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A new concept of q-calculus with respect to another function

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Abstract. This paper pioneers a functional approach to quantum calculus, providing a new perspective on its number-theoretic properties. By leveraging functional methods, we introduce a framework for modifying variable - order q-differential equations and their solutions. This work significantly advances the field of quantum calculus, particularly in the areas of functional quantum number theory and functional - order q-derivatives.

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1 Introduction

The q-calculus, also known as quantum calculus, is often referred to as "calculus without limits" [9] since it parallels traditional infinitesimal calculus but without relying on the concept of limits. In this work, we primarily follow the notation and approach introduced by Kac and Cheung [9]. Most definitions, concepts, and methods in q- calculus tend to adopt a number-theoretic or, more broadly, a discrete approach [4–10].

The q-calculus serves as a bridge between mathematics and physics [6], with intriguing applications in quantum groups, Quantum Field Theory, and General Relativity [12]. Many mathematicians have contributed to the development of calculus in a quantum framework [6,9,10], which is now known as q-calculus.

The main motivation towards extending the definition in functional analogue is that, It offers a powerful tool for modeling complex systems with time-varying dynamics, non-linear behavior, and memory effects. By providing a more flexible and adaptable framework, it has the potential to advance our understanding of a wide range of phenomena in various scientific and engineering disciplines in the future.

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The paper explores an alternative approach to defining q-calculus and its properties, with a focus on its functional formulation and applications across various fields. We have demonstrate how this approach reveals interesting and elegant properties in different contexts. Additionally, we highlight the functional q-integral, which builds upon the q-integral introduced by F.H. Jackson, who pioneered the systematic development of q-calculus in the early 20^{th} century.

In Section 2, we present the basic definitions and properties of quantum calculus, while Section 3 introduces new definitions that build upon the existing ones which include the functional approaches of q-calculus and h-calculus, including the q- derivative and h-derivative, along with the relationships between them and the fundamental properties of the newly defined functions. Section 4 discusses applications of these concepts across various fields, leading into the conclusion. In Section 5, we prove several theorems based on the new definitions and properties, and in Section 6, we explore applications in number theory. The chapter concludes with a summary and references.

2 Basic definition and properties

The concept of the derivative using q-differential and h-differential is similar to ordinary differentiation, but with some differences [9].

Definition 2.1. (q-derivative)
Given an arbitrary function $f(x)$ then its q-derivative is denoted by $D_q f(x)$ and is given by [9]

$$D_q f\left(x\right) = \frac{f\left(qx\right) - f(x)}{qx - x} = \frac{f\left(qx\right) - f(x)}{\left(q-1\right)x}. \tag{2.1}$$

Definition 2.2. (h- derivative)
Given an arbitrary function $f(x)$ then its h-derivative is denoted by $D_h f(x)$ and is given by [9]

$$D_h f\left(x\right) = \frac{f\left(x+h\right) - f(x)}{h}. \tag{2.2}$$

If $f(x)$ is differentiable then $\frac{df(x)}{dx} = \lim_{q\to 1} D_q f\left(x\right) = \lim_{h\to 0} D_h f\left(x\right)$ and both the operator satisfies linearity property [9].

Property 2.3. (q-analogue of n)
Given any positive integer n then its q-analogue is denoted by $[n]$ and given by [9]

$$[n] = \frac{q^n - 1}{q-1} = 1 + q + q^2 + \cdots + q^{n-1}. \tag{2.3}$$

Property 2.4. (q-derivative of product)
Given an arbitrary function $f\left(x\right)$ and $g(x)$ then its q-derivative of the product is denoted by $D_q(f\left(x\right) g\left(x\right))$ and is given by [9]

$$D_q\Big(f\left(x\right) g\left(x\right)\Big) = f\left(qx\right) D_q g\left(x\right) + g\left(x\right) D_q f\left(x\right) = f\left(x\right) D_q g\left(x\right) + g\left(qx\right) D_q f\left(x\right). \tag{2.4}$$

Property 2.5. **(q-derivative of quotient)**
Given an arbitrary function $f(x)$ and $g(x)$ then its q-derivative of the quotient is denoted by $D_q(f(x)\,g(x))$ and is given by [9].

$$D_q\Big(\frac{f(x)}{g(x)}\Big) = \frac{g(qx)\,D_q f(x) - f(qx)D_q g(x)}{g(x)\,g(qx)}. \tag{2.5}$$

In a similar manner, If one goes to check whether or not q-derivative satisfies the chain rule or not then there doesn't exist a general [9] chain rule for quantum version. In the h – derivative the letter [9] h is used as the remainder of a Planck's constant having relationship between q and h given by [9] $q = e^h$.

