)f\left(E\left(y\right)\right),$$

for each $x, y \in M$ and all $t \in [0,1]$.

**Definition 2.3.** [1] A function $f : M \to \mathbb{R}$ is said to be semi-$E$-convex on $M$ if $M$ is $E$-convex and

$$f\left(tE(x)+(1-t)E(y)\right) \leq tf\left(x\right)+(1-t)f\left(y\right),$$

for each $x, y \in M$ and all $t \in [0,1]$.

**Definition 2.4.** [1] We define a map $E \times I$ as follows:

$$\begin{array}{rcl} E \times I : \mathbb{R}^n \times \mathbb{R} & \to & \mathbb{R}^n \times \mathbb{R} \\ (x,t) & \to & (E \times I)(x,t) = (E(x),t). \end{array}$$

This Proposition gives a characterization of a semi-$E$-convex function in term of its $epi\left(f\right)$.

**Proposition 2.5.** *[1] The function $f : \mathbb{R}^n \to \mathbb{R}$ is semi-E-convex on $\mathbb{R}^n$ if and only if its epigraph $epi\left(f\right) = \{(x,\alpha) \in \mathbb{R}^n \times \mathbb{R} : f\left(x\right) \leq \alpha\}$ is $E \times I$-convex on $\mathbb{R}^n \times \mathbb{R}$.*

**Definition 2.6.** [2] A function $f : \mathbb{R}^n \to \mathbb{R}$ is lower semi-continuous if and only if, for every real number $\alpha$, the set $\{x \in \mathbb{R}^n : f\left(x\right) \leq \alpha\}$ is closed.

In the following, we introduce a Proposition about lower semi-continuous functions, which shall be used in the sequel. We refer to [2] for details and missing proofs.

**Proposition 2.7.** *[2] A function $f : \mathbb{R}^n \to \mathbb{R}$ is lower semi-continuous if and only if its epigraph is closed.*

**Definition 2.8.** Let $(x,s),(y,t) \in \mathbb{R}^{n+1}$, with $x, y \in \mathbb{R}^n$ and $s,t \in \mathbb{R}$. The line segment $[(x,s),(y,t)]$ (with endpoints $(x,s)$ and $(y,t)$) is the segment

$$\{\alpha(x,s)+(1-\alpha)(y,t) : 0 \leq \alpha \leq 1\}.$$

If $(x,s) \neq (y,t)$, the interior $](x,s),(y,t)[$ of $[(x,s),(y,t)]$ is the segment

$$\{\alpha(x,s)+(1-\alpha)(y,t) : 0 < \alpha < 1\}.$$

In a similar way, we can define $[(x,s),(y,t))$ and $((x,s),(y,t)]$.

## 3 Main results for semi-$E$-convex functions

**Lemma 3.1.** *Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a linear and idempotent map. Consider $(\overline{x},u) \in [(E\left(x\right),s),(E\left(y\right),t)]$. Then*

$$E\left(\overline{x}\right) = \overline{x}.$$

*Proof.* Let $(\overline{x},u) \in [(E\left(x\right),s),(E\left(y\right),t)]$, then there exist $\alpha \in [0,1]$, such that $(\overline{x},u) = \alpha(E\left(x\right),s)+(1-\alpha)(E\left(y\right),t)$. Using the fact that $E$ is a linear and idempotent map, we have

$$\begin{array}{rcl} (E \times I)\left(\overline{x},u\right) & = & \left(E\left(\alpha E\left(x\right)+(1-\alpha)E\left(y\right)\right),\alpha s+(1-\alpha)t\right) \\ & = & \left(\alpha E\left(x\right)+(1-\alpha)E\left(y\right),\alpha s+(1-\alpha)t\right) \\ & = & (\overline{x},u). \end{array}$$

On the other hand $(E \times I)\left(\overline{x},u\right) = \left(E\left(\overline{x}\right),u\right)$, therefore $E\left(\overline{x}\right) = \overline{x}$. □

