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Caputo fractional q-difference equations in Banach spaces

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Abstract. This paper aims to explore the existence results of a certain type of Caputo fractional q-difference equations in Banach spaces. To achieve this goal, we employ a fixed point theorem that relies on the concept of measure of noncompactness and the convex-power condensing operator. We give an illustrative example in the last section.

Keywords: Fractional q-difference equation, measure of noncompactness, solution, fixed point.
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1 Introduction

Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics, and other applied sciences. For more details on the applications of fractional calculus, the reader is directed to the books of Herrmann [18], Hilfer [19] and Tarasov [32]. For some fundamental results in the theory of fractional calculus and fractional differential equations we refer the reader to the monographs of Abbas *et al.* [1–3], Kilbas *et al.* [21], Samko *et al.* [31], and Zhou *et al.* [33].

The measure of noncompactness is a fundamental tool used in the theory of nonlinear analysis. This concept was first introduced by Alvàrez in his pioneering article [8], and later further developed by Mönch [24], Banaś and Goebel [10], and other researchers in the literature. The measure of noncompactness finds applications in various fields of applied mathematics, such as the theory of differential equations [5, 25]. In [14, 26, 30], Salim *et al.* applied

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the notion of measure of noncompactness to examine differential equations in Banach spaces.

In [22], the authors investigated the existence and Ulam-Hyers-Rassias stability of random solutions to the following random implicit fractional q-difference equation:

$$\begin{cases} ({}^cD_q^{\zeta}\alpha)(\vartheta,\delta) = \psi(\vartheta,\alpha(\vartheta,\delta),({}^cD_q^{\zeta}\alpha)(\vartheta,\delta),\delta);\ \vartheta \in \Theta := [0,\kappa],\ \delta \in \Psi, \\ \\ \alpha(0,\delta) = \alpha_0(\delta);\ \delta \in \Psi, \end{cases}$$

where $q \in (0,1)$, $\zeta \in (0,1]$, $\kappa > 0$, $(\Psi, \mathcal{A})$ is a measurable space, $\alpha_0 : \Psi \to \mathbb{R}$ is a measurable function, $\psi : \Theta \times \mathbb{R}^2 \times \Psi \to \mathbb{R}$ is a given function, and ${}^cD_q^{\zeta}$ is the Caputo fractional q-difference derivative of order ζ. The outcomes are given by the implementation of the fixed point theory, including Itoh's random fixed point theorem, the nonlinear alternative of Schaefer's type demonstrated by Dhage, and another random fixed point theorem of Dhage, specifically applied in Banach algebras. Furthermore, additional insights regarding the extremal and random extremal solutions are established based on the Carathéodory conditions and certain forms of monotonicity. The general theory of linear q-difference equations is investigated in the works of Adams [4] and Carmichael [13]. Meanwhile, Ahmad *et al.* conducted a study on several existence results for various types of nonlinear fractional q-difference equations in [6,7,16]. In [11], Boutiara et Benbachir studied some existence and uniqueness results to a fractional q-difference coupled system with integral boundary conditions via topological degree theory. The positive solutions of q-difference equations were examined by El-Shahed and Hassan [15]. Finally, the authors of [29] delved into the topological structure of solution sets for fractional q-difference inclusions, using Filippov's theorem.

In this paper, we consider the following fractional q-difference equation

$$({}^c\mathfrak{D}_q^{\varpi}\xi)(\vartheta) = \wp(\vartheta,\xi(\vartheta));\ \vartheta \in \Theta := [0,\varkappa], \tag{1.1}$$

with the initial condition

$$\xi(0) = \xi_0 \in \digamma, \tag{1.2}$$

where $q \in (0,1)$, $\varpi \in (0,1]$, $\varkappa > 0$, $\wp : \Theta \times \digamma \to \digamma$ is a given continuous function, $\digamma$ is a real (or complex) Banach space with norm $\|\cdot\|$, and ${}^c\mathfrak{D}_q^{\varpi}$ is the Caputo fractional q-difference derivative of order ϖ.

The present article has been organized as follows: In Section 2, some basic definitions and lemmas related to fractional calculus are recalled. In Section 3, by means of the fixed point theory combined with the concept of measure of noncompactness and the convex-power condensing operator, the existence of solutions for the problem (1.1)-(1.2) are obtained. At the end, we give an example to illustrate our main findings.

