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Existence of solutions for a class of Kirchhoff type problem with triple regime logarithmic nonlinearity

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Abstract. In this paper, we use variational methods to study the existence of nontrivial solutions for a class of Kirchhoff-type elliptic problems driven by the $p(x)$-Laplacian with triple regime and sign-changing nonlinearity. Specifically, we consider the following equation

$$\begin{cases} -m\left(\int_{B_R} \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\right) \Delta_{p(x)} u = |u|^{q(x)-2} u \log(|u|) + |u|^{q(x)-2} u & \text{in} \quad B_R, \\ u = 0, & \text{on} \quad \partial B_R. \end{cases}$$

Here, B_R represents the open ball in $\mathbb{R}^N$ $(N \geq 1)$ centered at zero with a radius of $R > 0$. The functions $m : \mathbb{R}^+ \to \mathbb{R}^+$, and $p, q : \overline{B}_R \to \mathbb{R}^+$ are continuous and satisfy certain conditions. The novelty of this paper lies in establishing the existence of solutions for a class of Kirchhoff-type problems characterized by a sign-changing reaction term and a triple regime (subcritical, critical, and supercritical).

Keywords: Variational methods, p(x)-Kirchhoff-type equation, Nonlocal problems.

2020 Mathematics Subject Classification: 35J10, 35J70 MSC2020

1 Introduction

Consider the open ball B_R in $\mathbb{R}^N$ $(N \geq 1)$, centered at zero with a radius of $R > 0$. This paper investigates the following Kirchhoff-type problem

$$\begin{cases} -m\left(\int_{B_R} \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\right) \Delta_{p(x)} u = |u|^{q(x)-2} u \log(|u|) + |u|^{q(x)-2} u & \text{in} \quad B_R, \\ u = 0, & \text{on} \quad \partial B_R, \end{cases} \tag{1.1}$$

where $m : \mathbb{R}^+ \to \mathbb{R}^+$ and $p, q : \overline{B}_R \to \mathbb{R}^+$ are continuous functions that satisfy certain conditions, which will be given later on. Let

$$C^+(B_R) = \left\{ h \in C(\overline{B}_R),\ h_- > 1 \right\}.$$

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For any $h \in C^+(B_R)$, we define

$$h_- = \min_{x \in \overline{B}_R} h(x) \quad \text{and} \quad h_+ = \max_{x \in \overline{B}_R} h(x).$$

For $h \in C^+(B_R)$, we define the $h-$Laplacian differential operator as follows

$$\Delta_{h(x)} u = \operatorname{div}\left(|\nabla u|^{h(x)-2} \nabla u\right) \quad \forall u \in W_0^{1,h(x)}(B_R).$$

This operator finds applications in various fields, including elasticity theory (refer to [22]) and the analysis of electrorheological and magnetorheological fluids, as exemplified in [19]. In the context of image restoration, as presented in [6], the authors proposed a framework based on the $p(x)$-Laplacian. Unlike equations involving the p-Laplacian operator, studying equations like those in (1.1) present additional challenges. Notably, equations in (1.1) lack scaling invariance, making many conventional approaches inapplicable.

In recent years, the two following problems have been vigorously studied by many authors, particularly when p and q are constants

$$(P_1) \qquad \begin{cases} -m\left(\int_{B_R} \frac{1}{p} |\nabla v|^p \, dx\right) \Delta_p v = |v|^{q-2} v & \text{in} \quad B_R, \\ v = 0, & \text{in} \quad \partial B_R, \end{cases}$$

and

$$(P_2) \qquad \begin{cases} -m\left(\int_{B_R} \frac{1}{p} |\nabla v|^p \, dx\right) \Delta_p v = |v|^{q-2} v \log(|v|) & \text{in} \quad B_R, \\ v = 0, & \text{in} \quad \partial B_R. \end{cases}$$

The approaches for solving problems (P_1) and (P_2) are based on the relationship between the parameter q and the Sobolev critical exponent p^* of p. This critical exponent is defined as

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } 1 < p < N \\ +\infty & \text{if } p \geq N. \end{cases}$$

Furthermore, only one of the situations below can occur

(i) $q < p^*$ (*subcritical case*).

(ii) $q = p^*$, provided that $1 < p < N$ (*critical case*).

(iii) $q > p^*$, provided that $1 < p < N$ (*supercritical case*).

The literature on problem (P_1) covering a single case of $(i) - (iii)$ is extensive and rich. Here, we provide a list of some papers where authors have studied the existence and multiplicity of solutions for (P_1), including Alves et al. [1], Bartsch and Liu [5], Bahri and Coron [4], Garcia Azorero and Peral Alonso [14], Gueda and Veron [15], De Napoli and Mariani [11], Dinca et al. [8], Egnell [12], Huang [16], and references therein.

