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Non-informative Bayesian dispersion particle filter

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Abstract. In this research paper, we attempt to introduce a new algorithm for filtering a state-space model. The observations of this algorithm follow an exponential dispersion model. The paper focuses here on the inclusion of non-informative prior knowledge in parameter estimation on nonlinear state-space models using an improper uniform prior measure. Therefore, a new particle filter is introduced. A conventional particle filter (PF) produces an incorrect sample from a discrete approximation distribution. This new algorithm is a regularized continuous distribution method that is obtained with the exponential dispersion model. A necessary and sufficient condition for the existence and convergence of the non-informative Bayesian estimator of dispersion parameters is established. This methodology extends the classical PF implemented by this new estimation method for the exponential dispersion model framework using a non-informative Bayesian approach. In order to evaluate the performance of the proposed algorithm, a case study with simulations and microscopic image restoration is carried out. The results exhibit a great performance improvement from the proposed approach.

Keywords: Exponential dispersion model, particle filter, non-informative Bayesian prior, Microscopic image restoration.

2020 Mathematics Subject Classification: 65C60, 62F15, 62M20. MSC2020

1 Introduction

In the realm of image restoration, we encounter a pervasive problem that manifests itself in diverse scenarios. Regardless of the provenance of a digital image, be it a photograph, a scanned document, satellite imagery, or any other source, it is imperative to enhance the quality of the specific image under consideration (see [21, 22, 32]). In order to tackle the challenge of image restoration, numerous algorithms have been devised, employing either deterministic or stochastic approaches. These algorithms have been meticulously crafted to address an array of image restoration issues (see [25,29]). Numerous methodologies have been devised to tackle the issue of noise, encompassing both linear statistical filtering techniques and non-linear statistical filtering techniques (see [3, 5, 10, 15, 22]). In the specific scope of this research investigation, our focus lies on the non-linear statistical filtering approach applied

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to a state-space process, which is exemplified by the particle filtering (PF) technique (see [5, 9, 31]). The PF has emerged as a superior method for addressing non-linear and non-Gaussian systems, and it has recently been adapted to address challenges encountered in image processing (see [3, 22, 25]). Within the realm of Bayesian state estimation, the PF has garnered widespread acceptance (see [1, 10]).

Nevertheless, it is crucial to acknowledge that the PF technique exhibits certain limitations when applied in real-world scenarios. Notably, the diversity of particles diminishes in situations where the noise process is constrained, while the variation of importance weights escalates to exceedingly high levels when the likelihood significantly deviates from the previous probability distribution (see [10, 13]). In order to confront these challenges, this research introduces an exceptional method for regularizing the particle filter, specifically tailored for system state estimation and forecasting.

Microscopy imaging plays a pivotal role in the analysis of biological cells, providing researchers with valuable insights at the single-cell level (see [11,21,24]). Despite its advantages, microscopic images often suffer from noise generated during data acquisition and instrument read-back procedures. This noise deteriorates image quality and poses challenges for subsequent analysis, particularly in the context of cell culture. Therefore, the restoration of microscopic images from noise-induced degradation becomes a critical step prior to segmentation.

In this research, we focus on de-noising images obtained through differential-interference-contrast (DIC) microscopy (see [2, 24]). DIC microscopy employs phase shifts in transparent samples to generate black-and-white representations, enabling differentiation of structures within the sample. Previous work has explored state measurement in optical images and phase function reconstruction using methods like the Wiener filter.

Our specific research objective is to estimate the measurement of state observations in relation to the state process using a non-informative Bayesian approach. We adopt an improper uniform prior distribution due to its near-flatness across the effective likelihood range. This non-informative Bayesian framework allows us to approximate state-space filtering in a Markov chain system, and we use Bayesian inference to estimate the model parameters. To validate our proposed methodology, we conduct simulations using both data and microscopic image restoration, comparing the results with classical filter methods.

The present paper aims to propose a novel approach for estimating and generalizing the state-space model in the context of image restoration, leveraging the non-informative Bayesian framework. By addressing the limitations of the particle filtering technique and applying it to the specific case of DIC microscopy, we aim to contribute to the advancement of image restoration techniques in the field of microscopy.

The rest of this paper is organized as follows: In Section 2, the basic concept concerning the exponential dispersion models is extended and some characteristics as well as properties are recalled. Additionally, a short theoretical review of Bayesian filtering is presented. A comprehensive background of particle filter is portrayed. In the next section, the extended particle filter using the non-informative Bayesian approach in order to estimate the variance of measurement noise and the dispersion parameter λ, is depicted. In Section 4, the effectiveness of the proposed methodology is demonstrated using simulation studies. Section 5 presented the experimental results with a microscopic image restoration case and demonstrated the accuracy of the proposed algorithm. The last section wraps up the conclusion and offers different perspectives for future works.

2 Background

2.1 Exponential dispersion models

Exponential Dispersion Models (EDMs) are a type of statistical model in which the probability distributions take on a unique form (see [6,16,25,26,28]). Actuality, this class of models is an extension of the Natural Exponential Families (NEFs) which play an essential part in the statistical theory as a consequence of their specific structure that enables drawing deductions dealing with suitable statistical inference. In the following Section, we shall recall some important characteristics of EDMs provided by Jørgensen [17]. Let ν be a σ-finite positive measures on $\mathbb{R}$. We denote

$$\Theta(\nu) = int\left\{\theta \in \mathbb{R} : L_\nu(\theta) = \int_{\mathbb{R}} e^{\theta x}\nu(dx) < +\infty\right\} \neq \emptyset,$$
$$K_\nu(\theta) = \log\left(L_\nu(\theta)\right),$$

where L_ν and K_ν are the Laplace transform and the cumulant function of ν, respectively. Let us denote by $\mathcal{M}(\mathbb{R})$ the set of σ-finite positive measures such that $\Theta(\nu)$ is not a Dirac measure.