Though the q-derivatives offer a powerful tool for modeling complex systems with quantum-like properties, they present certain challenges which include Mathematical Complexity, Physical Interpretation and Lack of Standardized Tools and Software.

To overcome these challenges, one should extend q-derivative towards its functional extension as it emphasizes the ability to model systems with time-varying quantum - like dynamics, which is not possible with fixed - order or constant q-parameter models and will encourage collaboration between researchers to develop standardized methods and tools to facilitate the adoption of variable order fractional q-calculus.

3 A functional extension of q-derivative

Now we approach the above definition using a functional manner i.e. q and h will be now functions which depend upon time *'t'* satisfying some conditions. We define the following sets of real-valued functions as:

$$Q = \{\, q(t)\ /\ q(t) \to 1 \ as\ t \to 0\,,\ and\ q(t) \neq 1\, \forall t\ \in\ \mathbb{R}\,\}.$$

The above set is non-empty i. e. $Q\ \neq \emptyset$ and are well defined. e.g., If $q(t) = 1 + t\,,\ t\ \in\ \mathbb{R}$ then $q(t)\ \in Q$.

$$H = \{\, h(t)\ /\ h(t) \to 0 \ as\ t \to 0\ and\ h(t) \neq 0\, \forall t\ \in\ \mathbb{R}\,\}.$$

The above set is non-empty i.e. $H\ \neq \emptyset$ and are well defined. e.g., If $h(t) = t\,,\ t\ \in\ \mathbb{R}$, then $h(t)\ \in H$.
Here onwards, throughout the remaining part, $q(t)\ \in Q$ and $h(t)\ \in H$.

Definition 3.1. **(Functional analogue of differential)**
Given an arbitrary function $f(x)$ then its $q(t)$-differential is

$$d_{q(t)}f(x) = f(q(t)\,x) - f(x),$$

and its $h(t)$-differential is

$$d_{h(t)}f(x) = f(x + h(t)) - f(x).$$

Definition 3.2. (**q(t)-derivative**)
Given an arbitrary function $f(x)$ then its q(t)-derivative is denoted by $D_{q(t)}f(x)$ and is given by

$$D_{q(t)}f\left(x\right) = \frac{f\left(q(t)x\right) - f(x)}{q(t)x - x} = \frac{f\left(q(t)x\right) - f(x)}{\left(q(t) - 1\right)x}. \tag{3.1}$$

Definition 3.3. (**h(t)-derivative**)
Given an arbitrary function $f(x)$ then its h(t)-derivative is denoted by $D_{h(t)}f(x)$ and is given by

$$D_{h(t)}f\left(x\right) = \frac{f\left(x + h(t)\right) - f(x)}{h(t)}. \tag{3.2}$$

From the definitions (2.1), (2.2), (3.1) and (3.2), one can say that functional q-derivative and functional h-derivative depends upon $q(t)$ and $h(t)$ respectively and both depends upon the parameter t.

Property 3.4. (**q(t)-analogue of *n***)
Given any positive integer n then its q(t)-analogue is denoted by $[n]_{q(t)}$ and given by

$$[n]_{q(t)} = \frac{[q(t)]^n - 1}{q(t) - 1} = 1 + q(t) + [q\left(t\right)]^2 + \cdots + [q\left(t\right)]^{n-1}. \tag{3.3}$$

Property 3.5. (q**(t)-analogue of *n*!**)
Given any positive integer n then its $q(t)$-analogue is denoted by $[n!]_{q(t)}$ and given by

$$[n!]_{q(t)} = [n]_{q(t)} \times [n-1]_{q(t)} \times \ldots. \times [3]_{q(t)} \times [2]_{q(t)} \times [1]_{q(t)}. \tag{3.4}$$

So, depending upon the choice of q**(t).** The value of $[n!]_{q(t)}$ and $[n]_{q(t)}$ varies. In other words, if $q_1\left(t\right) \to 1$ as $t \to 0$ much faster than $q_2\left(t\right) \to 1$ as $t \to 0$. Then $[n!]_{q_1(t)}$ and $[n]_{q_1(t)}$ tends much faster than $[n!]_{q_2(t)}$ and $[n]_{q_2(t)}$ to the value of $n!$and n as $t \to 0$.

