Journal of Innovative Applied Mathematics and Computational Sciences

J. Innov. Appl. Math. Comput. Sci. **4**(2) (2024), 188–200. DOI: 10.58205/jiamcs.v4i2.1884

http://jiamcs.centre-univ-mila.dz/

Formulas of the solutions of a non-autonomous difference equation and two systems of difference equations

Hiba Zabat [1], Nouressadat Touafek ✉[1] and Imane Dekkar [1]

[1]LMAM Laboratory, Faculty of Exact Sciences and Informatics, University of Jijel, 18000 Jijel, Algeria

Received 16 December 2024, Accepted 25 December 2024, Published 28 December 2024

Abstract. In this work, we explicitly solve the following:

- A higher-order non-autonomous difference equation:

$$x_{n+1} = \alpha_n x_{n-k} + \frac{\beta_n}{x_n x_{n-1} \cdots x_{n-k+1}},$$

where $n \in \mathbb{N}_0$, $k \in \mathbb{N}$, the sequences $(\alpha_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ are real, and the initial values $x_{-k}, x_{-k+1}, \ldots, x_0$ are nonzero real numbers.

- A three-dimensional system of second-order difference equations:

$$x_{n+1} = \frac{a_1 y_{n-1} z_{n-1}}{a x_{n-1} + b y_{n-1} + c z_{n-1}}, \quad y_{n+1} = \frac{a_2 x_{n-1} z_{n-1}}{a x_{n-1} + b y_{n-1} + c z_{n-1}},$$

$$z_{n+1} = \frac{a_3 x_{n-1} y_{n-1}}{a x_{n-1} + b y_{n-1} + c z_{n-1}},$$

where $n \in \mathbb{N}_0$, the parameters a, b, c, a_1, a_2, a_3 are real numbers, and the initial values $x_{-1}, x_0, y_{-1}, y_0, z_{-1}, z_0$ are nonzero real numbers.

- A three-dimensional system of first-order difference equations:

$$x_{n+1} = \frac{a_1 y_n z_n}{a x_n + b y_n + c z_n}, \quad y_{n+1} = \frac{a_2 x_n z_n}{a x_n + b y_n + c z_n}, \quad z_{n+1} = \frac{a_3 x_n y_n}{a x_n + b y_n + c z_n},$$

where $n \in \mathbb{N}_0$, the parameters a, b, c, a_1, a_2, a_3 are real numbers, and the initial values x_0, y_0, z_0 are nonzero real numbers.

Keywords: Difference equations and systems, Closed form of the solutions, Periodicity.
2020 Mathematics Subject Classification: 39A05, 39A10, 39A23. MSC2020

✉Corresponding author. Email: ntouafek@gmail.com

ISSN (electronic): 2773-4196

1 Introduction

Difference equations and their systems play a crucial role in modeling phenomena that evolve over discrete time steps. These models are widely applied across fields such as population dynamics, economics, and engineering. The search for explicit solutions to difference equations remains a key focus, drawing significant attention from researchers aiming to understand the behavior of discrete-time systems and facilitate informed decision-making. Numerous studies have contributed to this area (e.g., [1–6, 9–18, 20, 21]).

In [19], the authors studied the system of difference equations

$$u_{n+1} = \frac{a}{u_n} + \frac{b}{v_n}, \quad v_{n+1} = \frac{a}{u_n}, \quad n \in \mathbb{N}_0. \tag{1.1}$$

Substituting the second equation of this system into the first yields

$$u_{n+2} = \frac{a}{u_{n+1}} + \frac{b}{v_{n+1}} = \frac{a}{u_{n+1}} + \frac{b}{\frac{a}{u_n}} = \frac{a}{u_{n+1}} + \frac{b}{a}u_n, \quad n \in \mathbb{N}_0.$$

This equation can be expressed in the standard form

$$x_{n+1} = \frac{\alpha}{x_n} + \beta x_{n-1}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where x_{-1} and x_0 are the initial values, and α and β are parameters.

In the first part of this work, inspired by the above equation, we study the more general higher-order difference equation

$$x_{n+1} = \alpha_n x_{n-k} + \frac{\beta_n}{x_n x_{n-1} \cdots x_{n-k+1}}, \tag{1.3}$$

where $n \in \mathbb{N}_0$, $k \in \mathbb{N}$, the sequences $(\alpha_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ are real, and the initial values $x_{-k}, x_{-k+1}, \ldots, x_0$ are nonzero real numbers.