For all measures $\nu \in \mathcal{M}(\mathbb{R})$, the following set of probabilities

$$F = F(\nu) = \left\{p(\theta,\nu)(dy) = e^{\theta y - K_\nu(\theta)}\nu(dy)\,;\theta \in \Theta(\nu)\right\}, \tag{2.1}$$

is called the natural exponential family generated by ν. We refer the reader to [20], for more information.

Let us first point out that the map $\theta \longmapsto K'_\nu(\theta)$ is strictly convex, infinitely differentiable, and the value of its differential can be written as $K'_\nu(\theta) = \int_{\mathbb{R}} xp(\theta,\nu)(dx) = \mu$, which represents the mean of $p(\theta,\nu)$.

The first derivative K'_ν defines a diffeomorphism between $\Theta(\nu)$ and its image M_F, called the domain of means [16,25]. Let us denote by ψ_ν its inverse function.

On the other side, the map defined on M_F by $\mu \longmapsto V_F(\mu) = K''_\nu(\psi_\nu(\mu))$, is called the variance function of the NEF F (see [20]). It is easy to prove that for all $\mu \in M_F$, $V_F(\mu) = \left[\psi'_\nu(\mu)\right]^{-1}$. We come up with a new parametrization that we refer to as the mean parametrization. It should be noted that the NEF can be characterized by the variance function V_F. The Jørgensen set (see [17]) is defined as the following

$$\Lambda(\nu) = \left\{\lambda > 0, \exists \nu_\lambda \in \mathcal{M}(\mathbb{R}), L_\nu^\lambda = L_{\nu_\lambda}\right\}.$$

In this step, another useful class of models is presented, namely *exponential dispersion model EDM*. This class stands for a useful generalization of NEFs (see [6,25,26]). Its generated form is given by

$$f(x;\mu,\lambda)(dx) = e^{\psi_\nu\left(\frac{\mu}{\lambda}\right)x - \lambda K_\nu\left(\psi_\nu\left(\frac{\mu}{\lambda}\right)\right)}\nu_\lambda(dx)\,;\mu \in \lambda M_F, \lambda \in \Lambda(\nu). \tag{2.2}$$

It is called the additive version of the EDMs. The dispersion parameter $\sigma^2 = \frac{1}{\lambda}$ provides the richness that characterizes these models which in the case of NEFs is set to 1.

The Gaussian distribution is perhaps the most important distribution in statistics, particularly in regards to modelling applications like linear and non-linear regressions. This is because the Gaussian distribution is essential for both theoretical and practical reasons. Therefore, Jørgensen [16] suggested the reproductive EDMs as an extension of the Gaussian distribution.ăă According to Jørgensen [17], we took into consideration the following transformation: If $X \sim f(.;\mu,\lambda)$, where $\mu = \mathbb{E}(Y) = \lambda K'_{\nu}(\theta)$ is the expectation and λ is the dispersion parameter, then $Y = \frac{X}{\lambda}$ follows the reproductive distribution defined by

$$f(y;\mu,\lambda) = e^{\lambda[\psi(\mu)y - K_{\nu}(\psi(\mu))]}\nu^{*}_{\lambda}(dy), \tag{2.3}$$

and ν^{*}_{λ} denotes the image measure of ν_{λ} by the map $x \longmapsto \frac{x}{\lambda} = y$. This model is identified by both its expectation and its variance function, which are expressed as $\mathbb{E}(y) = K'(\theta) = \dfrac{\mu}{\lambda}$ and $Var(y) = \dfrac{V\left(\frac{\mu}{\lambda}\right)}{\lambda}$, where $\mu \in \lambda M_F$ and $\lambda \in \Lambda(\nu)$. To obtain additional details on the topic, we suggest that the reader refers to [25, 26]. This source may provide more comprehensive information and further insights regarding the subject matter at hand.

Suppose that we have a positive function $c(y,\lambda)$ and a σ-finite measure ζ such that $\nu^{*}_{\lambda} = c(y,\lambda)\zeta(dy)$. This assumption allows us to consider a broad class of models that can be formulated using the EDM framework. In particular, it enables us to model data that exhibit heterogeneity, overdispersion, and other complex phenomena that cannot be captured by traditional statistical models. Then, the reproductive version of the EDM is given by

$$f(y;\mu,\lambda) = e^{\lambda[\psi_{\nu}(\mu)y - K_{\nu}(\psi_{\nu}(\mu))]}c(y,\lambda)\zeta(dy). \tag{2.4}$$

In Table 2.1, we present necessary details of absolutely continuous PDFs of the EDM family specifying the normalizing constant ($c(y,\lambda)$), the cumulant function (K_{ν}), canonical parameter (θ), dispersion parameter (λ), mean (K'_{ν}), inverse function of the mean (ψ_{ν}) and variance function (V) of each distribution.

Table 2.1: Examples of some absolutely continuous PDF of EDMs.

	Gaussian	**Gamma**	**Inverse Gaussian**
$c(y,\lambda)$	$\frac{\sqrt{\lambda}}{\sqrt{2\pi}}e^{-\frac{\lambda y^2}{2}}$	$\frac{\lambda^{\lambda}y^{\lambda-1}}{\Gamma(\lambda)}$	$\frac{\sqrt{\lambda}}{\sqrt{2\pi}}y^{-\frac{3}{2}}e^{-\frac{\lambda}{2y}}$
K_{ν}	$\frac{\theta^2}{2}$	$-\log(-\theta)$	$-\sqrt{-2\theta}$
K'_{ν}	θ	$-\frac{1}{\theta}$	$(-2\theta)^{-1/2}$
ψ_{ν}	μ	$-\frac{1}{\mu}$	$-\frac{1}{2\mu^2}$
V	1	μ^2	μ^3

2.2 Bayesian filtering theory

Bayesian inference has been used extensively in the fields of numerical decision-making, classification and estimation, pattern recognition and machine learning and has become one of the most important divisions of statistics (see [3,5,9,15]). The work in [9], which laid down the theory and method of the Bayesian filtering, offers one of the first explorations of the iterative Bayesian estimation. It is a recursive algorithm and consists of two components: prediction and correction. Bayesian filter is equivalent to the Kalman filter (see [18]) when the parameters are normally distributed and if the transitions are linear.

