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Quasi-exact solvable Dirac equation for the generalized Cornell potential plus a novel angle-dependent potential

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Abstract. In this paper, we present the exact analytical solution of the Dirac equation with equal scalar and vector generalized Cornell potential plus a novel angle-dependent potential in the framework of quasi-exactly solvable problems. By applying the functional Bethe ansatz method, we derive the angular Dirac part solutions and by the biconfluent Heun differential equation, the radial Dirac part solutions are determined. The exact bound states and the corresponding energy eigenvalues are obtained. Overall, this paper is a general reference for many previous scientific researches because it includes many potentials, both central and non-central, which in turn adds a new addition to theoretical physics as well as modern mathematics.

Keywords: Dirac equation, Quasi-exactly solvable problems, Generalized Cornell potential, Angle-dependent potential, Bethe ansatz method, Biconfluent Heun equation.

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1 Introduction

The Dirac equation is a relativistic wave equation that describes the behavior of spin-$\frac{1}{2}$ particles, such as electrons, positrons, and quarks, within the framework of quantum mechanics and special relativity [15, 16, 27]. The intricate mathematical structure of the Dirac equation has garnered considerable scholarly interest, affirming its prominence as a significant field of study and application. Indeed, the Dirac equation has numerous applications across physics and related disciplines due to its fundamental role in describing relativistic particles. Some of its key applications include atomic and molecular physics, where it explains the fine structure and energy levels of hydrogen and hydrogen-like atoms; high-energy and particle physics, where it describes interactions between charged particles and electromagnetic

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fields; and condensed matter physics, particularly in the study of graphene and topological insulators, providing insights into the behavior of electrons in graphene and materials with Dirac-like quasiparticles.

In recent decades, the Dirac equation has attracted significant attention, and its importance has grown, particularly in the context of deriving solutions under different types of potentials, both central and non-central [1,4,18,20,24]. Non-central potentials are especially significant when analyzing systems where the interaction is not spherically symmetric. These potentials profoundly affect the behavior of relativistic particles and their energy spectra.

Various methods have been employed to solve quantum physics problems involving central and non-central potentials, as seen in works such as [3–6,9,11,17,19,22,23,25,28,31,36].

In this paper, we focus on solving the Dirac equation with a generalized Cornell potential combined with a novel angle-dependent potential in the framework of quasi-exactly solvable problems [32–35]. To achieve this, we adopt two different methods: the functional Bethe ansatz method [21,29,37] and the biconfluent Heun differential equation [30]. Recently, numerous studies have been published using the functional Bethe ansatz method, the biconfluent Heun differential equation, or a combination of both [2,7,8,10,12–14,26].

The structure of this paper is as follows: In Sect. 2, we present the Dirac equation under the quasi-exactly solvable generalized Cornell potential combined with a novel angle-dependent potential. In Sect. 3, the functional Bethe ansatz method is introduced. The polar and radial wave functions are determined by adopting the functional Bethe ansatz method and the approach of the biconfluent Heun differential equation in Sect. 4. Additionally, we provide the Dirac wave function along with the corresponding energy eigenvalues. Sect. 5 is dedicated to presenting numerical results.

2 Dirac equation with equal scalar and vector potentials

The time-independent Dirac equation with a scalar potential $S(\boldsymbol{r})$ and a vector potential $V(\boldsymbol{r})$, in natural units $\hbar = c = 1$, is expressed as [16]:

$$[\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta(M + S(\boldsymbol{r}))]\,\psi(\boldsymbol{r}) = [E - V(\boldsymbol{r})]\,\psi(\boldsymbol{r}), \tag{2.1}$$

where $\boldsymbol{p} = -i\boldsymbol{\nabla}$ is the momentum operator, M is the mass of the particle, E is the relativistic energy of the system, and $\boldsymbol{\alpha}$ and β are the 4×4 Dirac matrices given by

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & -\boldsymbol{I} \end{pmatrix}, \quad i = 1,2,3, \tag{2.2}$$

where $\boldsymbol{I}$ is the 2×2 unit matrix, and $\boldsymbol{\sigma}_i$ are the 2×2 Pauli matrices defined as:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3}$$

In the Pauli-Dirac representation:

$$\psi(\boldsymbol{r}) = \begin{pmatrix} \varphi(\boldsymbol{r}) \\ \chi(\boldsymbol{r}) \end{pmatrix}. \tag{2.4}$$