However, the literature concerning problem (P_2) involving a single case of $(i) - (iii)$ is limited. As far as we are aware, only a few papers have investigated the existence and multiplicity of solutions. Notably, Tian [21] studied the multiplicity of solutions for the following semilinear elliptic equations in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) with sign-changing logarithmic nonlinearity

$$\begin{cases} -\Delta v = a(x) v \log(|v|) & \text{in} \quad \Omega, \\ v = 0, & \text{in} \quad \partial\Omega. \end{cases}$$

Specifically, Tian proved that this problem has at least two nontrivial solutions, provided that $a \in C(\overline{\Omega})$, changes sign in Ω, and

$$\max_{x\in\overline{\Omega}} |a(x)| < 2\pi\left(2 - \frac{4|\Omega|}{Ne}\right).$$

Later on, in [18], the authors extended Tian's work to the following class of problems

$$\begin{cases} -\Delta^2 v + c\Delta v = v\log(|v|) & \text{in} \quad \Omega, \\ v = \Delta v = 0, & \text{in} \quad \partial\Omega. \end{cases}$$

Here, $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded smooth domain, and $c < \lambda_1$. In this work, the authors studied the existence of a ground solution using the linking theorem. In [10], de Queiroz considered the following elliptic problem with Neumann boundary conditions

$$\begin{cases} -\Delta v = \log v + h(x)v^p & \text{in} \quad B_R, \\ v > 0 & \text{in} \quad B_R, \\ \partial_\nu v = 0 & \text{in} \quad \partial B_R, \end{cases}$$

where $B_R = B_R(0) \subset \mathbb{R}^N$ $(N \geq 1)$, and $h \geq 0$ is a C^1 radial function, i.e., $h(x) = h(|x|)$ in B_R. Using a double perturbation argument, the author proved the existence of a positive radial solution $v \in C^2(\overline{B_R}\backslash\{0\}) \cap C(\overline{B}_R)$. The same double perturbation argument was also employed by Cui and Yang in [7] to study the existence of positive radial solutions for a class of quasilinear elliptic problems with logarithmic nonlinearity. It is beyond the scope of this discussion to list all the works on problem (P_2) here. However, it is worth noting that when p and q are not constants, problem (P_2) has not yet been extensively studied in the literature.

In the variable exponents' case, problem (P_2) becomes even richer, as it can exhibit a "subcritical-critical-supercritical" triple regime. This is illustrated by the decomposition $B_R = B_r \cup A_{R,r}$, where $A_{R,r}$ denotes the annulus centered at zero with radii r and R. The behavior is characterized by the conditions

$$\begin{array}{ll} q(x) < p^*(x) & \text{if } x \in B_r, \\ q(x) \geq p^*(x) & \text{if } x \in A_{R,r}, \end{array} \tag{1.2}$$

where $p^*(x) = \frac{p(x)}{N-p(x)}$ represents the critical Sobolev exponent corresponding to $p(x)$.

As far as we are aware, the first result concerning the existence of solutions for the following class of elliptic problems

$$(E) \qquad \begin{cases} -\Delta_{p(x)} v = |v|^{q(x)-2}v & \text{in} \quad \Omega, \\ v = 0 & \text{in} \quad \partial\Omega \end{cases}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ with the triple regime phenomenon (T) is discussed in [3]. Here, the triple regime phenomenon (T) can occur, where $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ and

$$(T) \qquad \begin{array}{lll} q(x) < p^*(x) & \text{in} & \Omega_1, \\ q(x) = p^*(x) & \text{in} & \Omega_2, \\ q(x) > p^*(x) & \text{in} & \Omega_3. \end{array}$$

Subsequently, Alves and Boudjeriou [2] extended this to a nonstationary case, where the triple regime phenomena (T) can also occur. They considered the problem

$$\begin{cases} v_t - \operatorname{div}\left(|\nabla v|^{p(x)-2}\nabla v\right) = |v|^{q(x)-2}v & \text{in} \quad \Omega,\ t > 0, \\ v = 0 & \text{in} \quad \partial\Omega,\ t > 0, \\ v(x,0) = v_0(x) & \text{in} \quad \Omega. \end{cases} \tag{1.3}$$

and established results regarding the local and global existence, as well as blow-up of solutions in finite time when (T) occurs.

Inspired by the works mentioned above, particularly ([3], [2]), our objective in this manuscript is to investigate problem (1.1), considering the triple regime phenomenon indicated in (1.2). We address this in situations where the reaction term is sign-changing and does not satisfy either the monotonicity condition or the Ambrosetti-Rabinowitz condition. This constitutes the main contribution of our paper.

Throughout this paper, we impose the following assumptions on the functions p and q :

$$p \in C^{0,1}(\overline{B}_R) \text{ and } 2 \leq p_- = \min_{x \in \overline{B}_R(0)} p(x) \leq \max_{x \in \overline{B}_R(0)} p(x) = p_+ < N, \tag{H_1}$$

$$p(x) = p(|x|) \quad \text{and} \quad q(x) = q(|x|) \quad \forall x \in \overline{B}_R(0), \tag{H_2}$$

there exists $0 < r < R$ such that

$$2p_+ < q_- = \min_{x \in \overline{B}_R(0)} q(x) \leq q(x) \leq \max_{x \in \overline{B}_r(0)} q(x) = q_+^r \leq \frac{p^*(x)}{2}. \tag{H_3}$$

Here, we denote by $C^{0,1}(\overline{B}R)$ the space of uniformly Hölder continuous functions in $\overline{B}R$.

We emphasize that no assumptions are placed on the function q within the annulus $A_{R,r} = \overline{B}_R(0) \backslash B_r(0)$. However, the function is subcritical in $\overline{B}_r(0)$, allowing for q to exhibit critical and supercritical growth near the boundary.