Bayesian filter consists essentially in estimating the state of a dynamic system from generally noisy observations (see [5, 10, 22, 25]). Let us consider the following non-linear system model

$$x_k = f(x_{k-1}) + v_k, \tag{2.5}$$

$$y_k = h(x_k) + w_k, \tag{2.6}$$

where the state of the process x_k is assumed to be a discrete state-space Markov model, and the measurement process y_k is the observed state of the hidden Markov model at time k. $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $h : \mathbb{R} \longrightarrow \mathbb{R}$ are possibly non-linear functions. $\{v_k, k \in \mathbb{N}\}$ is a process of stochastic noise identified by its specified probability density (PDF) function. Likewise, $\{w_k, k \in \mathbb{N}\}$ is also an additive noise with identified measurements. These noises are independently and identically distributed (i.i.d.). It is a more general model in some respects.

The Bayesian filter in non-linear systems (see [10, 27, 31]) is based on observations to evaluate or estimate the posterior probability density function for the state sequences recursively as follows:

1. **The initial probability density function:**

$$p(x_0|y_0) = p(x_0). \tag{2.7}$$

2. **The prediction stage:** the prediction step consists of determining the posterior density of the state $p(x_k|y_{1:k-1})$ given as presented in the transition probability $p(x_k|x_{k-1})$ and the sequence of observations accessible $y_{1:k-1}$, using the Chapman-Kolmogorov equation and Markov property. Therefore, we obtain

$$p(x_k|y_{1:k-1}) = \int p(x_k|x_{k-1})p(x_{k-1}|y_{1:k-1})dx_{k-1}, \tag{2.8}$$

where $y_{1:k-1} = (y_1, ..., y_{k-1})$ and $p(x_{k-1}|y_{1:k-1})$ is available. In this step, The distribution of the probability involved with the predicted state is the sum (integral) of the probability distribution products related to the transition between the time $k-1$ and the time k and the previous probability distributions over all possible x_{k-1}.

3. **The update stage:** New sequences y_k at time k and the observation model lend the likelihood $p(y_k|x_k)$ that the prediction can be corrected through the update process. The Bayes rule allows to express the posterior density

$$p(x_k|y_{1:k}) = \frac{p(y_k|x_k)p(x_{k-1}|y_{1:k-1})}{p(y_k|y_{1:k-1})}, \tag{2.9}$$

where the denominator

$$p(y_k|y_{1:k-1}) = \int p(y_k|x_k)p(x_k|y_{1:k-1})dx_k \tag{2.10}$$

is constant, so that it can always be replaced in practice by the α coefficient. In this step, the distribution of probability of the update stage is relative to the measurement likelihood distribution and the predicted state, the likelihood function $p(y_k|x_k)$ determines essentially the measurement noise model in the equation (2.6).

2.3 Particle filter algorithm

The primary purpose of the particle filter (PF) is to estimate the posterior density of the state variables based on the observation variables (see [1, 3, 10, 25, 31]). An approximation of the optimum sequential Bayesian state estimation is carried out by a PF. This approximation is based on Monte Carlo simulation and importance sampling. The posterior pdf $p(x_k|y_{1:k})$, is approximatively represented in a centralized setting by N_s randomly drawn samples or particles x_k^i with related weights ω_k^i, for all $i = 1, ..., N_s$ (see [5, 9, 13]). To be more specific, the propagation of the posterior pdf $p(x_{k-1}|y_{1:k-1}) \to p(x_k|y_{1:k})$ is substituted with the propagation of the particles and weights $\left\{\left(x_{k-1}^i, \omega_{k-1}^i\right)\right\}_{i=1}^{N_s} \to \left\{\left(x_k^i, \omega_k^i\right)\right\}_{i=1}^{N_s}$ at each time step k (see [27]). This propagation consists of a prediction step and an update or correction step, in the same way as the optimal sequential Bayesian estimation scenario that was discussed in Section 2.2. The posterior density at k can be approximated with N_s particles by using the Sequential Importance Sampling (SIS) algorithm as

$$p(x_k|y_{1:k}) \approx \sum_{i=1}^{N_s} \omega_k^i \delta\left(x_k - x_k^i\right), \tag{2.11}$$

where δ is the Dirac measure and the weights ω_k^i represented in equation (2.11) are determined by $\omega_k^i \propto \frac{p(x_{k-1}^i|y_{1:k})}{q(x_{k-1}^i|y_{1:k})}$ in such a way that $\left\{x_k^i\right\}_{i=1}^{N_s}$ is drawn from the importance function $q(x_k^i|y_{1:k})$ (see [22, 25, 29]).

The method produced with the SIS is influenced by the phenomena of degeneration, in virtually all the particles, besides a handful of weights; all of them must stay close to zero and become insignificant. It is obvious that the degeneracy issue has an unfavourable impact on PF. The use of resampling whenever a significant degeneracy is observed is the method that can be utilized to decrease the negative effects that are brought on by degeneracy. The resampled particles may be generated in the simplest scenario by performing a sampling procedure with replacement from the set $\left\{\left(x_k^i, \omega_k^i\right)\right\}_{i=1}^{N_s}$, where x_k^i is sampled with probability ω_k^i. The weights are recalculated as $\omega_k^i = 1/N_s$. In this case, it is called Sampling Importance Resampling (SIR) filter. For more details, we refer the readers to [3, 10].