Substituting Eqs. (2.2)–(2.4) into Eq. (2.1) yields the following two coupled first-order differential equations:

$$\boldsymbol{\sigma}\cdot\boldsymbol{p}\,\chi(\boldsymbol{r}) = [E - V(\boldsymbol{r}) - M - S(\boldsymbol{r})]\,\varphi(\boldsymbol{r}), \tag{2.5}$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{p}\,\varphi(\boldsymbol{r}) = [E - V(\boldsymbol{r}) + M + S(\boldsymbol{r})]\,\chi(\boldsymbol{r}). \tag{2.6}$$

Under the condition of equal scalar and vector potentials, Eqs. (4) and (5) reduce to:

$$\boldsymbol{\sigma}\cdot\boldsymbol{p}\,\chi(\boldsymbol{r}) = [E-M-2V(\boldsymbol{r})]\,\varphi(\boldsymbol{r}), \tag{2.7}$$

$$\chi(\boldsymbol{r}) = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+M}\,\varphi(\boldsymbol{r}). \tag{2.8}$$

Substituting Eq. (2.8) into Eq. (2.7) results in the following Schrödinger-like differential equation:

$$[\boldsymbol{p}^2+2(E+M)V(\boldsymbol{r})]\,\varphi(\boldsymbol{r}) = [E^2-M^2]\,\varphi(\boldsymbol{r}). \tag{2.9}$$

We consider $V(\boldsymbol{r})$ as the generalized Cornell potential plus a novel angle-dependent potential, written in spherical coordinates as:

$$V(\boldsymbol{r}) = V(r,\theta) = V_0(r)+\frac{V_1(\theta)}{r^2}, \tag{2.10}$$

where

$$V_0(r) = a_1r^2+a_2r+a_3+\frac{a_4}{r}+\frac{a_5}{r^2}, \tag{2.11}$$

$$V_1(\theta) = \frac{b_1+b_2\sin^2\theta}{\cos^2\theta}+\frac{b_3}{\sin^2\theta\cos^2\theta}, \tag{2.12}$$

where a_i and b_j $(i=1,2,\ldots,5,\ j=1,2,3)$ are arbitrary parameters.

3 Functional Bethe Ansatz method

In this section, we provide a comprehensive overview of the functional Bethe ansatz method [21,37]. Consider the general second-order linear ordinary differential equation:

$$\left[P(t)\frac{d^2}{dt^2}+Q(t)\frac{d}{dt}+W(t)\right]S(t)=0, \tag{3.1}$$

where $P(t)$, $Q(t)$, and $W(t)$ are polynomials of degree at most 4, 3, and 2, respectively:

$$P(t)=\sum_{k=0}^{4}p_kt^k,\quad Q(t)=\sum_{k=0}^{3}q_kt^k,\quad W(t)=\sum_{k=0}^{2}w_kt^k, \tag{3.2}$$

where p_k, q_k, and w_k are parameters. Let n be a non-negative integer. The differential equation (3.1) has nth-degree polynomial solutions $S(t)$ of the form [37]:

$$S(t)=\prod_{i=1}^{n}(t-t_i),\qquad S(t)\equiv 1\quad \text{for } n=0, \tag{3.3}$$

where the coefficients w_k of the polynomial $W(t)$ must satisfy the constraints:

$$w_2 = -n(n-1)p_4-nq_3, \tag{3.4}$$

$$w_1 = -[2(n-1)p_4+q_3]\sum_{i=1}^{n}t_i-n(n-1)p_3-nq_2, \tag{3.5}$$

$$w_0 = -[2(n-1)p_4+q_3]\sum_{i=1}^{n}t_i^2-2p_4\sum_{i<j}^{n}t_it_j-[2(n-1)p_3+q_2]\sum_{i=1}^{n}t_i -n(n-1)p_2-nq_1, \tag{3.6}$$

and the roots $t_1, t_2, \ldots, t_n$ are determined by the Bethe ansatz equations:

$$\sum_{j\neq i}^{n} \frac{2}{t_i - t_j} + \frac{q_3 t_i^3 + q_2 t_i^2 + q_1 t_i + q_0}{p_4 t_i^4 + p_3 t_i^3 + p_2 t_i^2 + p_1 t_i + p_0} = 0, \qquad i = 1, 2, \ldots, n. \tag{3.7}$$

The above equations (3.4)–(3.7) determine all polynomials $W(t)$ such that Eq. (3.1) admits a polynomial solution (3.3). Hence, the differential equation (3.1) is quasi-exactly solvable for certain parameter values.