Property 3.6. (q**(t)-shifting operator**)
The $q\left(t\right) -$ shifting operator [15] is defined as

$$\Phi^a_{q(t)}\left(m\right) = q\left(t\right)m + \left(1 - q\left(t\right)\right)a, \tag{3.5}$$

where,

$$D_{\Phi^a_{q(t)}(m)}f\left(x\right) = \frac{f\left(\Phi^a_{q(t)}\left(m\right)x\right) - f(x)}{(\Phi^a_{q(t)}\left(m\right) - 1)x}. \tag{3.6}$$

From the properties ((3.5) and (3.6)), it can be easily seen that,
1) $D_{\Phi^0_{q(t)}(1)}f\left(x\right) = D_{mq(t)}f\left(x\right)$.
2) $D_{\Phi^a_{q(t)}(a)}f\left(x\right) = D_a f\left(x\right)$ for $a > 0$.
3) $D_{\Phi^m_{q(t)}(m)}f\left(x\right) = D_m f\left(x\right)$ for $m > 0$.
4) $\Phi^a_{q(t)}\left(m\right) \to {}_a\Phi_q\left(m\right)$ as $t \to 0$.

Property 3.7. **(Linear operator)**
The definition (3.5) satisfies the property that;

$$\Phi^{a}_{q_1(t)+q_2(t)}(m) = \{q_1(t) + q_2(t)\}m + (1 - \{q_1(t) + q_2(t)\})\,a,$$

it follows that

$$\Phi^{a}_{q_1(t)+q_2(t)}(m) = \{q_1(t)\}\,m + \{q_2(t)\}\,m + \ (1 - q_1(t))a + (1 - q_2(t))\,a.$$

Thus,

$$\Phi^{a}_{q_1(t)+q_2(t)}(m) = \ \Phi^{a}_{q_1(t)}(m) + \ \Phi^{a}_{q_2(t)}(m).$$

Which means, the operator $\Phi^{a}_{q(t)}(m)$ is a linear operator.

Property 3.8. **(Functional quantum congruence)**
The way in which we have congruence relations among the integer. One can extend the same towards a functional approach of the congruence relation under the defined conditions:

$$\text{If } b \ \equiv a \ (mod\ m) \text{ then } [b]_{q(t)} \ \equiv [a]_{q(t)} \ (mod\ m).$$

Whenever there exist some $q(t) \in \ Q$ such that $\left|[b-a]_{q(t)} - km\right| \ \to 0$ where b, a, k and m are positive integers.

4 Theoretical applications

From a theoretical point of view; we have found out some properties of Q and H and different properties using $q(t)$-analogue of n as well as $q(t)$-analogue of $n!$ along with the topological properties of Q and H.

Theorem 4.1. *There is no common function between Q and H i.e. $Q \cap H = \emptyset$.*

Proof. As we know the limit of a function if it exists then it is unique. So, by construction of Q and H one can see that there is no function $f(t)$ such that $f(t) \ \to 1$ and $\ f(t) \to 0$ *as* $t \ \to 0$. Hence, $Q \ \cap H = \ \emptyset$. □

Theorem 4.2. *If $|q_1(t)| < \ |q_2(t)|$ then $[n]_{q_1(t)} < \ [n]_{q_2(t)}$ as $t \to 0$ where, $q_1(t)\ ,\ q_2(t) \ \in Q$.*