As a method to reduce the degeneration issues, which prevail in the proposed algorithm non-informative Bayesian Dispersion particle filter (NIBD-PF), the update of NIBD-PF recommended a resampling of the discrete distribution. On the other side, there have been other problems in the resampling method, particularly the question of losing diversity of particles. We suggested NIBD-PF in order to resolve this problem. This regularity method shall be determined at the point of the resampling phase, where the samples are taken from continuous instead of discrete distributions.

3 Proposed method

3.1 Non-informative Bayesian priors

In this section, we show that if we choose the prior of λ to be improperly uniform, then the resulting joint posterior distribution will be proper.

How to define the prior distribution is an important issue in the Bayesian analysis. If it has limited effects on the posterior distribution, a prior is non-informative. Despite, they are always difficult to find (see [9, 19, 28]), the non-informative prior in some applications are very popular. A large uniform distribution contains the usual non-informative priors (U(10000,10000),U(0,10000)), $(\mathcal{U}(-10000, 10000), \mathcal{U}(0, 10000))\,, or for distributed normal law\,(\mathcal{N}(0, 10000))\,.Alth$ the arguments of the authors who support the use of informative priors to overcome problems [12] and who convince the state that non-informational priors encourage model progress [14], many Bayesian Data processing textbooks include theoretical and computational examples of non-informative priors, avert making express recommendations for priors and do not demonstrate how informative priors can affect outcomes.

Let $y_{1:N} = (y_1, ..., y_N)$ be a sample of reproductive exponential dispersion distribution, the likelihood function is given by

$$L_N\,(y; \lambda, \mu) = \prod_{k=1}^{N} f\,(y_k; \lambda, \mu)\,. \tag{3.1}$$

Assume that the non-informative prior for λ based on improper distribution is $\pi(\lambda) = 1_{(0,+\infty)}(\lambda)$ and the non-informative Bayesian distribution of the parameter λ exists, if and only if, the integral

$$\int_0^{+\infty} L_N\,(y; \lambda, \mu)\, d\lambda < +\infty. \tag{3.2}$$

converges almost surely. In this case, the non-informative Bayesian density function $f\,(\lambda, \mu; y)$ of the parameter λ, is defined by

$$f\,(\lambda; \mu, y) = \frac{\prod_{k=1}^{N} f\,(y_k; \lambda, \mu)}{\displaystyle\int_0^{+\infty} \prod_{k=1}^{N} f\,(y_k; \lambda, \mu)\, d\lambda}. \tag{3.3}$$

If $\int_0^{+\infty} \lambda f\,(\lambda; \mu, y)\, d\lambda < +\infty$, then the non-informative Bayesian estimator $\hat{\lambda}$ of λ is given by

$$\hat{\lambda} = \mathbb{E}\,(\lambda | y, \mu) = \int_0^{+\infty} \lambda f\,(\lambda, \mu; y)\, .d\lambda \tag{3.4}$$

3.1.1 Parameter estimation

Now, we put forward a sufficient condition such that the non-informative Bayesian estimator of the parameter λ exists. The results are as follows

Theorem 1. Let $y_1, ..., y_N$ be N independent positive observations from reproductive exponential dispersion model $f\,(\lambda, \mu)$, then

$$\int_0^{+\infty} \lambda^s \prod_{k=1}^{N} f\,(y_k; \lambda, \mu)\, d\lambda < +\infty, \tag{3.5}$$

converges almost surely, for all $s \geqslant 0$. As a matter of fact, the non-informative Bayesian estimator $\hat{\lambda}$ of λ exists.

Before drawing in the proof of this theorem, the following lemma is introduced.

Lemma 1. Let ν be a probability measure concentrated on $\mathbb{R}^+$, then

1. $\exists t_0 \in \Theta(\nu)$ such that, $(-\infty, t_0] \subset \Theta(\nu)$.

2. $\lim_{t \to -\infty} K'_\nu(t) = 0$.

3. $\lim_{t \to -\infty} \frac{K_\nu(t)}{t} = 0.$

Proof 1. See [28].

Remark 1. Note that for $s = 0$, the non-informative Bayesian density function $f(\lambda; \mu, y)$ exists. If $s = 1$, the non-informative Bayesian estimator $\hat{\lambda}$ exists. The conditional variance of the non-informative Bayesian estimator $\mathbb{V}(\lambda | y, \mu)$ exists for $s = 2$.

Proposition 1. [17] Let $Y \sim f(., \mu, \lambda)$ be a continuous reproductive exponential dispersion model renormalized saddle point approximation. Then,

$$\frac{Y - \mu}{\sqrt{V(\mu)/\lambda}} \to \mathcal{N}(0,1), \quad when \quad \lambda \to +\infty. \tag{3.6}$$

Theorem 2. Let $y_1, ..., y_N$ be observations from $f(.; \mu, \lambda)$. Then

1. The posterior distribution of λ is given by

$$\lambda | y_1, ..., y_N \sim Ga\left(\frac{N}{2} + 1, \frac{1}{2}\sum_{k=1}^{N}\frac{(y_k - h(x_k))^2}{V(h(x_k))}\right).$$

2. The non-informative Bayesian estimator of λ is $\hat{\lambda} = \dfrac{N+2}{\sum_{k=1}^{N}\frac{(y_k - h(x_k))^2}{V(h(x_k))}}$.

