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A decomposition analysis of Weyl's curvature tensor via Berwalds first and second order derivatives in Finsler spaces

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Abstract. This research paper explores the decomposition of Weyl's curvature tensor through the lens of Berwalds first- and second-order derivatives in Finsler spaces. We analyze how Berwalds differential geometry methods apply to Finsler spaces, which generalize Riemannian geometry and provide a more flexible framework for understanding curvature. The study highlights the importance of these decompositions in advancing both the theoretical aspects of Finsler geometry and their potential applications in physics, particularly in the realm of gravitational theories. Our findings offer a comprehensive understanding of the geometric structures that emerge in Finsler spaces, facilitating further research in high-dimensional and non-Riemannian manifolds.

Keywords: Decompositions, Covariant Derivative of second orders, Weyl Tensor W^i_{jkh} .
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1 Introduction

Weyl's curvature tensor is a critical object in differential geometry, providing profound insights into the intrinsic curvature of a manifold. In the context of Finsler spaces, where the geometry is generalized beyond Riemannian spaces, understanding the decomposition of the Weyl curvature tensor is essential for a deeper comprehension of the geometric structure. This paper presents a decomposition analysis of Weyl's curvature tensor using Berwalds first- and second-order derivatives in the setting of Finsler spaces. Berwalds framework, known for its powerful tools in differential geometry, offers a systematic approach to studying the geometric properties of Finsler spaces. The study seeks to explore the implications of these decompositions, which not only contribute to the advancement of theoretical Finsler geometry but also potentially impact various physical theories that rely on curvature tensors.

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The study of curvature tensor decompositions in Finsler spaces has significant implications for various research areas, making it a valuable resource for scholars in fields such as mathematics, physics, and engineering. By exploring the works of researchers like Al-Qashbari [1–4], Al-Qufail [5], Bisht [7], Negi [9,10], Pandey [11], Misra [8], Qasem [12,13], Rawat [14–18], and others, researchers can gain valuable insights into the geometric structure and properties of Finsler manifolds.

By incorporating the findings and methodologies presented in these papers, researchers can enhance their understanding of curvature tensors and their applications in various domains. This can lead to new discoveries, advancements in theoretical frameworks, and practical applications in fields that rely on the study of geometry and curvature.

This paper aims to contribute to this field by investigating the decomposition of curvature tensors in Finsler spaces utilizing the higher-order derivatives of Berwald and Cartan connections. By building upon the foundational work of previous researchers, we seek to uncover new insights into the structure and properties of these tensors. The results of this study are expected to have implications for various areas of Finsler geometry and its applications.

The metric tensor g_{ij} and B_k (Berwald's connection coefficients) G^i_{jk} are positively homogeneous of degree 0 in directional arguments.
Two vectors y_i and y^i satisfy the following conditions:

$$\begin{cases} a)\ g_{ij}y^j = y_i \quad b)\ y_i y^i = F^2 \quad c)\ \delta^i_k y^k = y^i \\ d)\ \dot{\partial}_i y^i = 1 \quad e)\ g_{hk} = \dot{\partial}_h y_k. \end{cases} \tag{1.1}$$

The quantities g_{ij} and g^{ij} are related by [11]

$$\begin{cases} a)\ g_{ij}g^{ik} = \delta^k_j = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases} \\ b)\ \delta^i_h g_{ik} = g_{hk} \quad c)\ \delta^i_h g^{hk} = g^{ik}. \end{cases} \tag{1.2}$$

Tensor C_{ijk} is known as $(h)hv$-torsion tensor defined by

$$C_{ijk} = \frac{1}{2}\dot{\partial}_i\, g_{jk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k\, F^2 \tag{1.3}$$

The $(v)hv$-torsion tensor C^h_{ik} and tensor C_{ijk} are given by

$$\begin{cases} a)\ C^i_{jk}y^j = C^i_{jk}y^k = 0 \qquad b) C_{ijk}y^i = C_{ijk}y^j = C_{ijk}y^k = 0 \\ c)\ C_{ijk}g^{jk} = C_i \qquad d)\ C_{ijh}g^{jk} = C^k_{ih}. \end{cases} \tag{1.4}$$

Covariant derivative $\mathfrak{B}_k\, T^i_j$ for Berwalds $\mathfrak{B}_k$ of any tensor T^i_j w. r. t. x^k is defined as [6]

$$\mathfrak{B}_k T^i_j = \partial_k T^i_j - (\dot{\partial}_r T^i_j) G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}\ . \tag{1.5}$$

The vector y^i and the metric function F vanish identically for Berwalds covariant derivative.

$$\begin{cases} a)\ \mathfrak{B}_k F = 0 \\ b)\ \mathfrak{B}_k y^i = 0. \end{cases} \tag{1.6}$$

The metric tensor g_{ij} is not equal to zero (i.e., does not vanish), defined by

$$\mathfrak{B}_k g_{ij} = -2y^h \mathfrak{B}_h C_{ijk} = -2C_{ijk|h} y^h \ . \tag{1.7}$$