Proof. The $q(t)$**-analogue of** n is given by,

$$[n]_{q(t)} = \ 1 + q(t) + [q(t)]^2 + \cdots + [q(t)]^{n-1}.$$

It is given that, $|q_1(t)| < \ |q_2(t)|$.
Hence, we have

$$\left|1 + q_1(t) + [q_1(t)]^2 + \cdots + [q_1(t)]^{n-1}\right| < \ \left|1 + q_2(t) + [q_2(t)]^2 + \cdots + [q_2(t)]^{n-1}\right|,$$

it follows that $[n]_{q_1(t)} < \ [n]_{q_2(t)}$, as $t \to 0$. □

Theorem 4.3. *If $|q_1(t)| < |q_2(t)| \Rightarrow [n!]_{q_1(t)} < \ [n!]_{q_2(t)}$ as $t \to 0$ where, $q_1(t)\ ,\ q_2(t) \ \in Q$.*

Proof. The $q(t)$-**analogue of** $n!$ is given by,

$$[n!]_{q(t)} = [n]_{q(t)} \times [n-1]_{q(t)} \times \ldots . \times [3]_{q(t)} \times [2]_{q(t)} \times [1]_{q(t)}.$$

Using **(Theorem 2)** and the property given $|q_1(t)| < |q_2(t)|$.
It resulted into,

$$[n!]_{q_1(t)} < [n!]_{q_2(t)} \text{ as } t \to 0 \text{ where, } q_1(t)\ ,\ q_2(t)\ \in Q.$$

□

Theorem 4.4. *If* $H_1 = \{\ h(t)\ \in H/\ h(t)\ \in [0, b]$ and continuous $\}$ *then* H_1 *is non-empty. Moreover,* H_1 *is bounded, similarly If* $Q_1 = \{\ q(t)\ \in Q/\ q(t)\ \in [0, b]$ and continuous $\}$ *then* Q_1 *is non-empty. Moreover,* Q_1 *is bounded.*

Proof. **Step I)** By construction of H_1 *and* Q_1; it can be seen that it is non-empty.
Step II) We know that the set of all continuous functions with compact support is bounded, hence H_1 *and* Q_1 both are bounded sets.

□

5 Examples

In this section, we solve a few examples as an application of theorems 4.3, 3.1 and 3.2 along with the computation of $D_{q(t)}$ and $D_{h(t)}$ of the given functions.

5.1 Computation of $[n]_{q(t)}$

[5.1.A] $q(t) = (1+t)$ - analogue of $n = 3$ for $t = 0.1, 0.2, 0.3$,
[5.1.B] $q(t) = e^{-t}$ - analogue of $n = 3$ for $t = 0.1, 0.2, 0.3$.

Answer: In view of Definition [3.3]. We have,

$$[n]_{q(t)} = 1 + q(t) + [q(t)]^2 + ... + [q(t)]^{n-1}.$$

For $q(t) = (1+t)$ it gives

$$\begin{aligned} & [3]_{(1+t)} &=& \ 1 + (1+t) + (1+t)^2, \\ t = 0.1, \quad & [3]_{(1+t)} &=& \ 1 + 1.1 + +1.21 = 3.31, \\ t = 0.2, \quad & [3]_{(1+t)} &=& \ 1 + 1.2 + +1.44 = 3.64, \\ t = 0.3, \quad & [3]_{(1+t)} &=& \ 1 + 1.1 + +1.69 = 3.99. \end{aligned}$$

For $q(t) = e^{-t}$ it gives

$$\begin{aligned} & [3]_{(e^{-t})} &=& \ 1 + (e^{-t}) + (e^{-t})^2 \implies [3]_{(e^{-t})} = 1 + (e^{-t}) + (e^{-2t}), \\ t = 0.1, \quad & [3]_{(e^{-t})} &=& \ 1 + (e^{-0.1}) + (e^{-0.2}) = 1 + 0.9048 + 0.8187 = 2.7235, \\ t = 0.2, \quad & [3]_{(e^{-t})} &=& \ 1 + (e^{-0.2}) + (e^{-0.4}) = 1 + 0.8187 + 0.6703 = 2.489, \\ t = 0.3, \quad & [3]_{(e^{-t})} &=& \ 1 + (e^{-0.3}) + (e^{-0.6}) = 1 + 0.7408 + 0.5488 = 2.2896. \end{aligned}$$

As one can see, $[3]_{(e^{-t})} \to 3$ much faster than $[3]_{(1+t)} \to 3$ as $t \to 0$