Proof 2. According to Theorem 1 and Proposition 1, the non-informative Bayesian density function $f(\lambda, \mu; y_1, ..., y_N)$ (represented by the equation (3.3)) of the parameter λ can be evaluated as

$$\begin{aligned} f(\lambda; \mu, y_1, ..., y_N) &\propto \prod_{k=1}^{N} f(y_k; \lambda, \mu) \\ &\propto \prod_{k=1}^{N} \mathcal{N}\left(h(x_k), \frac{V(h(x_k))}{\lambda}\right) \\ &\propto \lambda^{\frac{N}{2}} e^{-\frac{\lambda}{2}\sum_{k=1}^{N}\frac{(y_k - h(x_k))^2}{V(h(x_k))}} \\ &\sim Ga\left(\frac{N}{2} + 1, \frac{1}{2}\sum_{k=1}^{N}\frac{(y_k - h(x_k))^2}{V(h(x_k))}\right). \end{aligned}$$

Therefore, for λ is large enough $\lambda \gg \lambda_0$, the non-informative Bayesian density function $f(\lambda, \mu; Y)$ is only the Gamma distribution with a shape parameter $\frac{N}{2} + 1$ and a scale one $\frac{1}{2}\sum_{k=1}^{N}\frac{(y_k - h(x_k))^2}{V(h(x_k))}$. Consequently, the non-informative Bayesian estimator $\hat{\lambda}$ of λ represented in equation (3.4), is obtained as $\hat{\lambda} = \dfrac{N+2}{\sum_{k=1}^{N}\frac{(y_k - h(x_k))^2}{V(h(x_k))}}$.

3.2 NIBD-PF Algorithm

In the following, we shall describe the NIBD-PF algorithm that we have presented and explain how it was derived from the SIR filter. In the context of recursive Bayesian filtering, the NIBD-PF is a Monte Carlo approach that may be used to solve these problems. The following algorithm describes the previous results

Algorithm 1 Non-informative Bayesian dispersion particle filter algorithm

1: **Initialization:** $x_0, \tau, \lambda^0, y_{1:N}, N_s, N, \epsilon$

2: **Generate:** $x_0^i \sim p(x_0)$; **for all** $i = 1 : N_s$

3: **Set:** $\omega_0^i = \dfrac{1}{N_s}$ **for all** $i = 1 : Ns$

4: **Iteration:** **for** $l = 1, 2, ...$
  for $k = 1 : N$

5: **Sample:** $\tilde{x}^i \sim \mathcal{N}\left(f\left(x_{k-1}^i\right), \tau\right)$; **for all** $i = 1 : N_s$

6: **Set:** $\tilde{\omega}^i = \omega^i \times \mathcal{N}\left(h\left(\tilde{x}_k^i\right), \frac{V\left(h\left(\tilde{x}_k^i\right)\right)}{\lambda^{(l)}}\right)(y_k)$; **for all** $i = 1 : N_s$

7: **Normalization:** $\tilde{\omega}^i \leftarrow \dfrac{\tilde{\omega}^i}{\sum_{i=1}^{N_s} \tilde{\omega}^i}$; **for all** $i = 1 : N_s$

8: **Compute:** $N_{eff} = \dfrac{1}{\sum_{i=1}^{N_s} \left(\tilde{\omega}^i\right)^2}$; **for all** $i = 1 : N_s$

9: **if** $N_{eff}/N_s \leqslant 0.75$
  Draw: $x^{1:N_s} \sim$ re-sample $\left(\tilde{\omega}^{1:N_s}, \tilde{x}^{1:N_s}\right)$
  Set: $\omega^{1:Ns} = \dfrac{1}{N_s}$
  else $x^{1:N_s} = \tilde{x}^{1:N_s}$
  $\omega^{1:N_s} = \tilde{\omega}^{1:N_s}$

10: **Set:** $\hat{x}_k = \sum_{i=1}^{N_s} \omega_k^i x_k^i$

11: **Update the dispersion parameter model of observation state, according to the following formula**

$$\lambda^{(l+1)} = \frac{N+2}{2\sum_{k=1}^{N} \frac{\lambda^{(l)}\left(y_k - h\left(x_k^i\right)\right)^2}{V\left(h\left(x_k^i\right)\right)}}$$

12: If $\left|\lambda^{(l+1)} - \lambda^{(l)}\right| < \epsilon$ is not satisfied return to step 4

4 Simulation Study

In this section, numerical simulation study results are highlighted to assess the proposed algorithm in this paper. Moreover, this brief section shows an example of how a non-linear and non-normal stochastic state space model can be evaluated using the proposed approach. Let us consider the following state model

$$x_k = \alpha x_{k-1} + \beta \cos(0.5\pi t) + v_k, \tag{4.1}$$

$$y_k = \gamma x_k^2 + w_k, \quad k = 1, ..., N, \tag{4.2}$$

where v_k follows zero-mean Gaussian distribution with variance equal to 0.01. The distribution of y_k given the state process x_k is a Gaussian dispersion model with power parameter $p = 1.5$. Data were generated for $\alpha = 0.5$, $\beta = 0.8$ and $\gamma = 0.3$. This model will explain adjustment at the measurement level of a non linearity in a system. According to the same parameter λ of the Inverse Gaussian dispersion model, multiple different filters have been used to approximate the hidden state x_t. The results are validated by comparing them with various algorithms such as the extended Kalman Filter (EKF) [5], the unscented Kalman Filter (UKF) [31] and the particle filter (PF) [9] in order to assess the effectiveness of the proposed methodology. The simulation results are summarized in Figure 4.1. The experiment has been generated for $N = 60$ data sets considering different sample sizes run $N_s = 100, 200, 500$.