4 Polar and radial Dirac wave functions

4.1 Separation of variables

In spherical coordinates, the wave function can be selected as

$$\varphi_{n,l,m}(r,\theta,\phi) = \frac{U_{n,l,m}(r)}{r} H_l(\theta) e^{im\phi}, \qquad m = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{4.1}$$

Substituting Eq. (4.1) into Eq. (2.9) leads to a set of second-order differential equations:

$$\frac{d^2 U_{n,l,m}(r)}{dr^2} + \left[-2(E_{n,l,m} + M) V_0(r) + E_{n,l,m}^2 - M^2 - \frac{\lambda}{r^2} \right] U_{n,l,m}(r) = 0, \tag{4.2}$$

$$\frac{d^2 H_l(\theta)}{d\theta^2} + \frac{1}{\tan\theta} \frac{dH_l(\theta)}{d\theta} + \left[-2(E_{n,l,m} + M) V_1(\theta) - \frac{\nu}{\sin^2\theta} + \lambda \right] H_l(\theta) = 0, \tag{4.3}$$

where $V_0(r)$ and $V_1(\theta)$ are given by Eqs. (2.11) and (2.12), respectively, and λ and ν are the separation constants.

4.2 Solution of the polar equation

To find polynomial solutions for the polar equation (4.3), we perform a variable change $t = \sin\theta$, resulting in the following second-order differential equation:

$$\begin{aligned} t^2(1-t^2)^2 \frac{d^2 H_l(t)}{dt^2} &+ t(2t^2 - 1)(1 - t^2) \frac{dH_l(t)}{dt} \\ &+ \Big[[-2(E_{n,l,m} + M) b_2 - \lambda]\, t^4 + [-2(E_{n,l,m} + M) b_1 + \nu + \lambda] t^2 \\ &\quad -2(E_{n,l,m} + M) b_3 - \nu \Big] H_l(t) = 0. \end{aligned} \tag{4.4}$$

Under the constraint $b_1 = -b_2 - b_3$, Eq. (4.4) reduces to

$$t^2(1-t^2) \frac{d^2 H_l(t)}{dt^2} + t(1-2t^2) \frac{dH_l(t)}{dt} + \left[(\lambda + 2(E_{n,l,m} + M) b_2)\, t^2 - 2(E_{n,l,m} + M) b_3 - \nu \right] H_l(t) = 0. \tag{4.5}$$

This equation is amenable to the functional Bethe ansatz method. According to this method, Eq. (4.5) has l-th degree polynomial solutions:

$$H_l(t) = \prod_{i=1}^{l} (t - t_i), \qquad H_0(t) \equiv 1 \quad \text{for } l = 0, \tag{4.6}$$

where the roots t_i are determined by the Bethe ansatz equations:

$$\sum_{j\neq i}^{l} \frac{2}{t_i - t_j} + \frac{2t_i^2 - 1}{t_i(t_i^2 - 1)} = 0, \qquad i = 1, 2, \ldots, l, \tag{4.7}$$

provided the following conditions are satisfied:

$$l^2 - 2l \sum_{i=1}^{l} t_i^2 - 2\sum_{i<j}^{l} t_i t_j = 2(E_{n,l,m} + M)b_3 + \nu, \tag{4.8}$$

$$l(l+1) = 2(E_{n,l,m} + M)b_2 + \lambda, \tag{4.9}$$

$$l \sum_{i=1}^{l} t_i = 0. \tag{4.10}$$

4.2.1 Specific cases for l

Case $l = 0$: The polar wave function is given by

$$H_0(\theta) = 1,$$

and the relations between the separation constants λ, ν and the potential parameters b_2, b_3 are:

$$\lambda = -2(E_{n,0,m} + M)b_2, \tag{4.11}$$

$$\nu = -2(E_{n,0,m} + M)b_3. \tag{4.12}$$

Case $l = 1$: The real roots of Eq. (4.7) are

$$t_1 = \pm\frac{\sqrt{2}}{2},$$

with the separation constant λ given by

$$\lambda = 2 - 2(E_{n,1,m} + M)b_2,$$

and the potential parameter b_3 constrained by

$$\nu + 2(E_{n,1,m} + M)b_3 = 0.$$

The corresponding polar wave function is

$$H_1(\theta) = \sin\theta \mp \frac{\sqrt{2}}{2}. \tag{4.13}$$

Case $l = 2$: The real roots are:

$$t_1 = \pm\frac{\sqrt{6}}{3}, \quad t_2 = \mp\frac{\sqrt{6}}{3},$$

or

$$t_1 = \pm\frac{\sqrt{6}}{8} \pm \frac{\sqrt{22}}{8}, \quad t_2 = \mp\frac{\sqrt{6}}{8} \pm \frac{\sqrt{22}}{8}.$$