Throughout this work, the function $m(t)$ is assumed to satisfy the following conditions :

$$\exists m_0 > 0 \text{ such that } \quad m(t) \geq m_0, \ \forall t \geq 0, \tag{m_1}$$

and

$$M(t) \geq \frac{m(t)t}{2}, \quad \forall t \geq 0, \tag{m_2}$$

where $M(t) = \int_0^t m(s)\, ds$.

For the sake of clarity, we present the definition of weak solutions for problem (1.1).

Definition 1.1. We say that $u \in W_0^{1,p(x)}(B_R)$ is a weak solution of (1.1), if for any $v \in W_0^{1,p(x)}(B_R)$

$$m\left(\int_{B_R} \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\right) \int_{B_R} |\nabla u|^{p(x)-2} \nabla u . \nabla v\, dx = \int_{B_R} |u|^{q(x)-2} u \log(|u|) v\, dx + \int_{B_R} |u|^{q(x)-2} uv\, dx.$$

Our main result is as follows:

Theorem 1.2. *Suppose that the assumptions $(H_1) - (H_3)$ and $(m_1) - (m_2)$ hold. Then problem (1.1) has a nontrivial weak solution $u \in W_0^{1,p(x)}(B_R)$.*

Remark 1.3. We are curious whether the result stated in Theorem 1.2 remains valid if we replace the open ball B_R with any open smooth set $\Omega \subset \mathbb{R}^N$ and without assuming the condition (m_1). This matter will be addressed in a separate paper.

The structure of the paper proceeds as follows: Section 2 introduces preliminary concepts, while Section 3 presents the proof of the main result.

In the subsequent sections, positive constants are denoted by $c, C, c_i, C_i, i = 1, 2, \ldots$.

2 Preliminary Results

In this section, we initiate by revisiting specific definitions and established results related to Lebesgue and Sobolev spaces with variable exponents. For a comprehensive study of these spaces, we refer the reader to ([?], [19]) and the relevant references cited therein

Let Ω be an open set in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$. By $M(\Omega)$ we denote the space of measurable functions. Given $h \in C^+(\Omega)$, the variable Lebesgue space $L^{h(x)}(\Omega)$ is defined by

$$L^{h(x)} = \left\{ u \in M(\Omega) : \int_\Omega |u|^{h(x)} \, dx < +\infty \right\}.$$

This space is equipped with the so-called Luxemburg norm, defined by

$$\|u\|_{h(x)} = \|u\|_{L^{h(x)}(\Omega)} = \inf \left\{ \lambda > 0, \int_\Omega \left(\frac{|u|}{\lambda} \right)^{h(x)} \leq 1 \right\},$$

The space $\left(L^{h(x)}(\Omega), \| \, . \, \|_{h(x)} \right)$ is reflexive and separable Banach space (see [?]). If

$$h'(x) = \frac{h(x)}{h(x) - 1}, \quad \forall x \in \overline{\Omega},$$

then $h' \in C^+(\Omega)$ and we have $L^{h(x)}(\Omega)^* = L^{h'(x)}(\Omega)$. In addition, we have the following Hölder inequality

$$\int_\Omega |uv| \, dx \leq \left(\frac{1}{h_-} + \frac{1}{h'_-} \right) \|u\|_{h(x)} \|v\|_{h'(x)}, \quad \forall u \in L^{h(x)}(\Omega), \, \forall v \in L^{h'(x)}(\Omega). \tag{2.1}$$

If $h_1, h_2 \in C^+(\Omega)$ and $h_1(x) \leq h_2(x)$ for all $x \in \overline{\Omega}$, then $L^{h_2(x)}(\Omega) \hookrightarrow L^{h_1(x)}(\Omega)$ and the embedding is continuous.

By the Lebesgue spaces with variable exponents, for a given $h \in C^+(\Omega)$, we can establish the definition of the Sobolev space with variable exponent $W^{1,h(x)}(\Omega)$ as follows

$$W^{1,h(x)}(\Omega) = \left\{ u \in L^{h(x)}(\Omega) \, \middle| \, |\nabla u| \in L^{h(x)}(\Omega) \right\},$$

endowed with the norm

$$\|u\|_{1,h(x)} = \|u\|_{h(x)} + \|\nabla u\|_{h(x)}.$$

When $h \in C^+(\Omega) \cap C^{0,1}(\overline{\Omega})$ (that is, $h \in C^+(\Omega)$ is Lipschtiz continuous on $\overline{\Omega}$), then we can also define

$$W_0^{1,h(x)}(\Omega) = \overline{C_c^\infty(\Omega)}^{\| \cdot \|_{1,h(x)}}.$$

The spaces $W^{1,h(x)}(\Omega)$ and $W_0^{1,h(x)}(\Omega)$ are separable reflexive Banach spaces. For the space $W_0^{1,h(x)}(\Omega)$, the well-known Poincaré inequality is still valid, namely, there exists $c > 0$ such that

$$\|u\|_{h(x)} \leq c \|\nabla u\|_{h(x)}, \quad \forall u \in W_0^{1,h(x)}(\Omega).$$

This inequality implies on $W_0^{1,h(x)}(\Omega)$ we can consider the equivalent norm

$$\|u\| = \|\nabla u\|_{h(x)}, \quad \forall u \in W_0^{1,h(x)}(\Omega).$$

We introduce the following modular function

$$\rho(u) = \int_\Omega |\nabla u|^{h(x)} \, dx, \;\; \forall u \in L^{h(x)}(\Omega).$$

We have the following property.