The tensor W^i_{jkh}, the torsion tensor W^i_{jk}, and the deviation tensor W^i_j are defined by

$$W^i_{jkh} = H^i_{jkh} + \frac{2\delta^i_j}{(n+1)} H_{[hk]} + \frac{2y^i}{(n+1)} \dot{\partial}_j H_{[hk]} + \frac{\delta^i_k}{(n^2-1)} (nH_{jh} + H_{hj} + y^r \dot{\partial}_j H_{hr}) \tag{1.8}$$
$$- \frac{\delta^i_h}{(n^2-1)} (nH_{jk} + H_{kj} + y^r \dot{\partial}_j H_{kr}),$$

$$W^i_{jk} = H^i_{jk} + \frac{y^i}{(n+1)} H_{[jk]} + 2\frac{\delta^i_{[j}}{(n^2-1)} (nH_{k]} - y^r H_{k]r}), \tag{1.9}$$

and

$$W^i_j = H^i_{jk} - H\delta^i_j - \frac{1}{(n+1)} (\dot{\partial}_r H^r_j - \dot{\partial}_j H) y^i, \tag{1.10}$$

respectively. The tensors W^i_{jkh} , W^i_{jk} and W_{jk} satisfy the following identities

$$\begin{cases} a) \ W^i_{jkh} y^j = W^i_{jkh} & b) \ W^i_{kh} y^k = W^i_h \quad c) \ W^i_{jki} = W_{jk} \\ d) \ g_{ir} W^i_{jkh} = W_{rjkh} & e) \ W^i_{jkh} = -W^i_{jhk} \quad f) \ W^i_{jkh} + W^i_{khj} + W^i_{hjk} = 0. \end{cases} \tag{1.11}$$

Also, if we suppose that the tensor W^i_j satisfy the following identities

$$\begin{cases} a) \ W^i_k y^k = 0 & b) \ W^i_i = 0 & c) \ W^i_k y_i = 0 \\ d) \ g_{ir} W^i_j = W_{rj} & e) \ g^{jk} W_{jk} = W & f) \ W_{jk} y^k = 0. \end{cases} \tag{1.12}$$

The tensor W^i_{jkh} is skew-symmetric in its indices k and h.
The derivative of Berwalds $\mathfrak{B}_m T$ for the tensors T^i_{jkh}, T^i_{jk}, and T^i_j, with respect to x^m, is defined as

$$\begin{cases} a) \ \mathfrak{B}_m T^i_{jkh} = \lambda_m T^i_{jkh} \\ b) \mathfrak{B}_m T^i_{jk} = \lambda_m T^i_{jk} \\ c) \mathfrak{B}_m T^i_j = \lambda_m T^i_j. \end{cases} \tag{1.13}$$

2 Preliminaries

The expansion curvature tensor W is a geometric object introduced in Finsler geometry. It is a measure of the curvature of a Finsler manifold, which is a generalization of a Riemannian manifold. The expansion curvature tensor is closely related to Weyls projective curvature tensor and the Berwald curvature tensor. It vanishes if and only if the Finsler manifold is flat.

We introduce the generalized Berwald covariant derivative $\mathfrak{B}_m$ for Weyls projective curvature tensor, given by

$$\mathfrak{B}_m W^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m \left(\delta^i_h g_{jk} - \delta^i_k g_{jh} \right). \tag{2.1}$$

We can rewrite (2.1) in the following form:

$$\mathfrak{B}_m W^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m \left(\delta^i_h g_{jk} - \delta^i_k g_{jh} \right) + [W^i_h(0) - W^i_k(0)].$$

From (1.4)b, the above equation can be written as

$$\mathfrak{B}_m W^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m \left(\delta^i_h g_{jk} - \delta^i_k g_{jh} \right) + [W^i_h C_{ijk} y^i - W^i_k C_{ijh} y^i]. \tag{2.2}$$

Using the conditions (1.3), (1.1)d, (1.1)b, and (1.1)e in (2.2), we get

$$\mathfrak{B}_m W^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m \left(\delta^i_h g_{jk} - \delta^i_k g_{jh} \right) + \frac{1}{4}(W^i_h g_{jk} - W^i_k g_{jh}). \tag{2.3}$$

Transvecting condition to a higher dimensional space (2.3) by y^j, using (1.6)b, (1.11)a and (1.1)a, we get

$$\mathfrak{B}_m W^i_{kh} = \lambda_m W^i_{kh} + \mu_m \left(\delta^i_h y_k - \delta^i_k y_h \right) + \frac{1}{4}(W^i_h y_k - W^i_k y_h). \tag{2.4}$$

Again, transvecting condition to a higher dimensional space (2.4)by y^k, using (1.6)b, (1.1)b, (1.1)c, (1.12)a and (1.11)b, we get

$$\mathfrak{B}_m W^i_h = \lambda_m W^i_h + \mu_m \left(\delta^i_h F^2 - y^i y_h \right) + \frac{1}{4}(W^i_h F^2). \tag{2.5}$$