2 Preliminaries

Let $C(\Theta) := C(\Theta,\digamma) = \{\wp : \Theta \to \digamma, \wp \text{ continuous}\}$ be the Banach space with norm

$$\|\xi\|_\infty := \sup_{\vartheta\in\Theta} \|\xi(\vartheta)\|,$$

$L^1(\Theta)$ denotes the space of measurable functions $\chi : \Theta \to \digamma$ which are Bochner integrable with the norm

$$\|\chi\|_1 = \int_\Theta \|\chi(t)\| dt.$$

For $\omega \in \mathbb{R}$, we set

$$[\omega]_q = \frac{q^\omega - 1}{q - 1}.$$

The q-analogue of the power $(\omega - \varpi)^n$ is

$$(\omega - \varpi)^{(0)} = 1, \ (\omega - \varpi)^{(n)} = \Pi_{k=0}^{n-1}(\omega - \varpi q^k); \ \omega, \varpi \in \mathbb{R}, \ n \in \mathbb{N}.$$

$$(\omega - \varpi)^{(m)} = \omega^m \Pi_{k=0}^{\infty}\left(\frac{\omega - \varpi q^k}{\omega - \varpi q^{k+m}}\right); \ \omega, \varpi, m \in \mathbb{R}.$$

Definition 2.1. [20] We define the q-gamma function by

$$\Gamma_q(\vartheta) = \frac{(1-q)^{(\vartheta-1)}}{(1-q)^{\vartheta-1}}; \ \vartheta \in \mathbb{R} - \{0, -1, -2, \ldots\}$$

Definition 2.2. [20] We define the q-beta function by

$$\beta_q(\phi, \varphi) = \int_0^1 (1-\vartheta)^{(\varphi-1)} \vartheta^{\phi-1} d_q\vartheta.$$

Notice that

$$\Gamma_q(1+\vartheta) = [\vartheta]_q \Gamma_q(\vartheta), \ \ and \ \ \beta_q(\phi, \varphi) = \frac{\Gamma_q(\phi)\Gamma_q(\varphi)}{\Gamma_q(\phi+\varphi)}.$$

Definition 2.3. [20] Let $\xi : \Theta \to \digamma$ a function. We define the q-derivative of order $n \in \mathbb{N}$ of ξ by $(\mathfrak{D}_q^0 \xi)(\vartheta) = \xi(\vartheta)$,

$$(\mathfrak{D}_q \xi)(\vartheta) := (\mathfrak{D}_q^1 \xi)(\vartheta) = \frac{\xi(\vartheta) - \xi(q\vartheta)}{(1-q)\vartheta}; \ \vartheta \neq 0, \ \ (\mathfrak{D}_q \xi)(0) = \lim_{\vartheta \to 0} (\mathfrak{D}_q \xi)(\vartheta),$$

and

$$(\mathfrak{D}_q^n \xi)(\vartheta) = (\mathfrak{D}_q \mathfrak{D}_q^{n-1} \xi)(\vartheta); \ \vartheta \in \Theta, \ n \in \{1, 2, \ldots\}.$$

Set $\Theta_\vartheta := \{\vartheta q^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 2.4. [20] Let $\xi : \Theta_\vartheta \to \digamma$ a function. We define the q-integral of ξ by

$$(I_q \xi)(\vartheta) = \int_0^\vartheta \xi(s) d_q s = \sum_{n=0}^{\infty} \vartheta(1-q) q^n \xi(\vartheta q^n).$$

$(\mathfrak{D}_q I_q \xi)(\vartheta) = \xi(\vartheta)$, while if ξ is continuous at 0, then

$$(I_q \mathfrak{D}_q \xi)(\vartheta) = \xi(\vartheta) - \xi(0).$$

Let $\xi : \Theta \to \digamma$ a function and $\varpi \in \mathbb{R}_+ := [0, \infty)$.

Definition 2.5. [5] We define the Riemann–Liouville fractional q-integral of order ϖ of a function ξ by $(I_q^0\xi)(\vartheta) = \xi(\vartheta)$, and

$$(I_q^{\varpi}\xi)(\vartheta) = \int_0^{\vartheta} \frac{(\vartheta - qs)^{(\varpi-1)}}{\Gamma_q(\varpi)}\xi(s)d_qs; \ \vartheta \in \Theta.$$

Lemma 2.6. *[27] For $\lambda \in (-1, \infty)$:*

$$(I_q^{\varpi}(\vartheta - a)^{(\lambda)})(\vartheta) = \frac{\Gamma_q(\lambda+1)}{\Gamma(\lambda+\varpi+1)}(\vartheta - a)^{(\lambda+\varpi)}; \ 0 \le a < \vartheta < \varkappa.$$

For $a = \lambda = 0$:

$$(I_q^{\varpi}1)(\vartheta) = \frac{1}{\Gamma_q(1+\varpi)}\vartheta^{(\varpi)}.$$

Definition 2.7. [28] We define the Riemann–Liouville fractional q-derivative of order ϖ of a function ξ by $(\mathfrak{D}_q^0\xi)(\vartheta) = \xi(\vartheta)$, and

$$(\mathfrak{D}_q^{\varpi}\xi)(\vartheta) = (\mathfrak{D}_q^{[\varpi]} I_q^{[\varpi]-\varpi}\xi)(\vartheta); \ \vartheta \in \Theta,$$

where $[\varpi]$ is the integer part of ϖ.