Proposition 2.1. *For $u \in W_0^{1,h(x)}(\Omega)$, $u \neq 0$, we have the following properties :*

(i) *If $\|u\| \geq 1$, then $\|u\|^{p_-} \leq \rho(u) \leq \|u\|^{p_+}$.*

(ii) *If $\|u\| \leq 1$, then $\|u\|^{p_+} \leq \rho(u) \leq \|u\|^{p_-}$.*

In particular, $\rho(u) = 1$ if and only if $\|u\| = 1$ and if $(u_n) \subset W_0^{1,p(x)}(\Omega)$, then $\|u_n\| \to 0$ if and only if $\rho(u_n) \to 0$.

Setting

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if} \quad p_+ < N, \\ +\infty & \text{if} \quad p_+ \geq N. \end{cases}$$

Related to the Sobolev space $W_0^{1,h(x)}(\Omega)$, the following embeddings are true.

Proposition 2.2. *([19]) Let $p, t \in C^+(\Omega)$ and $1 < p_- \leq p_+ < N$. Then there hold*

(i) *If $1 \leq t \leq p^*$, the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{t(x)}(\Omega)$ is continuous.*

(ii) *If $1 \leq t \ll p^*$, the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{t(x)}(\Omega)$ is compact.*

Here $t \ll p^*$ means $\inf\limits_{x \in \Omega} (p^*(x) - t(x)) > 0$.

Subsequently, we define

$$S = \inf \left\{ \frac{\|\nabla u\|_{p(x)}}{\|u\|_{p^*(x)}} : u \in W_0^{1,p(x)}(\Omega) \backslash \{0\} \right\}. \tag{2.2}$$

Clearly, one has that $S > 0$. Now, let us consider the operator $A : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ defined by

$$\langle A(u), v \rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u . \nabla v \, dx \quad \forall u, v \in W_0^{1,p(x)}(\Omega).$$

The proof of the following proposition is found in [13, Theorem 3.1].

Proposition 2.3. *The operator A is bounded, continuous, strictly monotone, and of type $(S)_+$ which means that*

$$\text{if } u_n \rightharpoonup u \text{ weakly in } W_0^{1,p(x)}(\Omega) \text{ and } \limsup_{n \to \infty} \langle A(u_n), u_n - u \rangle \leq 0, \text{ then } u_n \to u \text{ in } W_0^{1,p(x)}(\Omega).$$

To handle the logarithmic terms, we introduce two crucial lemmata that will prove useful in the following section. The proof of the upcoming lemma is straightforward.

Lemma 2.4. *The following inequality holds*

$$|log(t)| \leq \frac{1}{r(x)} t^{r(x)}, \tag{2.3}$$

for all $t \in [1, +\infty)$ and $r : \overline{\Omega} \to (0, +\infty)$ is a continuous function satisfies that

$$0 < r^- = \min_{x \in \overline{\Omega}} r(x) \leq r(x) \leq r^+ = \max_{x \in \overline{\Omega}} r(x) < +\infty.$$

The next lemma plays a crucial role in our approach to estimating the logarithmic terms. It is noteworthy that no existing results for the logarithmic Sobolev inequality in variable exponent Sobolev spaces $W^{1,p(x)}(\Omega)$ currently exist.

Lemma 2.5. *Let $u \in W_0^{1,p(x)}(\Omega)\backslash\{0\}$. Then there exists $\kappa > 0$ such that*

$$\int_\Omega \frac{1}{\sigma(x)}|u|^{\sigma(x)} \log(|u|)\, dx \leq \log(\|u\|) \int_\Omega \frac{1}{\sigma(x)}|u|^{\sigma(x)}\, dx + \kappa \left(\|u\|^{\sigma_-} + \|u\|^{\sigma_+}\right), \tag{2.4}$$

where $1 < \sigma_- \leq \sigma(x) \leq \sigma_+ < p^*(x)$, for all $x \in \overline{\Omega}$.

Proof. Let us prove the lemma through contradiction. Assume that inequality (2.4) does not hold. Therefore, for every $n \in \mathbb{N}$, there exists $u_n \in W_0^{1,p(x)}(\Omega)\backslash\{0\}$ such that

$$\int_\Omega \frac{1}{\sigma(x)}|u_n|^{\sigma(x)} \log(|u_n|)\, dx > \log(\|u_n\|) \int_\Omega \frac{1}{\sigma(x)}|u_n|^{\sigma(x)}\, dx + n\left(\|u_n\|^{\sigma_-} + \|u_n\|^{\sigma_+}\right),$$

which can be rewritten as

$$\int_\Omega \frac{1}{\sigma(x)} \frac{|u_n|^{\sigma(x)}}{\|u_n\|^{\sigma_-} + \|u_n\|^{\sigma_+}} \log\left(\frac{|u_n|}{\|u_n\|}\right) dx > n. \tag{2.5}$$