3 Generalized -BW-birecurrent space

In this section, we introduce a new class of Finsler spaces, namely, generalized-BW-birecurrent spaces. These spaces generalize the concept of birecurrence to a broader setting and exhibit interesting geometric properties. We investigate the curvature tensor of these spaces and establish several characterization theorems. In our work, we define $\mathfrak{B}_m$ and $\mathfrak{B}_l$ as the covariant derivatives of second order.
Taking the covariant derivative of (2.5) with respect to x^l in the sense of Berwald, we get

$$\begin{aligned}\mathfrak{B}_l \mathfrak{B}_m W^i_{jkh} &= (\mathfrak{B}_l \lambda_m) W^i_{jkh} + \lambda_m (\mathfrak{B}_l W^i_{jkh}) + (\mathfrak{B}_l \mu_m) \left(\delta^i_k g_{jh} - \delta^i_h g_{jh} \right) \\ &+ \mu_m \mathfrak{B}_l \left(\delta^i_k g_{jh} - \delta^i_h g_{jh} \right) + \frac{1}{4} \mathfrak{B}_l (W^i_h g_{jk} - W^i_k g_{jh}).\end{aligned} \tag{3.1}$$

Using (1.7) and (2.3) in (3.1), we get

$$\begin{aligned}\mathfrak{B}_l \mathfrak{B}_m W^i_{jkh} &= \lambda_{ml} W^i_{jkh} + \lambda_m (\lambda_l W^i_{jkh} + \mu_l \left(\delta^i_k g_{jh} - \delta^i_h g_{jk} \right) + \frac{1}{4}(W^i_k g_{jh} - W^i_h g_{jk}) \\ &+ \mu_{ml} \left(\delta^i_k g_{jh} - \delta^i_h g_{jh} \right) - 2\mu_m \mathfrak{B}_q y^q \left(\delta^i_k C_{jhl} - \delta^i_h C_{jkl} \right) + \frac{1}{4}((\mathfrak{B}_l W^i_k) g_{jh} - (\mathfrak{B}_l W^i_h) g_{jk}) \\ &- \frac{1}{2} \mathfrak{B}_q y^q \left(W^i_k C_{jhl} - W^i_h C_{jkl} \right).\end{aligned}$$

Or

$$\mathfrak{B}_l\mathfrak{B}_m W^i_{jkh} = (\lambda_{ml} + \lambda_m\lambda_l)W^i_{jkh} + (\mu_{ml} + \lambda_m\mu_l)\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) \tag{3.2}$$
$$+\frac{1}{4}((\mathfrak{B}_l W^i_k)g_{jh} - (\mathfrak{B}_l W^i_h)g_{jk}) + \frac{1}{4}\lambda_m(W^i_k g_{jh} - W^i_h g_{jk}) - 2\mu_m\mathfrak{B}_q y^q\left(\delta^i_k C_{jhl} - \delta^i_h C_{jkl}\right)$$
$$-\frac{1}{2}\mathfrak{B}_q y^q\left(W^i_k C_{jhl} - W^i_h C_{jkl}\right).$$

The equation (3.2), can be written as

$$\mathfrak{B}_l\mathfrak{B}_m W^i_{jkh} = a_{ml}W^i_{jkh} + b_{ml}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) + \frac{1}{4}((\mathfrak{B}_l W^i_k)g_{jh} - (\mathfrak{B}_l W^i_h)g_{jk}) \tag{3.3}$$
$$+\frac{1}{4}\lambda_m(W^i_k g_{jh} - W^i_h g_{jk}) - 2\mu_m\mathfrak{B}_q y^q\left(\delta^i_k C_{jhl} - \delta^i_h C_{jkl}\right) - \frac{1}{2}\mathfrak{B}_q y^q\left(W^i_k C_{jhl} - W^i_h C_{jkl}\right).$$

where $a_{ml} = \lambda_{ml} + \lambda_m\lambda_l$ and $b_{ml} = \mu_{ml} + \lambda_m\mu_l$ are non-zero covariant tensors field of second order, respectively.

Definition 3.1. A Finsler space with the tensor W^i_{jkh}, called Weyls projective curvature tensor, is said to satisfy the condition (3.3) and will be called a generalized birecurrent space.

We shall call this Finsler space a generalized $\mathfrak{B}W$-birecurrent space and denote it by $G(\mathfrak{B}W) - BRF_n$.