Definition 2.8. [28] We define the Caputo fractional q-derivative of order ϖ of a function ξ by $({}^C\mathfrak{D}_q^0\xi)(\vartheta) = \xi(\vartheta)$, and

$$({}^C\mathfrak{D}_q^{\varpi}\xi)(\vartheta) = (I_q^{[\varpi]-\varpi}\mathfrak{D}_q^{[\varpi]}\xi)(\vartheta); \ \vartheta \in \Theta.$$

Lemma 2.9. *[28] Let $\varpi \in \mathbb{R}_+$.*

$$(I_q^{\varpi}\, {}^CD_q^{\varpi}\xi)(\vartheta) = \xi(\vartheta) - \sum_{k=0}^{[\varpi]-1}\frac{\vartheta^k}{\Gamma_q(1+k)}(D_q^k\xi)(0).$$

In particular, if $\varpi \in (0,1)$, then

$$(I_q^{\varpi}\, {}^CD_q^{\varpi}\xi)(\vartheta) = \xi(\vartheta) - \xi(0).$$

Lemma 2.10. *(1.1)-(1.2) is equivalent to the integral equation*

$$\xi(\vartheta) = \xi_0 + (I_q^{\varpi}\wp)(\vartheta).$$

$\mathcal{M}_X$ denote the class of all bounded subsets of a metric space $\mathbb{X}$.

Definition 2.11. [10] Let $\mathbb{X}$ be a complete metric space and $\varrho : \mathcal{M}_{\mathbb{X}} \to \mathbb{R}_+$ a map. ϱ is called a measure of noncompactness on $\mathbb{X}$ if, for all $\mathbb{k}, \mathbb{k}_1, \mathbb{k}_2 \in \mathcal{M}_{\mathbb{X}}$,

(a) Regularity: $\varrho(\mathbb{k}) = 0$ if and only if $\mathbb{k}$ is precompact,

(b) Invariance under closure: $\varrho(\mathbb{k}) = \varrho(\overline{\mathbb{k}})$,

(c) Semi-additivity: $\varrho(\mathbb{k}_1 \cup \mathbb{k}_2) = \max\{\varrho(\mathbb{k}_1), \varrho(\mathbb{k}_2)\}$.

Definition 2.12. [10] Let $\digamma$ be a Banach space and $\Omega_\digamma$ be the family of bounded subsets of $\digamma$. The Kuratowski measure of noncompactness is the map $\varrho : \Omega_\digamma \to \mathbb{R}_+$ defined by

$$\varrho(\mathfrak{T}) = \inf\{\epsilon > 0 : \mathfrak{T} \subset \cup_{j=1}^{m} \mathfrak{T}_j, \operatorname{diam}(\mathfrak{T}_j) \leq \epsilon\},$$

where $\mathfrak{T} \in \Omega_\digamma$.

Properties. The map ϱ satisfies:

(1) $\varrho(\mathfrak{T}) = 0 \Leftrightarrow \overline{\mathfrak{T}}$ is compact ($\mathfrak{T}$ is relatively compact).

(2) $\varrho(\mathfrak{T}) = \varrho(\overline{\mathfrak{T}})$.

(3) $\mathfrak{T}_1 \subset \mathfrak{T}_2 \Rightarrow \varrho(\mathfrak{T}_1) \leq \varrho(\mathfrak{T}_2)$.

(4) $\varrho(\mathfrak{T}_1 + \mathfrak{T}_2) \leq \varrho(\mathfrak{T}_1) + \varrho(\mathfrak{T}_2)$.

(5) $\varrho(\lambda\mathfrak{T}) = |\lambda|\varrho(\mathfrak{T}),\ \lambda \in \mathbb{R}$.

(6) $\varrho(\operatorname{conv}\mathfrak{T}) = \varrho(\mathfrak{T})$.

Lemma 2.13. *[9] Let $\Bbbk \subset C(\Theta)$ be bounded and equicontinuous. Then $\varrho(\Bbbk(\vartheta))$ is continuous on Θ and $\varrho_C(\Bbbk) = \max_{\vartheta\in\Theta} \varrho(\Bbbk(\vartheta))$.*

Lemma 2.14. *[9] Let $\Bbbk \subset \digamma$ be bounded. Then for each $\epsilon > 0$, there exists a sequence $\{\xi_n\}_{n\geq 1} \subset \Bbbk$ such that*

$$\varrho(\Bbbk) \leq 2\varrho(\{\xi_n\}_{n\geq 1}) + \epsilon.$$

Definition 2.15. A subset $\Bbbk \subset L^1(\Theta)$ is uniformly integrable if there exists $\psi \in L^1(\Theta)$ such that

$$\xi(\vartheta) \leq \psi(\vartheta);\ \text{for all } \xi \in \Bbbk \text{ and a.e. } \vartheta \in \Theta.$$

Lemma 2.16. *[17] Let $\{\xi_n\}_{n\geq 1} \subset L^1(\digamma)$ be an uniformly integrable, then the function $\vartheta \mapsto \varrho(\{\xi_n\}_{n\geq 1})$ is measurable, and*

$$\varrho\left(\left\{\int_0^\vartheta \xi_n(s)ds\right\}_{n\geq 1}\right) \leq 2\int_0^\vartheta \varrho(\{\xi_n(s)\}_{n\geq 1})ds.$$

We denote by $\bar{co}$, the closure of convex hull.