We shall make use of the following three sets:

$$H_{Sci} = \{f : \mathbb{R}^n \to \mathbb{R},\ f \text{ is lower semi continuous}\}, \tag{3.1}$$
$$H_{L,I} = \{E : \mathbb{R}^n \to \mathbb{R}^n,\ E \text{ is linear and idempotent}\} \tag{3.2}$$

and for each $E \in H_{L,I}$ we define $H_E$ as follows:

$$H_E = \{f \in H_{Sci},\ f(E(x)) \le f(x) \text{ for all } x \in \mathbb{R}^n\} \tag{3.3}$$

**Theorem 3.2.** *Let $E \in H_{L,I}$, and $f \in H_E$. Suppose that there exists an $\alpha \in ]0,1[$ such that for all $x, y \in \mathbb{R}^n$, $s, t \in \mathbb{R}$ such that $f(x) < s$, $f(y) < t$,*

$$f(\alpha E(x) + (1-\alpha)E(y)) < \alpha s + (1-\alpha)t.$$

*Then $f$ is semi-E-convex.*

*Proof.* By Proposition (2.5), it is sufficient to show that $epi\,(f)$ is $E \times I$-convex as a subset of $\mathbb{R}^n \times \mathbb{R}$. By contradiction, suppose that there exist $(x_1, \alpha_1), (x_2, \alpha_2) \in epi\,(f)$ (with $x_1, x_2 \in \mathbb{R}^n$ and $\alpha_1, \alpha_2 \in \mathbb{R}$) and $\alpha_0 \in ]0,1[$ such that,
$(\alpha_0 E(x_1) + (1-\alpha_0)E(x_2), \alpha_0\alpha_1 + (1-\alpha_0)\alpha_2) \notin epi\,(f)$.
Let $x_0 = \alpha_0 E(x_1) + (1-\alpha_0)E(x_2)$ and $\lambda_0 = \alpha_0\alpha_1 + (1-\alpha_0)\alpha_2$, then $(x_0, \lambda_0) \notin epi\,(f)$. Using the fact that $f \in H_E$, we see that $(E(x_1), \alpha_1), (E(x_2), \alpha_2) \in epi\,(f)$. Let

$$A = epi\,(f) \cap [(E(x_1), \alpha_1), (x_0, \lambda_0)]$$

and

$$B = epi\,(f) \cap [(x_0, \lambda_0), (E(x_2), \alpha_2)].$$

Since $f \in H_{Sci}$, by Proposition (2.7), $epi\,(f)$ is a closed subset of $\mathbb{R}^n \times \mathbb{R}$. Consequently, $A$ and $B$ are bounded and closed subsets of $\mathbb{R}^n \times \mathbb{R}$.
Also we have $(x_0, \lambda_0) \notin A$ and $(x_0, \lambda_0) \notin B$. Thus there exist $Z_A = (x_3, \alpha_3) \in A$ and $Z_B = (x_4, \alpha_4) \in B$ such that,

$$\min_{Z \in A} \|Z - (x_0, \lambda_0)\| = \|Z_A - (x_0, \lambda_0)\|$$

and

$$\min_{Z \in B} \|Z - (x_0, \lambda_0)\| = \|Z_B - (x_0, \lambda_0)\|.$$

Hence, we have

$$]Z_A, Z_B[ \cap epi\,(f) = \emptyset. \tag{3.4}$$

On the other hand, since $Z_A \in epi\,(f)$ and $Z_B \in epi\,(f)$, we get
$f(x_3) < \alpha_3 + \varepsilon$, $f(x_4) < \alpha_4 + \varepsilon$ for each $\varepsilon > 0$.
Since $\alpha(\alpha_3 + \varepsilon) + (1-\alpha)(\alpha_4 + \varepsilon) = \alpha\alpha_3 + (1-\alpha)\alpha_4 + \varepsilon$. By the hypothesis of the Theorem, we obtain

$$f(\alpha E(x_3) + (1-\alpha)E(x_4)) < \alpha\alpha_3 + (1-\alpha)\alpha_4 + \varepsilon.$$