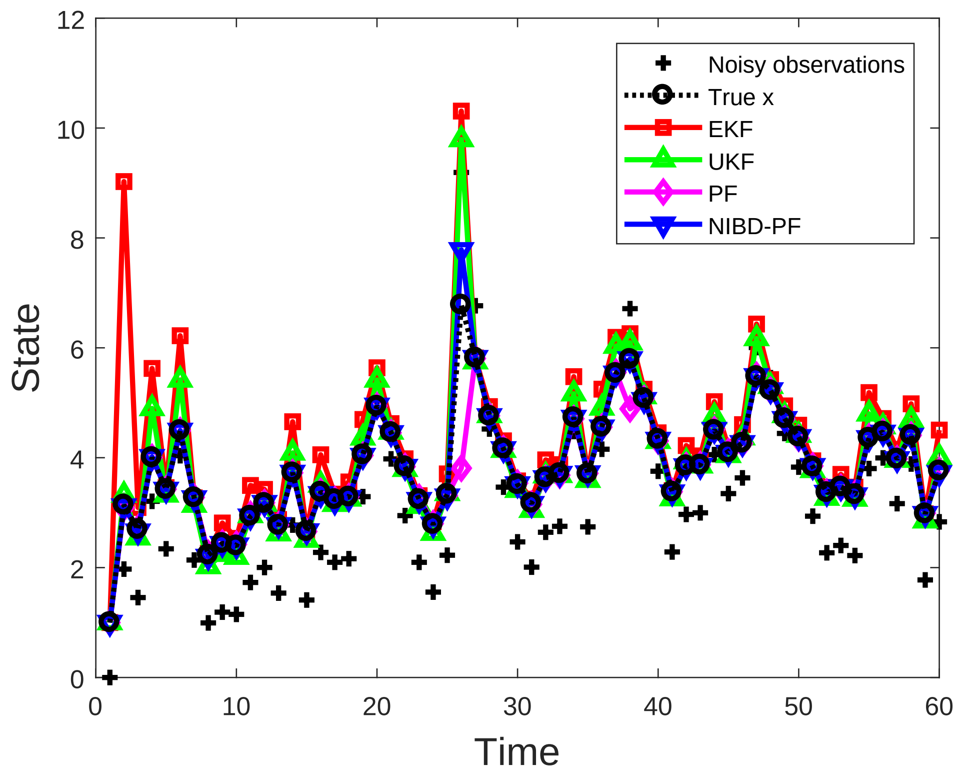

Figure 4.1: The evolution of the nonlinear model x_k as a function of k by different algorithms EKF, UKF, PF and NIBD-PF for $N = 60$, $N_s = 100$.

Figure 4.1 shows the estimated results of different nonlinear filters. Firstly, we can observe that the results of the proposed approach are similar to the original model and it is obvious that it has better performance.

The algorithm efficiency is the mean of the set of samples which can be determined by the formula $\hat{x}_k = \frac{1}{N_s} \sum_{i=1}^{N_s} x_k^i$, (see [31]). The Root Mean Square Errors (RMSE) of each run is expressed as $RMSE = \sqrt{\frac{1}{N} \sum_{k=1}^{N} (\hat{x}_k - x_k)^2}$.

Table 4.1 portrays the results of a comparison of the means and variance of the RMSE of the state estimate by different algorithms. It shows that the NIBD-PF filter offers excellent efficiency among the techniques provided and RMSE has a considerably number of independent available particles. This property is found here to be even more clearly displayed than for the conventional PF filter. Furthermore, we see that the UKF gives good performance while the EKF does not really function well.

Table 4.1: Mean and Variance of the RMSE of different algorithms

Algorithms	RMSE	
	Mean	Variance
EKF	0.9023	0.0031
UKF	0.6022	0.0024
PF (N_s=100,200,500)	0.7211, 0.3679, 0.3187	0.0271, 0.0187 , 0.0026
NIBD-PF (N_s=100,200,500)	0.1608, 0.1422, 0.0063	2.9224×10^{-3}, 1.3791×10^{-3}, 3.1902×10^{-4}

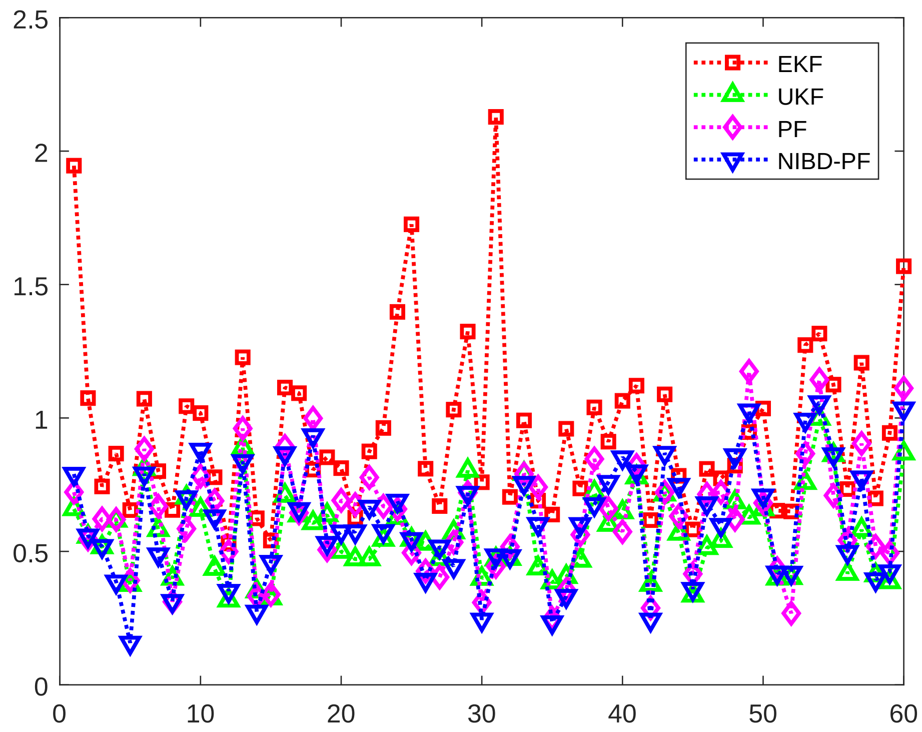

Figure 4.2: The RMSE of 60 times Monte Carlo simulations of different algorithms

Figure 4.2 displays the RMSE that occurred during 60 iterations of the Monte Carlo simulation. All algorithms exhibit a satisfactory performance. This figure shows that NIBD-PF have achieved greater accuracy than those calculated with the classical EKF, UKF and PF filters (blue curve). non-informative Bayesian estimate enables the filter to restrict its uncertainty, which ultimately leads to a lower error.