Definition 2.17. Let $\mathbb{X}$ be a real Banach space. An operator $\aleph : \mathbb{X} \to \mathbb{X}$ is a convex-power condensing about ξ_0 and n_0, if $\aleph$ is a continuous and bounded operator, and there exist $\xi_0 \in \mathbb{X}$ and a positive integer n_0 such that for any bounded and nonprecompact subset $S \subset \mathbb{X}$,

$$\varrho(\aleph^{(n_0,\xi_0)}(S)) < \varrho(S),$$

where

$$\aleph^{(1,\xi_0)}(S) = \aleph(S),\ \aleph^{(n,\xi_0)}(S) = \aleph(\bar{co}\{\aleph^{(n-1,\xi_0)}(S)\});\ n = 2,3,\cdots$$

Theorem 2.18. *[23] Let $\mathbb{X}$ be a real Banach space and $\Bbbk \subset \mathbb{X}$ be a bounded, closed and convex set in $\mathbb{X}$. If there exist $\xi_0 \in \Bbbk$ and a positive integer n_0 such that $\aleph : \Bbbk \to \Bbbk$ be a convex-power condensing operator about ξ_0 and n_0, then the operator $\aleph$ has at least one fixed point in $\Bbbk$.*

3 Main results

Definition 3.1. By a solution of the problem (1.1)-(1.2) we mean a continuous function ξ that satisfies the equation (1.1) on Θ and the initial condition (1.2).

Assumptions:

(H_1) The function $\vartheta \mapsto \wp(\vartheta,\xi)$ is measurable on Θ for each $\xi \in \digamma$, and the function $\xi \mapsto \wp(\vartheta,\xi,\chi)$ is continuous on $\digamma$ for a.e. $\vartheta \in \Theta$.

(H_2) There exists $p \in C(\Theta,\mathbb{R}_+)$, such that

$$\|\wp(\vartheta,\xi)\| \leq (1+\|\xi\|)p(\vartheta);\ \textit{for a.e. } \vartheta \in \Theta,\ \textit{and each } \xi \in \digamma,$$

and for some positive integer ν, we have

$$p^* = \sup_{\vartheta\in\Theta} p(\vartheta) < \frac{(\Gamma_q(1+\nu\varpi))^{\frac{1}{\nu}}}{4\varkappa^{\varpi}}.$$

(H_3) For each bounded set $\Bbbk \subset \digamma$ and for each $\vartheta \in \Theta$, we have

$$\varrho(\wp(\vartheta,\Bbbk)) \leq p(\vartheta)\varrho(\Bbbk).$$

Theorem 3.2. *Assume $(H_1)-(H_3)$ hold. If*

$$\ell := \frac{p^*\varkappa^{\varpi}}{\Gamma_q(\varpi+1)} < 1, \tag{3.1}$$

then the problem (1.1)-(1.2) admits at least one solution defined on Θ.

Proof. Consider the operator $\Xi : C(\Theta) \to C(\Theta)$ defined by

$$(\Xi\xi)(\vartheta) = \xi_0 + \int_0^{\vartheta} \frac{(\vartheta-qs)^{(\varpi-1)}}{\Gamma_q(\varpi)}\wp(s,\xi(s))d_qs;\ \vartheta \in \Theta. \tag{3.2}$$

Set

$$R > \frac{\|\xi_0\|+\ell}{1-\ell},$$

and $\nabla_R := \{w \in C(\Theta) : \|w\|_\infty \leq R\}$.
For any $\xi \in C(\Theta)$ and each $\vartheta \in \Theta$, we have

$$\begin{aligned}
\|(\Xi\xi)(\vartheta)\| &\leq \|\xi_0\| + \int_0^{\vartheta} \frac{(\vartheta-qs)^{(\varpi-1)}}{\Gamma_q(\varpi)}\|\wp(s,\xi(s))\|d_qs \\
&\leq \|\xi_0\| + \int_0^{\vartheta} \frac{(\vartheta-qs)^{(\varpi-1)}}{\Gamma_q(\varpi)}p(s)(1+\|\xi(s)\|)d_qs \\
&\leq \|\xi_0\| + p^*(1+R)\int_0^{\vartheta} \frac{(\vartheta-qs)^{(\varpi-1)}}{\Gamma_q(\varpi)}d_qs \\
&\leq \|\xi_0\| + \frac{p^*\varkappa^{\varpi}(R+1)}{\Gamma_q(1+\varpi)} \\
&= \|\xi_0\| + \ell(R+1) \\
&\leq R.
\end{aligned}$$

Thus

$$\|\Xi(\xi)\|_\infty \le R. \tag{3.3}$$

Then $\Xi(\nabla_R) \subset \nabla_R$.