The separation constant λ is given by

$$\lambda = 6 - 2(E_{n,2,m} + M)b_2,$$

with the potential parameter b_3 constrained by

$$\nu + 2(E_{n,2,m} + M)b_3 = 0.$$

The polar wave function takes the form:

$$H_2(\theta) = -\frac{\cos 2\theta}{2} - \frac{1}{6}, \tag{4.14}$$

$$H_2(\theta) = -\frac{\cos 2\theta}{2} \pm \frac{\sqrt{22}}{4}\sin\theta + \frac{3}{4}. \tag{4.15}$$

In conclusion, the results indicate that Eq. (4.5) is quasi-exactly solvable for specific parameter values.

4.3 Solution of the radial equation

We now turn to the derivation of polynomial solutions to the radial equation. To achieve this, we begin by transforming the differential equation (4.2) into a suitable form through substitution. Specifically, by setting the potential parameter $a_1 > 0$ and applying the transformation

$$r_1 = [2a_1(E_{n,l,m} + M)]^{\frac{1}{4}}\, r, \tag{4.16}$$

the differential equation (4.2) reduces to

$$\frac{d^2U_{n,l,m}(r_1)}{dr_1^2} + \left[c_1 + c_2 r_1 - r_1^2 + \frac{c_3}{r_1} - \frac{c_4}{r_1^2}\right]U_{n,l,m}(r_1) = 0, \tag{4.17}$$

where

$$\begin{cases} c_1 = \dfrac{-2a_3(E_{n,l,m}+M) + E_{n,l,m}^2 - M^2}{\sqrt{2a_1(E_{n,l,m}+M)}}, \\ c_2 = -2a_2(E_{n,l,m}+M)\,[2a_1(E_{n,l,m}+M)]^{-\frac{3}{4}}, \\ c_3 = -2a_4(E_{n,l,m}+M)\,[2a_1(E_{n,l,m}+M)]^{-\frac{1}{4}}, \\ c_4 = 2a_5(E_{n,l,m}+M) + \lambda. \end{cases} \tag{4.18}$$

Adopting the change of variable [2]

$$U_{n,l,m}(r_1) = r_1^{\frac{1+\sqrt{4c_4+1}}{2}} \exp\left(\frac{c_2 r_1 - r_1^2}{2}\right) u(r_1), \tag{4.19}$$

Eq. (4.17) transforms into

$$r_1 u''(r_1) + \left[1 + \sqrt{4c_4+1} + c_2 r_1 - 2r_1^2\right]u'(r_1) + \left[\frac{c_2}{2}\left(1 + \sqrt{4c_4+1}\right) + c_3 + \left(c_1 + \frac{c_2^2}{4} - \sqrt{4c_4+1} - 2\right)r_1\right]u(r_1) = 0. \tag{4.20}$$

This corresponds precisely to the canonical form of the biconfluent Heun differential equation [30]

$$r_1 u''(r_1) + \left(1+\alpha-\beta r_1 - 2r_1^2\right) u'(r_1) + \left[\left(\gamma-\alpha-2\right) r_1 - \frac{1}{2}\left(\delta + \left(1+\alpha\right)\beta\right)\right] u(r_1) = 0, \quad (4.21)$$

where the four Heun parameters are expressed as

$$\begin{cases} \alpha = \sqrt{4c_4+1}, & \beta = -c_2, \\ \gamma = c_1 + \frac{c_2^2}{4}, & \delta = -2c_3. \end{cases} \quad (4.22)$$

The polynomial solution of Eq. (4.21) is given by [14,30]

$$u(r_1) = N(\alpha,\beta,\gamma,\delta;r_1) = \sum_{\kappa=0}^{+\infty} \frac{A_\kappa}{(1+\alpha)_\kappa} \frac{r_1^\kappa}{\kappa!}, \quad (4.23)$$

where

$$\begin{aligned} A_0 &= 1, \\ A_1 &= \frac{(1+\alpha)\beta+\delta}{2}, \\ (\alpha)_\kappa &= \frac{\Gamma(\alpha+\kappa)}{\Gamma(\alpha)}, \quad \kappa \geq 0. \end{aligned}$$