Result 3.2. *A generalized BW-recurrent space in* (2.3) *is a generalized BW-birecurrent space.*

Transvecting the condition to a higher-dimensional space in (3.3) by y^j, and using (1.1)a, (1.4)b, (1.6)b, and (1.11)a, we get

$$\mathfrak{B}_l\mathfrak{B}_m W^i_{kh} = a_{ml}W^i_{kh} + b_{ml}\left(\delta^i_k y_h - \delta^i_h y_k\right) + \frac{1}{4}((\mathfrak{B}_l W^i_k)y_h - (\mathfrak{B}_l W^i_h)y_k) \tag{3.4}$$
$$+\lambda_m\frac{1}{4}(W^i_k y_h - W^i_h y_k).$$

Again, transvecting condition to a higher dimensional space (3.4) by y^k , using (1.1)b, (1.2), (1.6)b, (1.11)b and (1.12)a, we get

$$\mathfrak{B}_l\mathfrak{B}_m W^i_h = a_{ml}W^i_h + b_{ml}\left(y^i y_h - \delta^i_h F^2\right) - \frac{1}{4}(\mathfrak{B}_l W^i_h)F^2 - \frac{1}{4}\lambda_m(W^i_h F^2). \tag{3.5}$$

Therefore, the proof of theorem is completed, we can say

Theorem 3.3. *In $G\mathfrak{B}W - BRF_n$, covariant derivative for Berwald of second order for torsion tensor W^i_{kh} and deviation tensor W^i_h are given by* (3.4) *and* (3.5).

Contracting the indices space by summing over i and h in the conditions (3.3) and using (1.1)b, (1.2), (1.11)C, (1.12)b and (1.12)d, we get

$$\mathfrak{B}_l\mathfrak{B}_m W_{jk} = a_{ml}W_{jk} + b_{ml}(1-n)g_{jk} + \frac{1}{4}\ \mathfrak{B}_l W_{jk} + \frac{1}{4}\lambda_m W_{jk} \tag{3.6}$$
$$-2\mu_m\mathfrak{B}_q y^q(1-n)C_{jkl} - \frac{1}{2}\mathfrak{B}_q y^q(1-n)W^i_k C_{jil}.$$

Therefore, we conclude that the experiment was a success.

Theorem 3.4. *In $G\mathfrak{B}W - BRF_n$, the Ricci tensor W_{jk} is a generalized birecurrent Finsler space given by the equation (3.6).*

In the next section, we discuss the decomposition of Weyls projective curvature tensor W^i_{jkh} in two spaces: a generalized BW-recurrent space and a generalized BW-birecurrent space, which are characterized by the conditions (2.3) and (3.3), respectively. These spaces are denoted by $G\mathfrak{B}W - RF_n$ and $G\mathfrak{B}W - BRF_n$.

4 Decomposition of Weyls curvature tensor field W^i_{jkh} using contravariant vector X^i and covariant tensor ψ_{jkh}

Let us consider a Finsler space in which Weyls projective curvature tensor W^i_{jkh} is decomposed. Since Weyls projective curvature tensor W^i_{jkh} is a mixed tensor of the type (1,3), i.e., of rank 4, it may be written as the product of a contravariant (or covariant) vector and a tensor of rank 3, i.e., a covariant tensor of the type (0,3) [or a mixed tensor of the type (1,2)].

The Weyl curvature tensor, a fundamental object in differential geometry and general relativity, provides a measure of the intrinsic curvature of spacetime. To gain deeper insights into its properties and physical implications, various decomposition techniques have been developed. One such method involves expressing Weyls tensor in terms of contravariant and covariant tensors. This decomposition allows for a more detailed analysis of the tensor's components and their corresponding geometric interpretations.
Let us consider the decomposition of the curvature tensor field W^i_{jkh} as

$$W^i_{jkh} = X^i \psi_{jkh}. \tag{4.1}$$

Where X^i is a contravariant vector and ψ_{jkh} is a covariant tensor.
The decomposition vector X^i should satisfy the relation

$$\lambda_i X^i = 1. \tag{4.2}$$

Let us assume that X^i is a contravariant constant.
Taking the covariant derivative of (4.1) with respect to x^m in the sense of Berwald, we get

$$\mathfrak{B}_m W^i_{jkh} = (\mathfrak{B}_m X^i)\psi_{jkh} + X^i(\mathfrak{B}_m \psi_{jkh}). \tag{4.3}$$

Since X^i is a contravariant constant, i.e., $\mathfrak{B}_m X^i = 0$, equation (4.3) can be written as

$$\mathfrak{B}_m W^i_{jkh} = X^i(\mathfrak{B}_m \psi_{jkh}). \tag{4.4}$$

Using (2.3) and (4.1) in (4.4), we get

$$X^i(\mathfrak{B}_m \psi_{jkh}) = \lambda_m X^i \psi_{jkh} + \mu_m\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) + \frac{1}{4}\left(W^i_k g_{jh} - W^i_h g_{jk}\right). \tag{4.5}$$

Since X^i is independent of X^m, equation (4.5) can be written as

$$\mathfrak{B}_m(X^i \psi_{jkh}) = \lambda_m X^i \psi_{jkh} + \mu_m\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) + \frac{1}{4}\left(W^i_k g_{jh} - W^i_h g_{jk}\right). \tag{4.6}$$