Step 1. $\Xi : \nabla_R \to \nabla_R$ *is continuous.*

Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence such that $\xi_n \to \xi$ in ∇_R. Then, for each $\vartheta \in \Theta$, we have

$$\|(\Xi\xi_n)(\vartheta) - (\Xi\xi)(\vartheta)\| \le \int_0^\vartheta \frac{(\vartheta - qs)^{(\varpi-1)}}{\Gamma_q(\varpi)} \|(\wp(s,\xi_n(s)) - \wp(s,\xi(s))\| d_q s.$$

Since $\xi_n \to \xi$ as $n \to \infty$, then from Lebesgue's dominated convergence theorem we get

$$\|\Xi(\xi_n) - \Xi(\xi)\|_\infty \to 0 \ \ as \ \ n \to \infty.$$

Step 2. $\Xi(\nabla_R)$ *is bounded and equicontinuous.*

Since $\Xi(\nabla_R) \subset \nabla_R$ and ∇_R is bounded, then $\Xi(\nabla_R)$ is bounded.

Next, let $\vartheta_1, \vartheta_2 \in \Theta$, $\vartheta_1 < \vartheta_2$ and let $\xi \in \nabla_R$. Thus, we have

$$\begin{aligned}
&\|(\Xi\xi)(\vartheta_2) - (\Xi\xi)(\vartheta_1)\| \\
&\le \left\| \int_0^{\vartheta_2} \frac{(\vartheta_2 - qs)^{(\varpi-1)}}{\Gamma_q(\varpi)} \wp(s,\xi(s)) d_q s - \int_0^{\vartheta_1} \frac{(\vartheta_1 - qs)^{(\varpi-1)}}{\Gamma_q(\varpi)} \wp(s,\xi(s)) d_q s \right\|.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\|(\Xi\xi)(\vartheta_2) - (\Xi\xi)(\vartheta_1)\| &\le \int_{\vartheta_1}^{\vartheta_2} \frac{(\vartheta_2 - qs)^{(\varpi-1)}}{\Gamma_q(\varpi)} p(s)(1 + \|\xi(s)\|) d_q s \\
&+ \int_0^{\vartheta_1} \left| \frac{(\vartheta_2 - qs)^{(\varpi-1)}}{\Gamma_q(\varpi)} - \frac{(\vartheta_1 - qs)^{(\varpi-1)}}{\Gamma_q(\varpi)} \right| p(s)(1 + \|\xi(s)\|) d_q s \\
&\le p^*(1+R) \int_{\vartheta_1}^{\vartheta_2} \frac{(\vartheta_2 - qs)^{(\varpi-1)}}{\Gamma_q(\varpi)} d_q s \\
&+ p^*(1+R) \int_0^{\vartheta_1} \left| \frac{(\vartheta_2 - qs)^{(\varpi-1)}}{\Gamma_q(\varpi)} - \frac{(\vartheta_1 - qs)^{(\varpi-1)}}{\Gamma_q(\varpi)} \right| d_q s.
\end{aligned}$$

As $\vartheta_1 \longrightarrow \vartheta_2$, the right-hand side of the above inequality tends to zero.

Step 3. $\Xi : \bar{co}\Xi(\nabla_R) \to \bar{co}\Xi(\nabla_R)$ *is a convex-power condensing operator.*

Set $\Omega = \bar{co}\Xi(\nabla_R)$. Let $\chi \in \Omega$. We will prove that there exists a positive integer n_0 such that for any bounded and nonprecompact subset $\Bbbk \subset \Omega$,

$$\varrho_C(N^{(n_0,\chi)}(\Bbbk)) \le \varrho_C(\Bbbk).$$

For any $\Bbbk \subset \Omega$, and $\chi \in \Bbbk$, $N^{(n,\chi)}(\Bbbk) \subset \nabla_R$ is equicontinuous. Therefore, from Lemma 2.13 we have

$$\varrho_C(N^{(n,\chi)}(\Bbbk)) = \max_{\vartheta\in\Theta} \varrho(N^{(n,\chi)}(\Bbbk)(\vartheta)); \ n = 1, 2, \cdots \tag{3.4}$$