Using the following recurrence formulas:

$$A_1 + \eta A_0 = 0, \quad (4.24)$$

$$A_2 + (\eta-\beta) A_1 + (1+\alpha)(\gamma-\alpha-2) A_0 = 0, \quad (4.25)$$

$$A_{\kappa+2} + \left[\eta - (\kappa+1)\beta\right] A_{\kappa+1} + (\gamma-\alpha-2-2\kappa)(\kappa+1)(\alpha+\kappa+1) A_\kappa = 0, \quad \kappa \geq 1, \quad (4.26)$$

where $\eta = -\frac{(1+\alpha)\beta+\delta}{2}$, and A_κ is a polynomial of degree κ in η, the series solution (4.23) becomes a polynomial of degree n if and only if [30]

$$\gamma - \alpha - 2 = 2n \qquad \text{and} \qquad A_{n+1} = 0, \quad n = 0,1,2,\ldots \quad (4.27)$$

It is important to note that there are at most $(n+1)$ suitable values of η, denoted by η_σ^n with $0 \leq \sigma \leq n$. These discrete values of η_σ^n correspond only to discrete values of a_4 in the potential (2.11).

As a result, upon applying the recurrence relations (4.24)–(4.26) and the condition (4.27), the solution of the radial equation (4.17), which depends on the three quantum numbers n, l, and m, can be written as

$$U_{n,l,m}(r_1) = r_1^{\frac{1+\sqrt{1+4c_4}}{2}} \exp\left(\frac{c_2 r_1 - r_1^2}{2}\right) \sum_{\kappa=0}^{n} \frac{A_\kappa}{(1+\alpha)_\kappa} \frac{r_1^\kappa}{\kappa!}, \quad n,l,m = 0,1,\ldots \quad (4.28)$$

where $r_1 = \left[2a_1(E_{n,l,m}+M)\right]^{\frac{1}{4}} r$, $A_0 = 1$, $A_1 = -\eta_\sigma^n$, and A_κ for $\kappa = 2,3,\ldots,n$ are polynomials of degree κ in η_σ^n.

Moreover, considering Eqs. (4.18), (4.22), and the condition (4.27), the energy relation is given by

$$E_{n,l,m}^2 - M^2 - 2\sqrt{2(E_{n,l,m}+M)a_1}\left[n+1+\sqrt{2(E_{n,l,m}+M)a_5+\lambda+\frac{1}{4}}\right] + (E_{n,l,m}+M)\frac{a_2^2-4a_1a_3}{2a_1} = 0, \quad (4.29)$$

where the separation constant λ is given by Eq. (4.9).

4.4 Bound states and their associated energy eigenvalues

In the framework of the results obtained, it is evident that the bound states of the Dirac equation (2.1) are expressed as:

$$\psi_{n,l,m}(r,\theta,\phi) = [2a_1(E_{n,l,m}+M)]^{\frac{1}{4}} \begin{pmatrix} 1 \\ \frac{\sigma\cdot p}{E_{n,l,m}+M} \end{pmatrix} r_1^{\frac{-1+\sqrt{1+4c_4}}{2}} \exp\left(\frac{c_2 r_1 - r_1^2}{2} + im\phi\right) H_l(\theta) \sum_{\kappa=0}^{n} \frac{A_\kappa}{(1+\alpha)_\kappa} \frac{r_1^\kappa}{\kappa!},$$
$$n,l,m = 0,1,\ldots,$$

where

$$r_1 = [2a_1(E_{n,l,m}+M)]^{\frac{1}{4}}\, r,$$

and $H_l(\theta)$ represents the polynomial solutions of Eq. (4.3).

The corresponding discrete energy levels, under the constraint of the potential parameters $b_1 = -b_2 - b_3$, take the form:

$$E_{n,l,m}^2 - M^2 - 2\sqrt{2(E_{n,l,m}+M)a_1}\left[n+1+\sqrt{2(E_{n,l,m}+M)\,[a_5-b_2] + \left[l+\frac{1}{2}\right]^2}\right] + (E_{n,l,m}+M)\frac{a_2^2-4a_1a_3}{2a_1} = 0. \quad (4.30)$$

5 Numerical applications

In this section, we present examples of the wave function and the corresponding energy levels for specific values of the quantum numbers n, l, and m as numerical applications. To achieve this, the polynomials A_n and the corresponding solution $U_{n,l,m}(r)$ are computed for selected values of n.