5 Experimental Results

In this section, experimental results are provided to prove the efficiency of the algorithm suggested. In view of the large number of cellular images generated in bio-images (see [2,4,7,24]), the precise de-noising of an image in each cell is an important preliminary processing step to assess the content of cell type and size images in several pipelines (see [11,14,32]). Biological structure observation is a challenge, particularly in the field of live cell imagery. Optical microscopes are in general restricted by light diffract and the optical property of the object, such as spatial shifts in the refractive index, influences imaging, which contributes to aberrations while the light passes the object (see [4]). As the majority of cell components are transparent to visible light (see [11]) a high level of water may result in loss of contrast in conventional light microscopy, which is why staining is often used to create contrast with light absorption (see [8]). However, the living cells may be damaged by such a procedure. Differential interference contrast (DIC) microscopy developed by R. D. Allen, G. B. David and G. Nomarski [2] is the technology of our interest to address the inability to image untainted, visible biological specimens which are typical of bright field microscopes. One of the drawbacks of DIC microscopy is that the detected images can not be used conveniently for topography and morphology, because the light phase variations are latent within the intensity image. The restoration of the phase feature of the specimen from the observed DIC images is thus necessary (see [7,21,23]).

Previous information is supposed to describe the actual image structure by a 2D autoregression model with zero mean Gaussian noises $v(m,n)$. This is the only way to describe the image structure. The equation is expressed by

$$s(m,n) = a_1 s(m,n-1) + a_2 s(m-1,n) + a_3 s(m-1,n-1), \tag{5.1}$$

where a_1, a_2, a_3, are the image model coefficients (assumed stationary) which are addressed by the form of the Last Squares (see [6,25]), and $s(m,n)$ presents pixel estimated in the original image from past pixels at the m^{th} row and n^{th} column.

A general description of the state-space differences is of the form

$$x(m,n) = Cx(m-1,n) + Eu(m,n) + Dv(m,n), \tag{5.2}$$

$$y(m,n) = Hx(m,n) + w(m,n), \tag{5.3}$$

$$x(m,n) = \begin{pmatrix} s(m,n) \\ s(m,n-1) \\ s(m-1,n+1) \\ s(m-1,n) \end{pmatrix}, \quad C = \begin{pmatrix} a_1 & 0 & a_2 & a_3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$E = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}^T, \quad D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^T, \quad H = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix},$$

where $x(m,n)$ is the state at location (m,n); u denotes a deterministic input, The variables $v(m,n)$ and $w(m,n)$ denote the process noise and the measurement noise following the Normal distribution. According to the histogram of the original image, we fixed the power parameter $p = 2.3$. C, D, and E are system matrices. The deterministic input term, u, has not been used previously for the proposed algorithm and it is introduced as the recent estimate of pixel $s(m-1,n+1)$. $y(m,n)$ is the intensity in the degraded image at location (m,n).

Yeast is one of the most simple eukaryotic organisms, and therefore we selected this microscopic image in this simulation. However, in yeast and humans, certain critical cell functions (see [11]) are the same. This is why an analysis to understand basic molecular mechanisms in humans is an essential subject. Fission yeast or Schizosaccharomyces pombe 5.1(a) has emerged as a common cell growth and division model for investigation. It is effective partly because its cells have a constant size and only expand in volume (see [32]), it is very easy to record cell growth. Fission yeast chromosomes exchange many key elements with human chromosomes, which render the organism very beneficial in human genetics. The parameters of microscopic image of a fission yeast are identified using the Least Squares method as $a_1 = 0.8315$, $a_2 = 0.7128$ and $a_3 = -0.5443$.

Figure 5.1 shows the original Fission yeast cell image, the noisy image and restored image by Extended Kalman Filter (EKF) [5], restored image by Bootstrap Kernel-Diffeomorphism Filter (BKDF) [22] and restored image by non-informative Bayesian Dispersion particle filter (NIBD-PF) and their histograms, respectively. We can notice from the visual results for the microscopic image, the improvement of the proposed method over the other methods introduced.

Standard 325×260 pixels, microscopic gray image is used in this section to assess the efficiency of the proposed algorithm. From the original image x and the restored one $\hat{x}$, and four calculation criteria are presented to evaluate the performance of the proposed filter which we shall compare it to the EKF and BKDF. The metrics are given by:

- Mean absolute error (MAE), $MAE = \frac{1}{MN}\sum_{m,n=1}^{M,N} |s(m,n) - \hat{s}(m,n)|$.
- Mean square error (MSE), $MSE = \frac{1}{MN}\sum_{m,n=1}^{M,N} (s(m,n) - \hat{s}(m,n))^2$.
- Pic of signal to noise ratio (PSNR), $PSNR = 10\log_{10}\left(\frac{Max_I^2}{MSE}\right)$
- Image fidelity (IF), $IF = 1 - \frac{\sum_{m,n=1}^{M,N} (s(m,n) - \hat{s}(m,n))^2}{\sum_{m,n=1}^{M,N} (s(m,n))^2}$

Table 5.1: Results of comparison of EKF, BKDF and NIBD-PF with different criteria

Algorithms	MAE	MSE	PSNR	IF
EKF	35.4969	162.1583	27.1692	0.3641
BKDF	4.7656	47.9883	32.4572	0.8118
NIBD-PF	3.5052	30.7906	34.3844	0.8792

The obtained results for the Fission yeast cell image using NIBD-PF algorithm in Table 5.1 are shown along with the results from the reconstruction of the same image using the EKF and BKDF. It is clear from the statistical metrics values, that the proposed algorithm gives better results and more closer to the ideal microscopic image. The lower MAE, MSE, the increased PSNR value and the closing of IF to 1 show the performance of the NIBD-PF method which means fewer errors and higher quality of the Fission yeast cell image restoration than that obtained with EKF and BKDF.