Since $\varepsilon$ is an arbitrary positive real number, it follows that

$$f(\alpha E(x_3) + (1-\alpha)E(x_4)) \le \alpha\alpha_3 + (1-\alpha)\alpha_4. \tag{3.5}$$

Since $Z_A \in A \subset [(E(x_1), \alpha_1), (E(x_2), \alpha_2)]$ and
$Z_B \in B \subset [(E(x_1), \alpha_1), (E(x_2), \alpha_2)]$. By Lemma (3.1) we have $E(x_3) = x_3$ and
$E(x_4) = x_4$. Using (3.5) we get

$$(\alpha x_3 + (1-\alpha)x_4, \alpha\alpha_3 + (1-\alpha))\,\alpha_4) \in epi\,(f)\,.$$

Therfore

$$\alpha Z_A + (1-\alpha)Z_B \in epi\,(f)\,,$$

which contradicts (3.4). Thus, we conclude that $epi\,(f)$ is $E \times I$-convex. □

**Theorem 3.3.** *Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a linear and idempotent map, $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous and $f(E(x)) \leq f(x)$ for all $x \in \mathbb{R}^n$. Then $f$ is semi-E-convex if and only if there exists an $\alpha \in ]0,1[$ such that for all $x, y \in \mathbb{R}^n$*

$$f(\alpha E(x) + (1-\alpha)E(y)) \leq \alpha f(x) + (1-\alpha)f(y)\,.$$

*Proof.* Follows from Theorem (3.2) with $s = f(x) + \varepsilon$ and $t = f(y) + \varepsilon$ for each $\varepsilon > 0$, then taking $\varepsilon \to 0$. □

By taking $\alpha = \dfrac{1}{2}$, in Theorem 3.3 we'll find the following Corollary.

**Corollary 3.4.** *Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a linear and idempotent map, $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous and $f(E(x)) \leq f(x)$ for all $x \in \mathbb{R}^n$. Then $f$ is semi-E-convex if and only if for all $x, y \in \mathbb{R}^n$,*

$$f\left(\frac{1}{2}(E(x) + E(y))\right) \leq \frac{1}{2}[f(x) + f(y)]\,.$$

**Theorem 3.5.** *Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a linear and idempotent map, $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous and $f(E(x)) \leq f(x)$ for all $x \in \mathbb{R}^n$. Then $f$ is semi-E-convex if and only if for all $x, y \in \mathbb{R}^n$, there exists an $\alpha \in ]0,1[$ ($\alpha$ depends on $x, y$) such that*

$$f(\alpha E(x) + (1-\alpha)E(y)) \leq \alpha f(x) + (1-\alpha)f(y)\,. \tag{3.6}$$

*Proof.* In this case ($\alpha$ depends on $x, y$), the proof is similar to the Theorem 3.2 □

According to Theorems 3.3, 3.5 and Corollary 3.4 with $E = Id_{\mathbb{R}^n}$, we get $E \in H_{L,I}$, and $H_E = H_{Sci}$. Then we find results about convex functions.

**Theorem 3.6.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous.Then $f$ is convex if and only if there exists an $\alpha \in ]0,1[$ such that, for all $x, y \in \mathbb{R}^n$,*

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)\,.$$

**Theorem 3.7.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous. Then $f$ is convex if and only if for all $x, y \in \mathbb{R}^n$, there exists an $\alpha \in ]0,1[$ ($\alpha$ depends on $x, y$) such that*

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)\,.$$

**Corollary 3.8.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous. Then $f$ is convex if and only if for all $x, y \in \mathbb{R}^n$,*

$$f\left(\frac{1}{2}(x + y)\right) \leq \frac{1}{2}[f(x) + f(y)]\,.$$

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## Conflict of Interest

The authors have no conflicts of interest to declare.

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