According to the recurrence relations (4.24)-(4.26), the coefficients are given by:

$$\begin{aligned}
A_0 &= 1, \\
A_1 &= -\eta, \\
A_2 &= \eta^2 - \beta\eta - (1+\alpha)(\gamma-\alpha-2), \\
A_3 &= -\eta^3 + 3\beta\eta^2 - \left[2\beta^2 - \sum_{i=1}^{2} i(\gamma-\alpha-2i)(i+\alpha)\right]\eta - 2\beta(1+\alpha)(\gamma-\alpha-2), \\
A_4 &= \eta^4 - 6\beta\eta^3 + \left[11\beta^2 - \sum_{i=1}^{3} i(\gamma-\alpha-2i)(i+\alpha)\right]\eta^2 \\
&- \beta\left[6\beta^2 + 4(\gamma-\alpha-6)(3+\alpha) - \sum_{i=1}^{3}(i+4)(\gamma-\alpha-2i)(i+\alpha)\right]\eta \\
&+ 3(1+\alpha)(\gamma-\alpha-2)\left[(\gamma-\alpha-6)(3+\alpha) - 2\beta^2\right].
\end{aligned}$$

Consequently, the explicit form of the function $U_{n,l,m}(r)$ for $n = 1,2,3,4$ (valid for all l and

m) is:

$$\begin{aligned}
U_{0,l,m}(r) &= r_1^{\frac{1+\sqrt{1+4c_4}}{2}} \exp\left(\frac{c_2 r_1 - r_1^2}{2}\right), \\
U_{1,l,m}(r) &= r_1^{\frac{1+\sqrt{1+4c_4}}{2}} \exp\left(\frac{c_2 r_1 - r_1^2}{2}\right)\left[1 + \frac{A_1}{1+\alpha} r_1\right], \\
U_{2,l,m}(r) &= r_1^{\frac{1+\sqrt{1+4c_4}}{2}} \exp\left(\frac{c_2 r_1 - r_1^2}{2}\right)\left[1 + \frac{A_1}{1+\alpha} r_1 + \frac{A_2}{2!(1+\alpha)(2+\alpha)} r_1^2\right], \\
U_{3,l,m}(r) &= r_1^{\frac{1+\sqrt{1+4c_4}}{2}} \exp\left(\frac{c_2 r_1 - r_1^2}{2}\right)\left[1 + \frac{A_1}{1+\alpha} r_1 + \frac{A_2}{2!(1+\alpha)(2+\alpha)} r_1^2\right. \\
&+ \left.\frac{A_3}{3!(1+\alpha)(2+\alpha)(3+\alpha)} r_1^3\right], \\
U_{4,l,m}(r) &= r_1^{\frac{1+\sqrt{1+4c_4}}{2}} \exp\left(\frac{c_2 r_1 - r_1^2}{2}\right)\left[1 + \frac{A_1}{1+\alpha} r_1 + \frac{A_2}{2!(1+\alpha)(2+\alpha)} r_1^2\right. \\
&+ \frac{A_3}{3!(1+\alpha)(2+\alpha)(3+\alpha)} r_1^3 \\
&+ \left.\frac{A_4}{4!(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)} r_1^4\right].
\end{aligned}$$

In the case under consideration, we set $a_1 = 1$, $a_2^2 = 4a_3$, $a_5 = b_2$, and $M = 1$. Under these settings, the discrete energies can be expressed as:

$$E_{n,l,m}^3 - E_{n,l,m}^2 - E_{n,l,m} - 8\left[n + l + \frac{3}{2}\right]^2 + 1 = 0. \tag{5.1}$$

Based on the above relations, we provide explicit energy values for $n, l = \overline{0, \ldots, 5}$ and $\forall m \in \mathbb{N}$:

$$\begin{aligned}
\boldsymbol{E_0} &= E_{0,0,m} = \frac{4}{9\sqrt[3]{\frac{\sqrt{681}}{3} + \frac{235}{27}}} + \sqrt[3]{\frac{\sqrt{681}}{3} + \frac{235}{27}} + \frac{1}{3}, \\
\boldsymbol{E_1} &= E_{0,1,m} = E_{1,0,m} = \frac{4}{9\sqrt[3]{\frac{\sqrt{16475}}{\sqrt{27}} + \frac{667}{27}}} + \sqrt[3]{\frac{\sqrt{16475}}{\sqrt{27}} + \frac{667}{27}} + \frac{1}{3}, \\
\boldsymbol{E_2} &= E_{0,2,m} = E_{1,1,m} = E_{2,0,m} = \frac{4}{9\sqrt[3]{\frac{\sqrt{64043}}{\sqrt{27}} + \frac{1315}{27}}} + \sqrt[3]{\frac{\sqrt{64043}}{\sqrt{27}} + \frac{1315}{27}} + \frac{1}{3}, \\
\boldsymbol{E_3} &= E_{0,3,m} = E_{1,2,m} = E_{2,1,m} = E_{3,0,m} = \frac{4}{9\sqrt[3]{\sqrt{6513} + \frac{2179}{27}}} + \sqrt[3]{\sqrt{6513} + \frac{2179}{27}} + \frac{1}{3}, \\
\boldsymbol{E_4} &= E_{0,4,m} = E_{1,3,m} = E_{2,2,m} = E_{3,1,m} = E_{4,0,m} = \frac{4}{9\sqrt[3]{\frac{\sqrt{393371}}{\sqrt{27}} + \frac{3259}{27}}} + \sqrt[3]{\frac{\sqrt{393371}}{\sqrt{27}} + \frac{3259}{27}} + \frac{1}{3}, \\
\boldsymbol{E_5} &= E_{0,5,m} = E_{1,4,m} = E_{2,3,m} = E_{3,2,m} = E_{4,1,m} = E_{5,0,m} = \frac{4}{9\sqrt[3]{\frac{\sqrt{768443}}{\sqrt{27}} + \frac{4555}{27}}} + \sqrt[3]{\frac{\sqrt{768443}}{\sqrt{27}} + \frac{4555}{27}} + \frac{1}{3}.
\end{aligned}$$

When we set $a_1 = a_3 = 1$, $a_2 = 2$ and $a_5 = b_2 = \frac{1}{2}$ with the constraint of the potential parameters $b_1 = -b_2 - b_3$, the potential $V(\boldsymbol{r})$ expressed in Eq. (2.10) turns into

$$V(\boldsymbol{r}) = r^2 + 2r + 1 + \frac{a_4}{r} + \frac{1}{2r^2} + \frac{V_1(\theta)}{r^2}. \tag{5.2}$$

Now, using the above results, we introduce the different values of the energy eigenvalues $E_{n,l,m}$ and η which are the roots of $A_{n+1}, n = 1,2,3,4$ as shown in the Tables 1, 2 and 3.

Values of n	Values of $E_{n,0,m}$	Values of η_σ^n
1	E_1	$-0.89431,\ 4.4727$
2	E_2	$-1.8606,\ 3.2846,\ 9.7580$
3	E_3	$-2.9270,\ 2.0448,\ 8.2995,\ 15.678$
4	E_4	$-4.1203,\ 0.75310,\ 6.7914,\ 13.974,\ 22.133$

Table 1. Values of the discrete energies $E_{n,0,m}$ and the roots η_σ^n for $n = 1,2,3,4$.

Values of n	Values of $E_{n,1,m}$	Values of η_σ^n
1	E_2	$-1.5236,\ 5.2508$
2	E_3	$-3.1228,\ 3.5457,\ 11.124$
3	E_4	$-4.8164,\ 1.7799,\ 9.2243,\ 17.531$
4	E_5	$-6.6202,\ -5.3368 \times 10^{-2},\ 7.2724,\ 15.437,\ 24.405$

Table 2. Values of the discrete energies $E_{n,1,m}$ and the roots η_σ^n for $n = 1,2,3,4$.

Values of n	Values of $E_{n,2,m}$	Values of η_σ^n
1	E_3	$-2.0383,\ 5.8874$
2	E_4	$-4.151,\ 3.7192,\ 12.291$
3	E_5	$-6.3518,\ 1.4871,\ 9.9740,\ 19.156$
4	E_6	$-8.6522,\ -0.81601,\ 7.6019,\ 16.683,\ 26.437$

Table 3. Values of the discrete energies $E_{n,2,m}$ and the roots η_σ^n for $n = 1,2,3,4$.

To conclude, it is straightforward to provide numerical examples of the explicit form of the Dirac wave function $\psi_{n,l,m}(r,\theta,\phi)$ and their corresponding energy levels $E_{n,l,m}$:

1. ***For $n = l = m = 0$***, we have

$$\psi_{0,0,0}(r,\theta,\phi) = (2+2E_0)^{\frac{1}{4}} \begin{pmatrix} 1 \\ \frac{\sigma\cdot p}{1+E_0} \end{pmatrix} \exp\left[\frac{-r_1^2 - 2^{\frac{5}{4}}(1+E_0)^{\frac{1}{4}} r_1}{2}\right] \quad \text{with} \quad E_0 = E_{0,0,0}.$$

2. ***For $n = l = 1$ and $m = 2$:***

   - When $\eta \approx -1.5236$ is set, we get

$$\psi_{1,1,2}(r,\theta,\phi) \approx (2+2E_2)^{\frac{1}{4}} \begin{pmatrix} 1 \\ \frac{\sigma\cdot p}{1+E_2} \end{pmatrix} \left[0.5078 r_1^2 + r_1\right] \exp\left[\frac{-r_1^2 - 2^{\frac{5}{4}}(1+E_2)^{\frac{1}{4}} r_1 + 4i\phi}{2}\right] H_1(\theta),$$

with the corresponding energy level $E_2 = E_{1,1,2}$.