On the other hand, clearly we have

$$\|u_n\|^{\sigma(x)} \leq \|u_n\|^{\sigma_+} + \|u_n\|^{\sigma_-}.$$

This combined with (2.5) yields

$$\int_\Omega \frac{1}{\sigma(x)} \frac{|u_n|^{\sigma(x)}}{\|u_n\|^{\sigma(x)}} \log^+\left(\frac{|u_n|}{\|u_n\|}\right) dx > n,$$

where $\log^+(|x|) = \max\{\log|x|, 0\}$. Considering $v_n = \frac{u_n}{\|u_n\|}$ we have exhibited a sequence v_n such that

$$\int_\Omega \frac{1}{\sigma(x)}|v_n|^{\sigma(x)} \log^+(|v_n|)\, dx > n, \quad \|v_n\| = 1. \tag{2.6}$$

From Proposition 2.2, it follows, up to a subsequence we can assume that there exists a $v \in W_0^{1,p(x)}(\Omega)$ such that

$$\begin{cases} v_n \rightharpoonup v & \text{weakly in} \quad W_0^{1,p(x)}(\Omega), \\ v_n \to v & \text{strongly in} \quad L^{\sigma(x)}(\Omega), \\ v_n \to v & \text{a.e. in} \quad \Omega. \end{cases}$$

From which it follows that

$$\frac{1}{\sigma(x)}|v_n(x)|^{\sigma(x)} \log^+(|v_n(x)|) \to \frac{1}{\sigma(x)}|v(x)|^{\sigma(x)} \log^+(|v(x)|) \ \text{ a.e. in } \ \Omega. \tag{2.7}$$

Now we claim that

$$\lim_{n\to\infty} \int_\Omega \frac{1}{\sigma(x)}|v_n|^{\sigma(x)} \log^+(|v_n|)\, dx = \int_\Omega \frac{1}{\sigma(x)}|v|^{\sigma(x)} \log^+(|v|)\, dx. \tag{2.8}$$

Set

$$U = \{x \in \Omega : |v_n(x)| > 1\}.$$

Then, according to Lemma 2.4 and Proposition 2.2, for $0 < r(x) < p^*(x) - \sigma_+$ and for any measurable subset $V \subset U \subset \Omega$ we have

$$\int_V \frac{1}{\sigma(x)} \left| |v_n|^{\sigma(x)} \log^+(|v_n|) \right| dx \quad \leq \quad \frac{1}{r_-} \int_V \frac{1}{\sigma(x)} |v_n|^{\sigma(x)+r(x)}\, dx \leq \frac{CS^{-r_+-\sigma_+} + CS^{-r_--\sigma_-}}{r_-}.$$

Hence, this shows that $\frac{1}{\sigma(x)}|v_n|^{\sigma(x)} \log^+(|v_n|)_n$ is uniformly bounded and equi-integrable in $L^1(\Omega)$. Consequently, by combining this fact with (2.7), and subsequently employing the Vitali convergence theorem, we conclude that claim (2.8) holds. However, this contradicts (2.6). Therefore, this concludes the proof of our lemma □

3 Proof of Theorem 1.2

This section is devoted to establishing Theorem 1.2. To this end, we introduce several technical results that will be utilized in the proof. For any $r_0 \in (0, R)$, we have the continuous embedding

$$W^{1,p(x)}(B_R(0)) \hookrightarrow W^{1,p_-}(A_{R,r_0}),$$

and the compact embedding

$$W^{1,p_-}_{rad}(A_{R,r_0}) \hookrightarrow C(\overline{A}_{R,r_0}),$$

which is due to Strauss [20]. Therefore the embedding

$$W^{1,p(x)}_{rad}(B_R(0)) \hookrightarrow C(\overline{A}_{R,r_0}), \tag{3.1}$$

is compact, where

$$W^{1,p(x)}_{rad}(B_R(0)) = \{u \in W^{1,p(x)}(B_R(0)) \ \ u(x) = u(|x|) \quad \text{a.e} \quad x \in B_R(0)\}.$$

Hence, it follows that the embedding

$$W^{1,p(x)}_{0,rad}(B_R(0)) \hookrightarrow L^{q(x)}(B_R(0)), \tag{3.2}$$

is also compact, which is crucial in our approach.

We will establish the existence of nontrivial weak solutions for (1.1) by seeking critical points of the associated Euler functional:

$$E(u) = M\left(\int_{B_R} \frac{1}{p(x)}|\nabla u|^{p(x)}, dx\right) + \int_{B_R} \left(\frac{1}{q^2(x)} - \frac{1}{q(x)}\right) |u|^{q(x)}\, dx - \int_{B_R} |u|^{q(x)} \log(|u|), dx.$$

It's important to note that, due to the condition (H_3), this functional is not well-defined over the entire space $W_0^{1,p(x)}(B_R)$. Nevertheless, using (3.2), it can be verified that E belongs to the class $C^1(W^{1,p(x)}_{0,rad}(B_R), \mathbb{R})$, and

$$\begin{aligned} E'(u)v =& m\left(\int_{B_R} \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx\right) \int_{B_R} |\nabla u|^{p(x)-2}\nabla u.\nabla v\, dx - \int_{B_R} |u|^{q(x)-2}uv \log(|u|)\, dx \\ & - \int_{B_R} |u|^{q(x)-2} uv\, dx \end{aligned}$$

for all $u, v \in W^{1,p(x)}_{0,rad}(B_R)$. Now, we proceed to demonstrate that the functional E satisfies the Mountain Pass geometry.