Let $\epsilon > 0$. By Lemma 2.14, there exists a sequence $\{\xi_n\}_{n\ge1} \subset \Bbbk$ such that

$$\begin{aligned}
\varrho(\Xi^{(1,\chi)}(\Bbbk)(\vartheta)) &= \varrho(\Xi(\Bbbk)(\vartheta)) \\
&\le 2\varrho\left\{ \int_0^\vartheta \frac{(\vartheta q - s)^{(\varpi-1)}}{\Gamma_q(\varpi)} \wp(s, \{\xi_n(s)\}_{n\ge1}) d_q s \right\} + \epsilon.
\end{aligned}$$

Now, by Lemma 2.16, and (H_3) we have

$$\begin{aligned}
\varrho(\Xi^{(1,\chi)}(\mathbb{k})(\vartheta)) &\leq 4\left\{\int_0^\vartheta \frac{(\vartheta q - s)^{(\varpi-1)}}{\Gamma_q(\varpi)}\varrho(\wp(s,\{\xi_n(s)\}_{n\geq 1}))d_qs\right\}+\epsilon \\
&\leq 4p^*\left\{\int_0^\vartheta \frac{(\vartheta q - s)^{(\varpi-1)}}{\Gamma_q(\varpi)}\varrho(\{\xi_n(s)\}_{n\geq 1})d_qs\right\}+\epsilon \\
&\leq 4p^*\varrho_C(\mathbb{k})\left\{\int_0^\vartheta \frac{(\vartheta q - s)^{(\varpi-1)}}{\Gamma_q(\varpi)}d_qs\right\}+\epsilon \\
&\leq \frac{4p^*\vartheta^\varpi}{\Gamma_q(1+\varpi)}\varrho_C(\mathbb{k})+\epsilon.
\end{aligned}$$

Since the last inequality is true for every $\epsilon > 0$, we infer that

$$\varrho(\Xi^{(1,\chi)}(\mathbb{k})(\vartheta)) \leq \frac{4p^*\vartheta^\varpi}{\Gamma_q(1+\varpi)}\varrho_C(\mathbb{k}).$$

Again by using Lemma 2.14, for any $\epsilon > 0$, there exists a sequence $\{w_n\}_{n\geq 1} \subset \bar{co}\{\Xi^{(1,\chi)}(\mathbb{k})\}$ such that

$$\begin{aligned}
\varrho(\Xi^{(2,\chi)}(\mathbb{k})(\vartheta)) &= \varrho(\Xi(\bar{co}\{\Xi^{(1,\chi)}(\mathbb{k})\})(\vartheta)) \\
&\leq 2\varrho\left\{\int_0^\vartheta \frac{(\vartheta q - s)^{(\varpi-1)}}{\Gamma_q(\varpi)}\wp(s,\{w_n(s)\}_{n\geq 1})d_qs\right\}+\epsilon \\
&\leq 4\left\{\int_0^\vartheta \frac{(\vartheta q - s)^{(\varpi-1)}}{\Gamma_q(\varpi)}\varrho(\wp(s,\{w_n(s)\}_{n\geq 1}))d_qs\right\}+\epsilon \\
&\leq 4p^*\left\{\int_0^\vartheta \frac{(\vartheta q - s)^{(\varpi-1)}}{\Gamma_q(\varpi)}\varrho(\{w_n(s)\}_{n\geq 1})d_qs\right\}+\epsilon \\
&\leq 4p^*\left\{\int_0^\vartheta \frac{(\vartheta q - s)^{(\varpi-1)}}{\Gamma_q(\varpi)}\varrho(\bar{co}\{N^{(1,\chi)}(\mathbb{k})\}(s))d_qs\right\}+\epsilon \\
&\leq 4p^*\left\{\int_0^\vartheta \frac{(\vartheta q - s)^{(\varpi-1)}}{\Gamma_q(\varpi)}\varrho(N^{(1,\chi)}(\mathbb{k})(s))d_qs\right\}+\epsilon \\
&\leq 4p^*\frac{4p^*}{\Gamma_q(1+\varpi)}\varrho_C(\mathbb{k})\left\{\int_0^\vartheta \frac{(\vartheta q - s)^{(\varpi-1)}s^\varpi}{\Gamma_q(\varpi)}d_qs\right\}+\epsilon \\
&\leq \frac{(4p^*)^2}{\Gamma_q(1+\varpi)}\varrho_C(\mathbb{k})\left\{\int_0^\vartheta \frac{(\vartheta q - s)^{(\varpi-1)}s^\varpi}{\Gamma_q(\varpi)}d_qs\right\}+\epsilon.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\int_0^{\vartheta} \frac{(\vartheta q - s)^{(\varpi-1)} s^{\varpi}}{\Gamma_q(\varpi)} d_q s &= \frac{(\vartheta q)^{2\varpi}}{\Gamma_q(\varpi)} \int_0^{\vartheta} \left(1 - \frac{s}{\vartheta q}\right)^{(\varpi-1)} \left(\frac{s}{\vartheta q}\right)^{\varpi} \frac{d_q s}{\vartheta q} \\
&\leq \frac{\vartheta^{2\varpi}}{\Gamma_q(\varpi)} \int_0^1 (1-x)^{(\varpi-1)} x^{\varpi} d_q x \\
&= \frac{\vartheta^{2\varpi}}{\Gamma_q(\varpi)} \beta_q(\varpi, 1+\varpi) \\
&= \frac{\vartheta^{2\varpi}}{\Gamma_q(\varpi)} \frac{\Gamma_q(\varpi)\Gamma_q(1+\varpi)}{\Gamma_q(1+2\varpi)} \\
&= \vartheta^{2\varpi} \frac{\Gamma_q(1+\varpi)}{\Gamma_q(1+2\varpi)}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\varrho(\Xi^{(2,\chi)}(\Bbbk)(\vartheta)) &\leq \frac{(4p^*)^2}{\Gamma_q(1+\varpi)} \varrho_C(\Bbbk) \left\{ \vartheta^{2\varpi} \frac{\Gamma_q(1+\varpi)}{\Gamma_q(1+2\varpi)} \right\} + \epsilon \\
&\leq \frac{(4p^*)^2 \vartheta^{2\varpi}}{\Gamma_q(1+2\varpi)} \varrho_C(\Bbbk) + \epsilon.
\end{aligned}$$