3. ***For n = 2, l = 1 and m = 2:***

   - If taking $\eta \approx 11.124$, we are able to obtain

$$\psi_{2,1,2}(r,\theta,\phi) \approx (2+2E_3)^{\frac{1}{4}} \begin{pmatrix} 1 \\ \frac{\sigma\cdot p}{1+E_3} \end{pmatrix} \left[1.6232\, r_1^3 - 2.781\, r_1^2 + r_1\right]$$
$$\times \exp\left[\frac{-r_1^2 - 2^{\frac{5}{4}}(1+E_3)^{\frac{1}{4}} r_1 + 4i\phi}{2}\right] H_1(\theta) \quad \text{with} \quad E_3 = E_{2,1,2}.$$

4. ***For n = 3, l = 1 and m = 2:***

   - When we take $\eta \approx 9.224$, we can show that

$$\psi_{3,1,2}(r,\theta,\phi) \approx (2+2E_4)^{\frac{1}{4}} \begin{pmatrix} 1 \\ \frac{\sigma\cdot p}{1+E_4} \end{pmatrix} \left[0.4673\, r_1^4 + 0.6155\, r_1^3 - 2.3061\, r_1^2 + r_1\right]$$
$$\times \exp\left[\frac{-r_1^2 - 2^{\frac{5}{4}}(1+E_4)^{\frac{1}{4}} r_1 + 4i\phi}{2}\right] H_1(\theta).$$

The corresponding energy level $E_4 = E_{3,1,2}$.

5. ***For n = 4, l = 1 and m = 2:***

   - In the case $\eta \approx 24.405$, we find that

$$\psi_{4,1,2}(r,\theta,\phi) \approx (2+2E_5)^{\frac{1}{4}} \begin{pmatrix} 1 \\ \frac{\sigma\cdot p}{1+E_5} \end{pmatrix} \left[2.0661\, r_1^5 - 8.5021\, r_1^4 + 11.623\, r_1^3 - 6.1013\, r_1^2 + r_1\right]$$
$$\times \exp\left[\frac{-r_1^2 - 2^{\frac{5}{4}}(1+E_5)^{\frac{1}{4}} r_1 + 4i\phi}{2}\right] H_1(\theta),$$

with the energy level $E_5 = E_{4,1,2}$.

6. ***For n = 4, l = 2 and m = 3:***

   - When $\eta \approx -0.8160$, it follows that

$$\psi_{4,2,3}(r,\theta,\phi) \approx (2+2E_6)^{\frac{1}{4}} \begin{pmatrix} 1 \\ \frac{\sigma\cdot p}{1+E_6} \end{pmatrix} \left[-0.0267\, r_1^6 - 0.2317\, r_1^5 - 0.5234\, r_1^4 + 0.136\, r_1^3 + r_1^2\right]$$
$$\times \exp\left[\frac{-r_1^2 - 2^{\frac{5}{4}}(1+E_6)^{\frac{1}{4}} r_1 + 6i\phi}{2}\right] H_2(\theta).$$

The corresponding energy level is $E_6 = E_{4,2,3}$.

It should be noted that the potential parameters for all these cases are constrained by $b_1 = -b_2$ and $b_3 = 0$. Furthermore, $r_1 = [2a_1(E_{n,l,m} + M)]^{\frac{1}{4}}\, r$, where $H_1(\theta)$ is given by Eq. (4.13) and $H_2(\theta)$ is provided in Eqs. (4.14)–(4.15).

6 Conclusion

In this work, we solved the Dirac equation with non-central scalar and vector potentials, including the generalized Cornell potential combined with a novel angle-dependent potential, within the framework of quasi-exactly solvable problems. The bound states and energy levels associated with the problem have been determined.

By employing the functional Bethe ansatz method to derive the polynomial solutions of the polar equation and utilizing the biconfluent Heun differential equation to solve the radial equation, we presented the Dirac wave function. To validate our findings, numerical results were also provided.

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Conflict of interest

The authors have no conflicts of interest to declare.

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