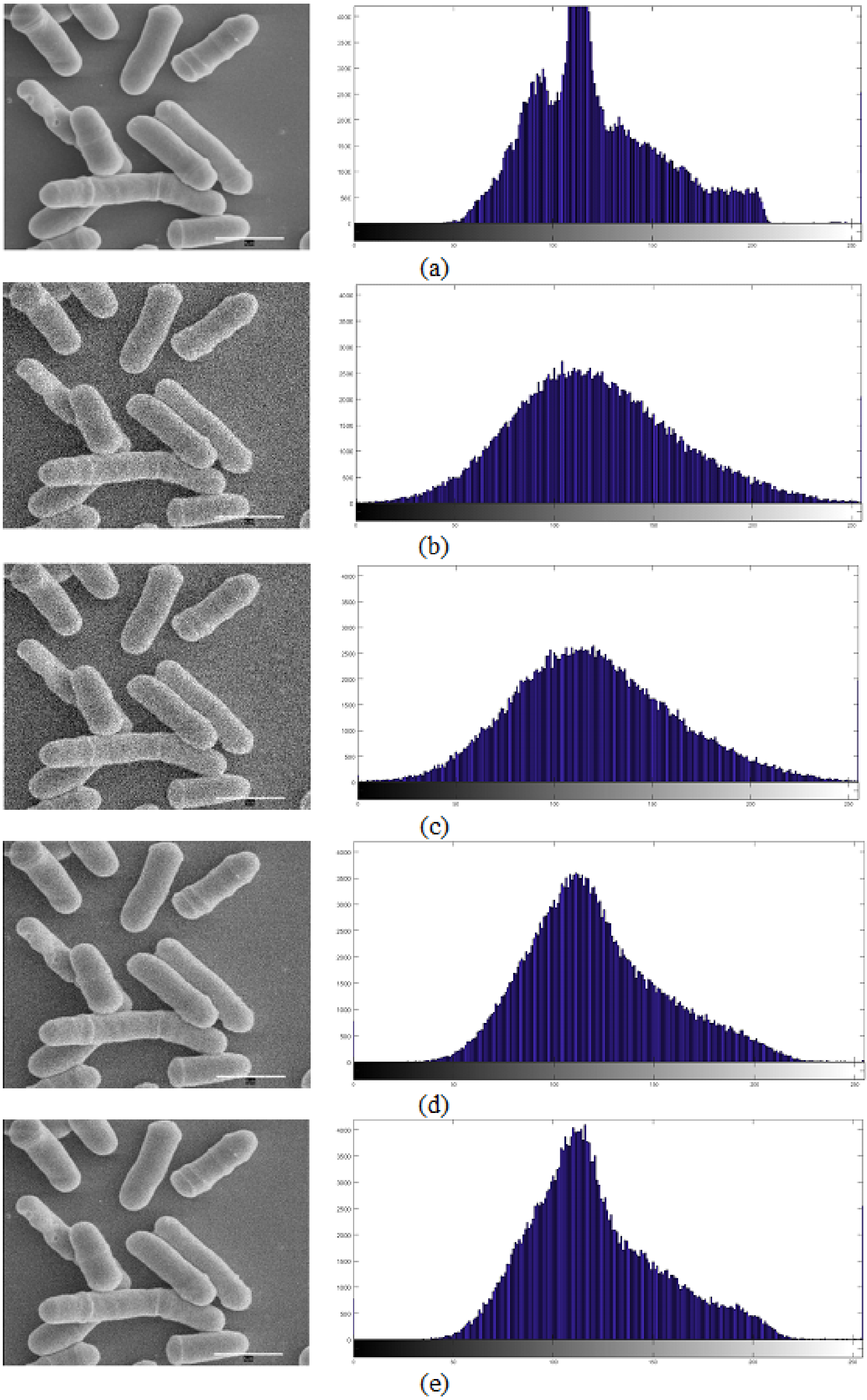

Figure 5.1: The results of the Original image (a), Noised image (b), Restored image by EKF (c), Restored image by BKDF (d), Restored image by NIBD-PF (e)

6 Conclusion and discussion

At this stage of analysis, we would assert that the basic objective of this study is to enhance and extend our knowledge on the state-space model. For this reason, the exponential dispersion distribution, acting as a measurement noise, was introduced. In addition, the dispersion parameter model were estimated by the non-informative Bayesian approach. Our ultimate purpose was to overcome the deficiency of the former method and estimate the dispersion parameter λ. Besides, the existence of this parameter's estimator is proved. The results from the simulation study show that the current particulate lter performs better than the conventional particle lter when equivalent numbers of the particles are used. This suggests that this algorithm will provide significant enhancements compared to other non-linear filtering algorithms. Therefore, we adopted the proposed algorithm to the area of microscopic image (Fission yeast) restoration. This paper contains a comparison with other restoration techniques (like the EKF and BKDF). The criteria of MAE, MSE, PSNR and IF results were used to prove the efficiency of NIBD-PF in Table 5.1. The findings of our graphics and metrics reveal the high performance of NIBD-PF in restoring noised microscopic images according to EKF and BKDF methods. To this extent, our work is a step that may be taken further as it paves the way and lays the ground for future works to explore further this new approach and investigate its characteristics. In future research, we aspire to extend the proposed methodology using multivariate Tweedie distributions to colour images. We also seek to converge the estimated parameter to the optimal value. This is currently being investigated.

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