As the last inequality is true for every $\epsilon > 0$, we get

$$\varrho(\Xi^{(2,\chi)}(\Bbbk)(\vartheta)) \leq \frac{(4p^*)^2 \vartheta^{2\varpi}}{\Gamma_q(1+2\varpi)} \varrho_C(\Bbbk).$$

Repeating the process for $n = 3, 4, \cdots$, for each $\vartheta \in \Theta$, we can shown by mathematical induction that

$$\varrho(\Xi^{(n,\chi)}(\Bbbk)(\vartheta)) \leq \frac{(4p^*)^n \vartheta^{n\varpi}}{\Gamma_q(1+n\varpi)} \varrho_C(\Bbbk). \tag{3.5}$$

By induction, suppose that (3.5) holds for some n and check (3.5) for $n+1$.
By using Lemma 2.14, for any $\epsilon > 0$, there exists a sequence $\{y_n\}_{n \geq 1} \subset \bar{co}\{\Xi^{(n,\chi)}(\Bbbk)\}$ such

that

$$\begin{aligned}
\varrho(\Xi^{(n+1,\chi)}(\Bbbk)(\vartheta)) &= \varrho(\Xi(\bar{co}\{\Xi^{(n,\chi)}(\Bbbk)\})(\vartheta)) \\
&\leq 2\varrho\left\{\int_0^\vartheta \frac{(\vartheta q-s)^{(\varpi-1)}}{\Gamma_q(\varpi)}\wp(s,\{y_n(s)\}_{n\geq1})d_qs\right\}+\epsilon \\
&\leq 4\left\{\int_0^\vartheta \frac{(\vartheta q-s)^{(\varpi-1)}}{\Gamma_q(\varpi)}\varrho(\wp(s,\{y_n(s)\}_{n\geq1}))d_qs\right\}+\epsilon \\
&\leq 4p^*\left\{\int_0^\vartheta \frac{(\vartheta q-s)^{(\varpi-1)}}{\Gamma_q(\varpi)}\varrho(\{y_n(s)\}_{n\geq1})d_qs\right\}+\epsilon \\
&\leq 4p^*\left\{\int_0^\vartheta \frac{(\vartheta q-s)^{(\varpi-1)}}{\Gamma_q(\varpi)}\varrho(\bar{co}\{y^{(n,\chi)}(\Bbbk)\}(s))d_qs\right\}+\epsilon \\
&\leq 4p^*\left\{\int_0^\vartheta \frac{(\vartheta q-s)^{(\varpi-1)}}{\Gamma_q(\varpi)}\varrho(y^{(n,\chi)}(\Bbbk)(s))d_qs\right\}+\epsilon \\
&\leq 4p^*\frac{(4p^*)^n}{\Gamma_q(1+n\varpi)}\left\{\int_0^\vartheta \frac{(\vartheta q-s)^{(\varpi-1)}s^{n\varpi}}{\Gamma_q(\varpi)}d_qs\right\}+\epsilon \\
&\leq \frac{(4p^*)^{n+1}\vartheta^{(n+1)\varpi}}{\Gamma_q(1+(n+1)\varpi)}\varrho_C(\Bbbk)+\epsilon.
\end{aligned}$$

Thus, as the last inequality is true for every $\epsilon > 0$, we get

$$\varrho(\Xi^{(n+1,\chi)}(\Bbbk)(\vartheta)) \leq \frac{(4p^*)^{n+1}\vartheta^{(n+1)\varpi}}{\Gamma_q(1+(n+1)\varpi)}\varrho_C(\Bbbk).$$