5.2 Computation of derivatives:

[5.2.A] $D_{q(t)=(1+t)}$ - Derivative of $f(x) = x^n, n \in \mathbb{N}$
[5.2.B] $D_{h(t)=t}$ - Derivative of $f(x) = x^n, n \in \mathbb{N}$

Answer:

5.2.A] In view of definition 3.1, we have

$$D_{q(t)}f(x) = \frac{f(q(t)x) - f(x)}{(q(t)-1)x}.$$

New we substitute $f(x)$ by x^n. It gives

$$\begin{aligned} D_{(1+t)}(x^n) &= \frac{(1+t)^n x^n - x^n}{(1+t-1)x} \\ &= \frac{\left((1+t)^n - 1\right) x^n}{tx} \\ &= \frac{\left((1+t)^n - 1\right) x^{n-1}}{t}. \end{aligned}$$

The value $D_{(1+t)}(x^n) \to nx^{n-1}$ as $t \to 0$ i.e $D_{(1+t)}(x^n) \to \frac{d}{dx}(x^n)$ as $t \to 0$.

For example if we take $n = 2$ *i.e.* for $f(x) = x^2$ and $t = 0.1$, 0.01, 0.001 with $q(t) = 1 + t$

$$D_{1+t}(x^2) = \frac{\left((1+t)^2 - 1\right) x}{t}.$$

$$\begin{aligned} D_{1+t}(x^2) &= \frac{\left((1+t)^2 - 1\right)(x)}{t}, \\ t = 0.1, \quad D_{1+t}(x^2) &= \frac{\left((1+0.1)^2 - 1\right)(x)}{0.1} = \frac{0.21(x)}{0.1} = 2.1x, \\ t = 0.01, \quad D_{1+t}(x^2) &= \frac{\left((1+0.01)^2 - 1\right)(x)}{0.01} = \frac{0.0201(x)}{0.01} = 2.01x, \\ t = 0.001, \quad D_{1+t}(x^2) &= \frac{\left((1+0.001)^2 - 1\right)(x)}{0.001} = \frac{0.0020011x}{0.001} = 2.001x. \end{aligned}$$

one can see, how the value of $D_{q(t)}[f(x)] \to 2x$ as $t \to 0andq(t) \to 1$.

5.2.B] From the definition [3.2]. We have;

$$D_{h(t)}f(x) = \frac{f(x+h(t)) - f(x)}{h(t)}.$$

So, for $h(t) = t$ and $f(x) = x^n$ we have

$$\begin{aligned} D_{(t)}(x^n) &= \frac{f(x+t) - f(x)}{t} \\ &= \frac{(x+t)^n - x^n}{(t)} = nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}(t) + ... + (t)^{n-1} \\ &= nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}(t) + ... + (t)^{n-1}. \end{aligned}$$

The value $D_{(1+t)}(x^n) \to nx^{n-1}$ as $t \to 0$ i.e $D_{(1+t)}(x^n) \to \frac{d}{dx}(x^n)$ as $t \to 0$.

For example if we take $n = 2$ i. e. for $f(x) = x^2$ and $t = 0.1, 0.01, 0.001$ with $h(t) = t$.

$$D_t x^2 = 2x + t.$$

So, for different values of t we have

$$\begin{aligned} t = 0.1, \quad D_t(x^2) &= 2x + 0.1, \\ t = 0.01, \quad D_t(x^2) &= 2x + 0.01, \\ t = 0.001, \quad D_t(x^2) &= 2x + 0.001. \end{aligned}$$

The value of $D_t(x^2) \to 2x$ as $t \to 0$. *i.e* $D_{(t)}(x^2) \to 2x$ as $t \to 0$ and $h(t) \to 0$.

6 Conclusion

This paper presents a new framework for quantum calculus based on functional properties. By defining quantum properties in terms of function conditions, we introduce a novel approach to quantum derivatives and congruence relations. This functional perspective opens up new avenues for research, particularly in the intersection of quantum calculus and number theory. While this is an initial exploration, further investigation is necessary to fully realize the potential of this approach.

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Authors' contributions

All the authors have contributed equally to this paper.

Conflict of interest

The authors have no conflicts of interest to declare.

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