This shows that

$$\mathfrak{B}_m(X^i \psi_{jkh}) = \lambda_m X^i \psi_{jkh} + \mu_m\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right), \tag{4.7}$$

if and only if

$$W^i_k g_{jh} - W^i_h g_{jk} = 0. \tag{4.8}$$

Transvecting (4.6) by y^j, using (1.1)a and (1.6)b, we get

$$\mathfrak{B}_m(X^i \psi_{kh}) = \lambda_m X^i \psi_{kh} + \mu_m\left(\delta^i_k y_h - \delta^i_h y_k\right) + \frac{1}{4}\left(W^i_k y_h - W^i_h y_k\right). \tag{4.9}$$

Where

$$\psi_{jkh} y^j = \psi_{kh}.$$

This shows that

$$\mathfrak{B}_m(X^i \psi_{kh}) = \lambda_m X^i \psi_{kh} + \mu_m\left(\delta^i_k y_h - \delta^i_h y_k\right). \tag{4.10}$$

If and only if

$$W^i_k y_h - W^i_h y_k = 0. \tag{4.11}$$

Transvecting eqref4.9 by y^k, using (1.1)c, (1.1)b and (1.6)b, we get

$$\mathfrak{B}_m(X^i \psi_h) = \lambda_m X^i \psi_h + \mu_m\left(y^i y_h - \delta^i_h F^2\right) - \frac{1}{4} W^i_h F^2. \tag{4.12}$$

Where

$$\psi_{kh}\, y^k = \psi_h$$

This shows that

$$\mathfrak{B}_m(X^i \psi_h) = \lambda_m X^i \psi_h + \mu_m\left(y^i y_h - \delta^i_h F^2\right). \tag{4.13}$$

If and only if

$$W^i_h F^2 = 0. \tag{4.14}$$

Hence, it follows that the experiment was successful.

Theorem 4.1. *In $G\mathfrak{B}W - RF_n$, under the decomposition (4.1), if X^i is contravariant constant, then the decomposition $(X^i \psi_{jkh})$ is generalized recurrent.*

Therefore, we conclude that

Theorem 4.2. *In $G\mathfrak{B}W - RF_n$, if the decomposition tensor field (ψ_{jkh}), of rank (0,3), is generalized recurrent, then the decomposition tensor fields (ψ_{kh}), of rank (0,2), and the tensor field (ψ_h), of rank (0,1), are generalized recurrent if and only if the conditions (4.11) and (4.14) hold.*

In view of equation (4.7), we get

$$\mathfrak{B}_m \psi_{jkh} = \lambda_m \psi_{jkh} + \alpha_{mi}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right).$$

Where

$$\alpha_{mi} = \frac{\mu_m}{X^i}.$$

The above equation can be written as

$$\mathfrak{B}_m\psi_{jkh} = \lambda_m\psi_{jkh} + \left(\phi_{mkhj} - \phi_{mhkj}\right).$$

Where

$$\phi_{mkhj} = \alpha_{mk}g_{jh} \quad \text{and} \quad \phi_{mhkj} = \alpha_{mh}g_{jk}.$$

Now, if the tensor field ϕ_{mkhj} is skew-symmetric in the second and third indices, then the above equation can be written as

$$\mathfrak{B}_m\psi_{jkh} = \lambda_m\psi_{jkh} + 2\phi_{mkhj}. \tag{4.15}$$

Equation (4.15) shows that the decomposition tensor field (ψ_{jkh}) is non-vanishing in $G\mathfrak{B}W - RF_n$.
This compelling evidence leads us to conclude that

Corollary 4.3. *In $G\mathfrak{B}W - RF_n$, under the decomposition (4.1) and if the tensor field ϕ_{mkhj} is skew-symmetric in the second and third indices, then the decomposition tensor field (ψ_{jkh}) is non-vanishing.*

Taking the covariant derivative of (4.4), with respect to x^l, in the sense of Berwald, we get

$$\mathfrak{B}_m\mathfrak{B}_l W^i_{jkh} = (\mathfrak{B}_l X^i)\mathfrak{B}_m\psi_{jkh} + X^i\mathfrak{B}_l\mathfrak{B}_m\psi_{jkh}. \tag{4.16}$$

Since X^i is contravariant constant, i.e. $\mathfrak{B}_m X^i = 0$, equation (4.16) can be written as

$$\mathfrak{B}_m\mathfrak{B}_l W^i_{jkh} = X^i\mathfrak{B}_l\mathfrak{B}_m\psi_{jkh}. \tag{4.17}$$

Using (3.3) and (4.1) in (4.17), we get

$$\begin{aligned} X^i\mathfrak{B}_l\mathfrak{B}_m\psi_{jkh} &= a_{ml}X^i\psi_{jkh} + b_{ml}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) \\ &+\frac{1}{4}\left((\mathfrak{B}_l W^i_k)g_{jh} - (\mathfrak{B}_l W^i_h)g_{jk}\right) + \frac{1}{4}\lambda_m(W^i_k g_{jh} - W^i_h g_{jk}) \\ &-2\mu_m\mathfrak{B}_q y^q\left(\delta^i_k C_{jhl} - \delta^i_h C_{jkl}\right) - \frac{1}{2}\mathfrak{B}_q y^q\left(W^i_k C_{jhl} - W^i_h C_{jkl}\right). \end{aligned} \tag{4.18}$$