From (3.4), we get

$$\varrho_C(\Xi^{(n,\chi)}(\Bbbk)) = \max_{\vartheta\in\Theta}\varrho(\Xi^{(n,\chi)}(\Bbbk)(\vartheta)) \leq \frac{(4p^*)^n\varkappa^{n\varpi}}{\Gamma_q(1+n\varpi)}\varrho_C(\Bbbk).$$

Since

$$\frac{(4p^*)^n\varkappa^{n\varpi}}{\Gamma_q(1+n\varpi)} = \frac{(4p^*)^n\varkappa^{n\varpi}}{[n\varpi]_q\frac{(1-q)^{(n\varpi-1)}}{(1-q)^{n\varpi-1}}} \to 0 \; as \;\; n\to\infty,$$

then, there exists a positive integer $n_0 = \nu$, such that

$$\frac{(4p^*)^{n_0}\varkappa^{n_0\varpi}}{\Gamma_q(1+n_0\varpi)} < 1.$$

Hence, for any bounded and nonprecompact subset $\Bbbk \subset \Omega$, we have

$$\varrho_C(\Xi^{(n_0,\chi)}(\Bbbk)) < \varrho_C(\Bbbk).$$

Therefore, $\Xi : \Omega \to \Omega$ is a convex-power condensing operator. Theorem 2.18 implies that Ξ has a fixed point which is a solution of problem (1.1)-(1.2).

4 Example

Let the Banach space

$$l^1 = \left\{ \xi = (\xi_1, \xi_2, \dots, \xi_n, \dots) : \sum_{n=1}^{\infty} |\xi_n| < \infty \right\}$$

under the norm

$$\|\xi\|_{l^1} = \sum_{n=1}^{\infty} |\xi_n|.$$

Consider the problem

$$\begin{cases} ({}^c\mathfrak{D}_{\frac{1}{4}}^{\frac{1}{2}} \xi_n)(\vartheta) = \wp_n(\vartheta, \xi(\vartheta)); \ \vartheta \in [0,1], \\ \xi(0) = (0, 0, \dots, 0, \dots), \end{cases} \tag{4.1}$$

where

$$\begin{cases} \wp_n(\vartheta, \xi) = \dfrac{\vartheta^{\frac{-1}{4}} (2^{-n} + \xi_n(\vartheta)) \sin \vartheta}{64L(1 + \|\xi\|_{l^1} + \sqrt{\vartheta})(1 + \|\xi\|_{l^1})}; \ \vartheta \in (0,1], \\ \quad \wp_n(0, \xi) = 0, \end{cases} .$$

with

$$L > \frac{1}{8\Gamma_{\frac{1}{4}}(\frac{1}{2})}, \ \wp = (\wp_1, \wp_2, \dots, \wp_n, \dots), \ and \ \xi = (\xi_1, \xi_2, \dots, \xi_n, \dots).$$

For each $\vartheta \in (0, 1]$, we have

$$\begin{aligned} \|\wp(\vartheta, \xi(\vartheta))\|_{l^1} &= \sum_{n=1}^{\infty} |\wp_n(s, \xi_n(s))| \\ &\le \frac{\vartheta^{\frac{-1}{4}} |\sin \vartheta|}{64L(1 + \|\xi\|_{l^1} + \sqrt{\vartheta})(1 + \|\xi\|_{l^1})} (1 + \|\xi\|_{l^1}). \end{aligned}$$

Thus, the hypothesis (H_2) is satisfied with

$$\begin{cases} p(\vartheta) = \dfrac{\vartheta^{\frac{-1}{4}} |\sin \vartheta|}{64L}; \ \vartheta \in (0,1], \\ \quad p(0) = 0. \end{cases}$$

So; we have $p^* \le \frac{1}{64L}$, and then

$$\ell = \frac{p^* \varkappa^{\varpi}}{\Gamma_q(1 + \varpi)} \le \frac{1}{64L\Gamma_{\frac{1}{4}}(1 + \frac{1}{2})} < \frac{1}{64} < 1.$$

Hence, by Theorem 3.2 the problem (4.1) has at least one solution defined on $[0, 1]$.

5 Conclusions

In the present research, we have investigated existence criteria for the solutions of Caputo fractional q-difference equations in Banach spaces. To achieve the desired results for the given problem, the fixed-point approach was used with the concept of measure of noncompactness and the convex-power condensing operator. An example is provided to demonstrate how the major results can be applied. Our results in the given configuration are novel and substantially contribute to the literature on this new field of study. We feel that there are multiple potential study avenues such as coupled systems, problems with infinite delays, and many more. We hope that this article will serve as a starting point for such an undertaking.

Declarations

Availability of data and materials

Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

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Authors' contributions

The study was carried out in collaboration of all authors. All authors read and approved the final manuscript.

Conflict of interest

The authors have no conflicts of interest to declare.

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