Lemma 3.1. *Assume that $(H_1)-(H_3)$ hold. Then the function E satisfies that*

1. *There exist $\eta > 0$ and $\delta > 0$ such that*

$$E(u) \geq \delta, \quad \textit{for all} \;\; u \in W^{1,p(x)}_{0,rad}(B_R) \; \textit{with} \; \|u\| = \eta.$$

2. *There exists $e \in W^{1,p(x)}_{0,rad}(B_R)$ with $\|e\| > \eta$ such that $E(e) < 0$.*

Proof. First, we establish that E satisfies condition (1). Using Lemma 2.5 and the assumptions $(m_1)-(m_2)$, we deduce that

$$\begin{aligned} E(u) &= M\left(\int_{B_R} \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right) + \int_{B_R}\left(\frac{1}{q^2(x)} - \frac{1}{q(x)}\right)|u|^{q(x)}\,dx - \int_{B_R}\frac{1}{q(x)}|u|^{q(x)}\log(|u|)\,dx \\ &\geq \frac{m_0}{2p_+}\|u\|^{2p_+} - 2\kappa\|u\|^{q_-}, \end{aligned}$$

for all $u \in W^{1,p(x)}_{0,rad}(B_R)$ with $0 < \|u\| < 1$. Choose η to be sufficiently small such that

$$0 < \eta < \min\left\{1, \left(\frac{m_0}{4p_+\kappa}\right)^{\frac{1}{q_- - 2p_+}}\right\},$$

and since $q_- > 2p_+$, it follows that

$$E(u) \;\; \geq \;\; \eta^{2p_+}\left(\frac{m_0}{2p_+} - 2\kappa\eta^{q_- - 2p_+}\right) := \delta > 0,$$

for all $u \in W^{1,p(x)}_{0,rad}(B_R)$ with $\|u\| = \eta$, thus this shows (1).

For $t > 1$, we can easily deduce from (m_2) that

$$M(t) \leq M(1)t^2 = ct^2.$$

Fix $w \in W^{1,p(x)}_{0,rad}(B_R)$. Then for all $t > 1$, we have

$$E\,(tw) \leq c_1 t^{2p_+} - c_2 t^{q_-} - c_3 t^{q_-}\log(t) + t^{q_-}\int_{B_R}|w|^{q(x)}\log^-(|w|)\,dx,$$

where $\log^-|x| = \min\{\log|x|, 0\}$, and $c_1, c_2, c_3 > 0$. Since $q_- > 2p_+$, $E(tw) \to -\infty$ as $t \to +\infty$. Taking $e = \bar{t}w$ with $\bar{t} > 1$ sufficiently large, hence we conclude that (2) holds. □

By the Mountain Pass Theorem (see [19]), there exists a Palais Smale sequence $\{u_n\} \subset W^{1,p(x)}_{0,rad}(B_R)$ for E, that is

$$E(u_n) \to d \quad \text{and} \quad E'(u_n) \to 0, \quad \text{as } n \to +\infty. \tag{3.3}$$

where

$$d = \inf_{g\in\Gamma}\max_{t\in[0,1]} E(g(t))$$

with

$$\Gamma = \left\{g \in C([0,1], W^{1,p(x)}_{0,rad}(B_R)) : \, g(0) = 0, \; g(1) = e\right\}.$$

Lemma 3.2. *The functional E satisfies the $(PS)_d$ at any level $d \in \mathbb{R}$.*

Proof. Let $u_n \subset W^{1,p(x)}_{0,rad}(B_R)$ be a $(\mathrm{PS})_d$ sequence satisfying (3.3). In the following, we assert that un is bounded in $W^{1,p(x)}_{0,rad}(B_R)$. Assuming the opposite, let's argue by contradiction and suppose that u_n is unbounded in $W^{1,p(x)}_{0,rad}(B_R)$. Thus, we can choose a subsequence, still denoted as u_n, such that $\|u_n\| \to \infty$ and let $\delta = \inf_{n\geq 1}\|u_n\| > 0$. Considering (3.3), we arrive at the following

$$\begin{aligned} C + \|u_n\| &\geq E(u_n) - \frac{1}{q_-}E'(u_n)u_n, \\ &\geq \frac{m_0(q_- - 2p_+)}{2p_+q_-}\left(\int_{B_R}|\nabla u_n|^{p(x)}\,dx\right) + \int_{B_R}\left(\frac{1}{q_-} - \frac{1}{q(x)}\right)|u_n|^{q(x)}\log(|u_n|)\,dx \\ &\quad + \int_{B_R}\left(\frac{1}{q_-} - \frac{1}{q(x)}\right)|u_n|^{q(x)}\,dx. \end{aligned}$$

According to Proposition 2.1, we conclude that

$$C + \|u_n\| \geq \frac{m_0(q_- - 2p_+)}{2p_+q_-}\|u_n\|^{2p_-} + \int_{B_R}\left(\frac{1}{q_-} - \frac{1}{q(x)}\right)|u_n|^{q(x)}\log(|u_n|)\,dx. \tag{3.4}$$