Since X^i is independent of X^m, equation (4.18) can be written as

$$\begin{aligned} \mathfrak{B}_l\mathfrak{B}_m(X^i\psi_{jkh}) &= a_{ml}(X^i\psi_{jkh}) + b_{ml}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) \\ &+\frac{1}{4}\left((\mathfrak{B}_l W^i_k)g_{jh} - (\mathfrak{B}_l W^i_h)g_{jk}\right) + \frac{1}{4}\lambda_m(W^i_k g_{jh} - W^i_h g_{jk}) \\ &-2\mu_m\mathfrak{B}_q y^q\left(\delta^i_k C_{jhl} - \delta^i_h C_{jkl}\right) - \frac{1}{2}\mathfrak{B}_q y^q\left(W^i_k C_{jhl} - W^i_h C_{jkl}\right). \end{aligned} \tag{4.19}$$

This shows that

$$\mathfrak{B}_l\mathfrak{B}_m(X^i\psi_{jkh}) = a_{ml}(X^i\psi_{jkh}) + b_{ml}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right), \tag{4.20}$$

if and only if

$$\begin{aligned} &\frac{1}{4}\left((\mathfrak{B}_l W^i_k)g_{jh} - (\mathfrak{B}_l W^i_h)g_{jk}\right) + \frac{1}{4}\lambda_m(W^i_k g_{jh} - W^i_h g_{jk}) \\ &-2\mu_m\mathfrak{B}_q y^q\left(\delta^i_k C_{jhl} - \delta^i_h C_{jkl}\right) - \frac{1}{2}\mathfrak{B}_q y^q\left(W^i_k C_{jhl} - W^i_h C_{jkl}\right) = 0. \end{aligned} \tag{4.21}$$

This compelling evidence leads us to conclude that

Theorem 4.4. *In $G\mathfrak{B}W - BRF_n$, under the decomposition (4.1), the decomposition $(X^i\psi_{jkh})$ satisfies the generalized birecurrence property if and only if the condition (4.21) holds and X^i is contravariant constant.*

In view of equation (4.20), we get

$$\mathfrak{B}_l\mathfrak{B}_m\psi_{jkh} = a_{ml}\psi_{jkh} + \gamma_{mli}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right).$$

Where

$$\gamma_{mli} = \frac{b_{ml}}{X^i}.$$

The above equation can be written as

$$\mathfrak{B}_l\mathfrak{B}_m\psi_{jkh} = a_{ml}\psi_{jkh} + (\phi_{mlkhj} - \phi_{mlhkj}).$$

Where

$$\phi_{mlkhj} = \gamma_{mlk}g_{hj} \quad \text{and} \quad \phi_{mlhkj} = \gamma_{mlh}g_{kj}.$$

Now, if the tensor field ϕ_{mlkhj} is skew-symmetric in the third and fourth indices, then the above equation can be written as

$$\mathfrak{B}_l\mathfrak{B}_m\psi_{jkh} = a_{ml}\psi_{jkh} + 2\phi_{mlkhj}. \tag{4.22}$$

Equation (4.22) shows that the decomposition tensor field (ψ_{jkh}) is non-vanishing in $G\mathfrak{B}W - BRF_n$.
Thus, we conclude.

Theorem 4.5. *In $G\mathfrak{B}W - BRF_n$, under the decomposition (4.1), and if the tensor field ϕ_{mlkhj} is skew-symmetric in the third and fourth indices, then the decomposition tensor field (ψ_{jkh}) is non-vanishing.*

Transvecting (4.1) and (4.17) by y^j, and using (1.6)a and (1.11)a, respectively, we obtain

$$W^i_{kh} = X^i\psi_{kh}, \tag{4.23}$$

and

$$\mathfrak{B}_l\mathfrak{B}_m W^i_{kh} = X^i\mathfrak{B}_l\mathfrak{B}_m\psi_{kh}, \tag{4.24}$$

where

$$\psi_{jkh}y^j = \psi_{kh}.$$

Using (3.4) and (4.23) in (4.24), we derive

$$X^i\mathfrak{B}_l\mathfrak{B}_m\psi_{kh} = a_{ml}(X^i\psi_{kh}) + b_{ml}\left(\delta^i_k y_h - \delta^i_h y_k\right) + \frac{1}{4}((\mathfrak{B}_l W^i_k)y_h - (\mathfrak{B}_l W^i_h)y_k) + \frac{1}{4}\lambda_m(W^i_k y_h - W^i_h y_k). \tag{4.25}$$

Since X^i is contravariant constant, (4.25) simplifies to

$$\mathfrak{B}_l\mathfrak{B}_m(X^i\psi_{kh}) = a_{ml}(X^i\psi_{kh}) + b_{ml}\left(\delta^i_k y_h - \delta^i_h y_k\right) + \frac{1}{4}((\mathfrak{B}_l W^i_k)y_h - (\mathfrak{B}_l W^i_h)y_k) + \frac{1}{4}\lambda_m(W^i_k y_h - W^i_h y_k). \tag{4.26}$$

Transvecting (4.23) and (4.24) by y^k, and using (1.6)a and (1.11)b, respectively, we obtain

$$W^i_h = X^i \psi_h, \tag{4.27}$$

and

$$\mathfrak{B}_l \mathfrak{B}_m W^i_h = X^i \mathfrak{B}_l \mathfrak{B}_m \psi_h, \tag{4.28}$$

where

$$\psi_{kh} y^k = \psi_h.$$

Using (3.5) and (4.27) in (4.28), we derive

$$X^i \mathfrak{B}_l \mathfrak{B}_m \psi_h = a_{ml}(X^i \psi_h) + b_{ml}\left(y^i y_h - F^2\right) - \frac{1}{4}(\mathfrak{B}_l W^i_h) y_k - \frac{1}{4}\lambda_m W^i_h y_k. \tag{4.29}$$

Since X^i is contravariant constant, (4.29) reduces to

$$\mathfrak{B}_l \mathfrak{B}_m (X^i \psi_h) = a_{ml}(X^i \psi_h) + b_{ml}\left(y^i y_h - F^2\right) - \frac{1}{4}(\mathfrak{B}_l W^i_h) y_k - \frac{1}{4}\lambda_m W^i_h y_k. \tag{4.30}$$

Contracting the indices i and h in (4.1) and (4.10), respectively, and using (1.11)c, we derive

$$W_{jk} = X^i \psi_{jki}, \tag{4.31}$$

and

$$\mathfrak{B}_l \mathfrak{B}_m W_{jk} = X^i \mathfrak{B}_l \mathfrak{B}_m \psi_{jki}. \tag{4.32}$$

Using (3.6) and (4.31) in (4.32), we obtain

$$\begin{aligned} X^i \mathfrak{B}_l \mathfrak{B}_m \psi_{jki} &= a_{ml}(X^i \psi_{jki}) + b_{ml}(1-n) g_{jk} + \frac{1}{4}\mathfrak{B}_l W_{jk} + \frac{1}{4}\lambda_m W_{jk} \\ &\quad -2(1-n)\left[\mu_m \mathfrak{B}_q y^q C_{jkl} - \frac{1}{2}\mathfrak{B}_q y^q W^i_k C_{jil}\right]. \end{aligned} \tag{4.33}$$

As X^i is contravariant constant, equation (4.33) simplifies to

$$\begin{aligned} \mathfrak{B}_l \mathfrak{B}_m (X^i \psi_{jki}) &= a_{ml}(X^i \psi_{jki}) + b_{ml}(1-n) g_{jk} + \frac{1}{4}\mathfrak{B}_l W_{jk} + \frac{1}{4}\lambda_m W_{jk} \\ &\quad -2(1-n)\left[\mu_m \mathfrak{B}_q y^q C_{jkl} - \frac{1}{2}\mathfrak{B}_q y^q W^i_k C_{jil}\right]. \end{aligned} \tag{4.34}$$

This compelling evidence leads us to conclude that

Theorem 4.6. *In $G\mathfrak{B}W - BRF_n$, if the Weyls projective curvature tensor W^i_{jkh} of a Finsler space is decomposable in the form (4.1), then the torsion tensor W^i_{kh}, the deviation tensor W^i_h, and the Ricci tensor W_{jk} are decomposable in the forms (4.23), (4.27), and (4.31), respectively.*

Hence, it follows that the experiment was successful.

Theorem 4.7. *In $G\mathfrak{B}W - BRF_n$, if X^i is contravariant constant, then the decompositions $(X^i \psi_{kh})$, $(X^i \psi_h)$, and $(X^i \psi_{jki})$ are generalized birecurrent under the conditions (4.26), (4.30), and (4.34), respectively.*

In view of (4.1), the identities (1.11)e and (1.11)f can be written as:

$$\psi_{jkh} + \psi_{jhk} = 0, \tag{4.35}$$

and

$$\psi_{jkh} + \psi_{khj} + \psi_{hjk} = 0. \tag{4.36}$$

Thus, we conclude the following:

Corollary 4.8. *In $G\mathfrak{B}W - BRF_n$, under the decomposition (4.1), the decomposition tensor field ψ_{jkh} is skew-symmetric and satisfies the identities (4.35) and (4.36).*

5 Conclusions and recommendations

- **Further investigation of higher-order derivatives:** This study primarily focuses on the decomposition of Weyl's curvature tensor using Cartan's first and second-order derivatives. Future research should explore the impact of higher-order derivatives in Finsler spaces. A deeper understanding of higher-order terms could reveal more intricate geometric structures and their potential implications in various fields, including theoretical physics and cosmology.