On the other hand, through a direct calculation it follows that

$$\begin{aligned} \int_{B_R}\left(\frac{1}{q_-} - \frac{1}{q(x)}\right)|u_n|^{q(x)}\log(|u_n|)\,dx &= \int_{B_R^{(1)}}\left(\frac{1}{q_-} - \frac{1}{q(x)}\right)|u_n|^{q(x)}\log(|u_n|)\,dx \\ &\quad + \int_{B_R^{(2)}}\left(\frac{1}{q_-} - \frac{1}{q(x)}\right)|u_n|^{q(x)}\log(|u_n|)\,dx. \end{aligned} \tag{3.5}$$

where

$$B_R^{(1)} = \{x \in B_R : |u_n(x)| \leq 1\} \quad \text{and} \quad B_R^{(2)} = \{x \in B_R : |u_n(x)| > 1\}.$$

Since

$$\inf_{t\in(0,1)} t^{q(x)}\log(t) = t^{q(x)}\log(t)\Big|_{t=e^{-\frac{1}{q(x)}}} = \frac{-1}{eq(x)}. \tag{3.6}$$

This combined with (3.5) yields

$$\int_{B_R}\left(\frac{1}{q_-} - \frac{1}{q(x)}\right)|u_n|^{q(x)}\log(|u_n|)\,dx \geq \int_{B_R^{(1)}}\left(\frac{1}{q_-} - \frac{1}{q(x)}\right)\frac{-1}{eq(x)}\,dx. \tag{3.7}$$

Then it follows from (3.4) and (3.7) that

$$\frac{m_0(q_- - 2p_+)}{2p_+q_-}\|u_n\|^{2p_-} \leq C + \|u_n\| + C_1R^N. \tag{3.8}$$

Dividing (3.8) by $\|u_n\|^{2p_-}$ and letting $n \to \infty$, we obtain

$$0 \geq \frac{m_0(q_- - 2p_+)}{2p_+q_-} > 0,$$

which is absurd. Thus $\{u_n\}$ is bounded in $W^{1,p(x)}_{0,rad}(B_R)$. Going if necessary up to a subsequence, we may assume that there exists $u \in W^{1,p(x)}_{0,rad}(B_R)$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in} \quad W^{1,p(x)}_{0,rad}(B_R), \\ u_n \to u & \text{strongly in} \quad L^{q(x)}(B_R), \\ u_n \to u & \text{a.e. in} \quad \Omega. \end{cases} \tag{3.9}$$

We now establish the claim

$$\lim_{n\to\infty}\int_{B_R}|u_n|^{q(x)-2}u_n(u_n-u)\log(|u_n|)\,dx=0. \tag{3.10}$$

To do this, we need to show that $|u_n|^{q(x)-1}\log(|u_n|)$ is bounded in $L^{q'(x)}(B_R)$ where $q'(x)=\frac{q(x)}{q(x)-1}$. Indeed, by Lemma 2.4 and (3.6) we obtain

$$\int_{B_R}\left||u_n|^{q(x)-1}\log(|u_n|)\right|^{q'(x)}\,dx\le c\left(\frac{1}{e(q_{-1}-1)}\right)R^N+\int_{B_R}|u_n|^{2q(x)}\,dx. \tag{3.11}$$

Notice that by (3.1) and (H_3), we have

$$\begin{aligned}\int_{B_R}|u_n|^{2q(x)}\,dx &= \int_{B_r}|u_n|^{2q(x)}\,dx+\int_{A_{R,r}}|u_n|^{2q(x)}\,dx\\ &\le cR^N+\|u_n\|_{2q_+^r}^{2q_+^r}+cR^N\left(\|u_n\|^{2q_-}+\|u_n\|^{2q_+}\right)\\ &\le cR^N+\|u_n\|^{2q_+^r}+cR^N\left(\|u_n\|^{2q_-}+\|u_n\|^{2q_+}\right)\le C_2.\end{aligned}$$

Here we have used the fact that $W^{1,p(x)}_{0,rad}(B_R)\hookrightarrow L^{2q_+^r}(B_R)$. Combining this inequality with (3.11), we get

$$\int_{B_R}\left||u_n|^{q(x)-1}\log(|u_n|)\right|^{q'(x)}\,dx\le C_3. \tag{3.12}$$

Using (3.9), (3.12) and Hölder's inequality, we deduce that claim (3.10) is valid. Since

$$\begin{aligned}\langle E'(u_n),u_n-u\rangle =&m\left(\int_{B_R}\frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right)\langle A(u_n),u_n-u\rangle+\int_{B_R}|u_n|^{q(x)-2}u_n(u_n-u)\log(|u_n|)\,dx\\ &+\int_{B_R}|u_n|^{q(x)-2}u_n(u_n-u)\,dx.\end{aligned}$$

Then, combining the fact that $E'(u_n)\to 0$ in $(W^{1,p(x)}_{0,rad}(B_R))^*$ with $u_n\rightharpoonup u$ weakly in $W^{1,p(x)}_{0,rad}(B_R)$ and (3.9)-(3.10), we conclude

$$\begin{aligned}m_0\limsup_{n\to\infty}\langle A(u_n),u_n-u\rangle &= m_0\limsup_{n\to\infty}\langle A(u_n)-A(u),u_n-u\rangle\\ &\le \limsup_{n\to\infty}m\left(\int_{B_R}\frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right)\langle A(u_n),u_n-u\rangle=0.\end{aligned}$$