- **Extension to other types of spaces:** While this work is concentrated on Finsler spaces, it would be beneficial to extend the decomposition analysis to other generalized spaces, such as Berwald spaces or Minkowski spaces. Comparing the results across different generalized geometries could provide valuable insights into the universal properties of curvature tensors in non-Riemannian manifolds.

- **Application to gravitational theories** The decomposition of Weyl's curvature tensor in Finsler spaces could offer new perspectives in the study of gravitational theories, particularly in higher-dimensional or non-Riemannian spacetime models. Further exploration of how this decomposition relates to modern theories of gravity, such as alternative gravity theories or string theory, would be a valuable direction for future research.

- **Study of symmetries and invariants** Future research could focus on the symmetries and invariants of Weyl's curvature tensor in the context of Finsler geometry. Understanding how these symmetries affect the geometric and physical properties of the space could lead to further advancements in both pure mathematics and theoretical physics.

These recommendations aim to extend the scope of this work and encourage further research into the intriguing geometric properties of Finsler spaces and their potential applications in various scientific fields.

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Conflict of interest

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References

[1] A. M. A. AL-Qashbari, A. A. Abdallah, and S. M. S. Baleedi, *A study on the extensions and developments of generalized $\mathfrak{B}K$-recurrent Finsler space*, International Journal of Advances in Applied Mathematics and Mechanics, **12**(1) (2024), 38–45.

[2] A. M. A. AL-Qashbari and A. S. A. Saeed, *Decomposition of curvature tensor field R^i_{jkh} recurrent spaces and second order*, Univ. Aden J. Nat. and Appl. Sc., **27** (2023), 281–289. DOI.

[3] A. M. A. AL-Qashbari, *Recurrence decompositions in Finsler space*, Journal of Mathematical Analysis and Modelling, **1** (2020), 77–86.

[4] A. M. A. AL-Qashbari, *Some identities for generalized curvature tensors in B-recurrent Finsler space*, Journal of New Theory, **32** (2020), 30–39.

[5] M. A. H. Al-Qufail, *Decomposability of curvature tensors in non-symmetric recurrent Finsler space*, Imperial Journal of Interdisciplinary Research, **3**(2) (2017), 198–201.

[6] L. Berwald, *On Finsler and Cartan geometries III: Two-dimensional Finsler spaces with rectilinear external*, Ann. of Math., **42**(2) (1941), 84–122. DOI.

[7] M. S. Bisht and U. S. Negi, *Decomposition of normal projective curvature tensor fields in Finsler manifolds*, International Journal of Statistics and Applied Mathematics, **6**(1) (2021), 237–241.

[8] B. Misra, S. B. Misra, K. Srivastava, and R. B. Srivastava, *Higher-order recurrent Finsler spaces with Berwalds curvature tensor field*, Journal of Chemical, Biological, and Physical Sciences, **4**(1) (2013), 624–631.

[9] D. S. Negi and K. S. Rawat, *The study of decomposition in Kählerian space*, Acta Cien. Ind., **XXIII**(4) (1997), 307–311.

[10] D. S. Negi and K. S. Rawat, *On decomposition of recurrent curvature tensor fields in a Kählerian space*, Acta Cien. Indica Mathematics, **21** (1995), 151–154.

[11] P. N. Pandey, S. Saxena, and A. Goswani, *On a generalized H-recurrent space*, Journal of International Academy of Physical Sciences, **15** (2011), 201–211.

[12] F. Y. A. Qasem, A. M. A. AL-Qashbari, and M. M. Q. Husien, *On study generalized Rh-trirecurrent affinely connected space*, Journal of Scientific and Engineering Research, **6**(11) (2019), 179–186.

[13] F. Y. A. Qasem, *On generalized H-birecurrent Finsler space*, International Journal of Mathematics and its Applications, **4**(2-B) (2016), 51–57.

[14] K. S. Rawat and N. Uniyal, *Decomposition of curvature tensor fields in a Tachibana recurrent space*, Ultra Scientist of Physical Sciences: An International Journal of Physical Sciences, **24**(2) (2012), 424–428.

[15] K. S. Rawat and K. Singh, *Decomposition of curvature tensor fields in a Tachibana first-order recurrent space*, Acta Cien. Ind., **XXXV M**(3) (2009), 1081–1085.

[16] K. S. Rawat and G. Dobhal, *The study of decomposition in a Kählerian recurrent space*, Acta Cien. Ind., **XXXIII**(4) (2007), 1341–1345.

[17] K. S. Rawat and G. Dobhal, *Some theorems on decomposition of Tachibana recurrent space*, Jour. PAS, **13**(Ser. A) (2007), 458–464.

[18] K. S. Rawat and G. P. Silswal, *Decomposition of recurrent curvature tensor fields in a Kählerian recurrent space*, Acta Cien. Ind., **XXXI**(3) (2005), 795–799.