Therefore, using Proposition 2.3, we arrive at the desired result, that is as $n\to\infty$

$$u_n\to u \text{ strongly in } W^{1,p(x)}_{0,rad}(B_R).$$

Hence, E satisfies $(\mathrm{PS})_d$ condition. Thus, the proof is now complete. □

Consequently, using Lemmas 3.1 and 3.2, we find a nontrivial solution $u\in W^{1,p(x)}_{0,rad}(B_R)$ for problem (1.1) that satisfies $E'(u)=0$ and $E(u)=d$. In other words, we have obtained a critical point of the functional E at the level d, that is for any $v\in W^{1,p(x)}_{0,rad}(B_R)$

$$m\left(\frac{1}{p(x)}\int_{B_R}|\nabla u|^{p(x)}\,dx\right)\int_{B_R}|\nabla u|^{p(x)-2}\nabla u.\nabla v\,dx=\int_{B_R}|u|^{q(x)-2}u\log(|u|)v\,dx+\int_{B_R}|u|^{q(x)-2}uv\,dx. \tag{3.13}$$

In the subsequent discussion, we will prove that these solutions are indeed weak solutions in $W_0^{1,p(x)}(B_R)$. In order to establish this, we need to verify that the equality (3.13) holds throughout $W_0^{1,p(x)}(B_R)$. An essential tool in this direction is the principle of symmetric criticality developed by Kobayashi and Otani [17, Theorem 2.2]. However, direct application of this theorem is not feasible in our case, as the energy functional associated with problem (1.1) is given by

$$E(u) = M\left(\frac{1}{p(x)}\int_{B_R}|\nabla u|^{p(x)}\,dx\right) + \int_{B_R}\left(\frac{1}{q^2(x)} - \frac{1}{q(x)}\right)|u|^{q(x)}\,dx - \int_{B_R}\frac{1}{q(x)}|u|^{q(x)}\log(|u|)\,dx,$$

which is not well defined in whole $W_0^{1,p(x)}(B_R)$. In order to overcome this difficulty, we borrow some ideas from [3]. Consider the function

$$\begin{aligned} g(x,t) =& \xi(|x|)\left(|t|^{q(x)-2}t\log(|t|) + |t|^{q(x)-2}t\right) + (1-\xi(|x|))\left(|u(x)|^{q(x)-2}u(x)\log(|u|)\right.\\ &\left.+|u(x)|^{q(x)-2}u(x)\right), \end{aligned}$$

for any $x \in B_R$, where $\xi \in C^\infty([0,R];\mathbb{R})$ satisfies

$$\xi(x) = \begin{cases} 1, & x \in \overline{B}_r(0), \\ 0, & x \in \overline{B}_R(0) \setminus \overline{B}_{\frac{3r}{2}}(0). \end{cases}$$

Since $u \in C(\overline{A}_{R,r})$ (see (3.1)), it follows from (H_3) that

$$|g(x,t)| \le C(|t|^{p^*(x)-1} + 1), \quad \forall (x,t) \in B_R \times \mathbb{R}. \tag{3.14}$$

Associated with the function g, we have the problem

$$\begin{cases} -m\left(\int_{B_R}\frac{1}{p(x)}|\nabla w|^{p(x)}\,dx\right)\Delta_{p(x)}w = g(x,w) \quad \text{in} \quad B_R, \\ w = 0 \quad \text{on} \quad \partial B_R, \end{cases} \tag{P_g}$$

whose associated energy is given by

$$E(w) = M\left(\frac{1}{p(x)}\int_{B_R}|\nabla w|^{p(x)}\,dx\right) - \int_{B_R}G(x,w)\,dx,$$

where $G(x,t) = \int_0^t g(x,s)\,ds$. From (3.14), E is well defined in the whole space $W_0^{1,p(x)}(B_R)$, $E \in C^1(W_0^{1,p(x)}(B_R);\mathbb{R})$ and

$$E'(w)v = m\left(\frac{1}{p(x)}\int_{B_R}|\nabla w|^{p(x)}\,dx\right)\int_{B_R}|\nabla w|^{p(x)-2}\nabla w\nabla v\,dx - \int_{B_R}g(x,w)v\,dx,\ \forall v \in W_0^{1,p(x)}(B_R).$$

Since

$$g(x,u(x)) = |u|^{q(x)-2}u(x)\log(|u|) + |u|^{q(x)-2}u(x), \quad \forall x \in B_R,$$

we see that u is a critical point of E restricted to $W_{0,rad}^{1,p(x)}(B_R)$. Now we can apply the Palais principle of symmetric criticality developed by Kobayashi and Otani [17, Theorem 2.2] to conclude that u is a nontrivial critical point of E in the whole $W_0^{1,p(x)}(B_R)$, that is for any $v \in W_0^{1,p(x)}(B_R)$,

$$\begin{aligned} m\left(\frac{1}{p(x)}\int_{B_R}|\nabla u|^{p(x)}\,dx\right)\int_{B_R}|\nabla u|^{p(x)-2}\nabla u\nabla v\,dx =& \int_{B_R}|u|^{q(x)-2}u(x)\log(|u|)v\,dx \\ &+ \int_{B_R}|u|^{q(x)-2}u(x)v\,dx. \end{aligned}$$

Thus, the proof is now complete.

Conflict of interest

The authors have no conflicts of interest to disclose.

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