In the second part of this work, we analyze a three-dimensional system of second-order difference equations given by

$$x_{n+1} = \frac{a_1 y_{n-1} z_{n-1}}{a x_{n-1} + b y_{n-1} + c z_{n-1}}, \quad y_{n+1} = \frac{a_2 x_{n-1} z_{n-1}}{a x_{n-1} + b y_{n-1} + c z_{n-1}}, \quad z_{n+1} = \frac{a_3 x_{n-1} y_{n-1}}{a x_{n-1} + b y_{n-1} + c z_{n-1}}, \tag{1.4}$$

where $n \in \mathbb{N}_0$, a, b, c, a_1, a_2, a_3 are real parameters, and the initial values $x_{-1}, x_0, y_{-1}, y_0, z_{-1}, z_0$ are nonzero real numbers. For $a = b = c = 1$, this system simplifies to

$$x_{n+1} = \frac{a_1 y_{n-1} z_{n-1}}{x_{n-1} + y_{n-1} + z_{n-1}}, \quad y_{n+1} = \frac{a_2 x_{n-1} z_{n-1}}{x_{n-1} + y_{n-1} + z_{n-1}}, \quad z_{n+1} = \frac{a_3 x_{n-1} y_{n-1}}{x_{n-1} + y_{n-1} + z_{n-1}}. \tag{1.5}$$

This particular system (1.5) was solved by Elsayed et al. in [8]. Motivated by their work, we derive closed-form solutions for System (1.4), generalizing the results from [8] and providing further insights.

In the third part, we explicitly solve the following three-dimensional system of first-order difference equations:

$$x_{n+1} = \frac{a_1 y_n z_n}{a x_n + b y_n + c z_n}, \quad y_{n+1} = \frac{a_2 x_n z_n}{a x_n + b y_n + c z_n}, \quad z_{n+1} = \frac{a_3 x_n y_n}{a x_n + b y_n + c z_n}, \tag{1.6}$$

where $n \in \mathbb{N}_0$, a, b, c, a_1, a_2, a_3 are real parameters, and x_0, y_0, z_0 are nonzero real initial values. At the end of this work, we propose an open problem extending Systems (1.4) and (1.6).

2 The equation $x_{n+1} = \alpha_n x_{n-k} + \frac{\beta_n}{x_n x_{n-1} \ldots x_{n-k+1}}$

In this part we give explicit formulas for well defined solutions of the equation (1.3) and we will give a special attention to the autonomous case.

Definition 2.1. A well defined solution $(x_n)_{n=-k}^{+\infty}$ of Equation (1.3), is a solution such that

$$x_n \neq 0,\ n = -k, -k+1, \cdots .$$

The following result is devoted to the closed form of the solutions of Equation (1.3).

Theorem 2.2. *Let $(x_n)_{n=-k}^{+\infty}$ be a well defined solution of Equation (1.3), then for $n = 0, 1, \ldots$, and $l = 0, \ldots, k$, we have*

$$x_{(k+1)n+l+1} = x_{-k+l} \prod_{r=0}^{n} \left(\frac{\left(\prod_{i=0}^{(k+1)r+l} \alpha_i\right) x_0 x_{-1} \ldots x_{-k} + \sum_{j=0}^{(k+1)r+l} \left(\prod_{i=j+1}^{(k+1)r+l} \alpha_i\right) \beta_j}{\left(\prod_{i=0}^{(k+1)r+l-1} \alpha_i\right) x_0 x_{-1} \ldots x_{-k} + \sum_{j=0}^{(k+1)r+l-1} \left(\prod_{i=j+1}^{(k+1)r+l-1} \alpha_i\right) \beta_j} \right),$$

and in the autonomous case, that is when $\alpha_n = \alpha$, $\beta_n = \beta$ we have

$$x_{(k+1)n+l+1} = x_{-k+l} \prod_{r=0}^{n} \left(\frac{\alpha^{(k+1)r+l+1} x_0 x_{-1} \ldots x_{-k} + \left(\frac{\alpha^{(k+1)r+l+1}-1}{\alpha-1}\right) \beta}{\alpha^{(k+1)r+l} x_0 x_{-1} \ldots x_{-k} + \left(\frac{\alpha^{(k+1)r+l}-1}{\alpha-1}\right) \beta} \right),$$

if $\alpha \neq 1$, and

$$x_{(k+1)n+l+1} = x_{-k+l} \prod_{r=0}^{n} \left(\frac{x_0 x_{-1} \ldots x_{-k} + ((k+1)r+l+1)\,\beta}{x_0 x_{-1} \ldots x_{-k} + ((k+1)r+l)\,\beta} \right),$$

if $\alpha = 1$.

Proof. For every well-defined solution $(x_n)_{n=-k}^{+\infty}$ of Equation (1.3), we can write

$$x_n x_{n-1} \ldots x_{n-k+1} x_{n+1} = \alpha_n x_n x_{n-1} \ldots x_{n-k+1} x_{n-k} + \beta_n. \tag{2.1}$$

Putting

$$y_n = x_n x_{n-1} \ldots x_{n-k+1} x_{n-k},\ n = 0, 1, \ldots, \tag{2.2}$$

then from (2.1), we get the following well-known non-autonomous first-order linear difference equation

$$y_{n+1} = \alpha_n y_n + \beta_n,\ n = 0, 1, \ldots, \tag{2.3}$$

the solutions of Equation (2.3), see for example [7], are given by

$$y_n = \left(\prod_{i=0}^{n-1} \alpha_i\right) y_0 + \sum_{j=0}^{n-1} \left(\prod_{i=j+1}^{n-1} \alpha_i\right) \beta_j, \tag{2.4}$$

where

$$y_0 = x_0 x_{-1} \ldots x_{-k}. \tag{2.5}$$

It follows from (2.2) that,

$$x_{n+1} = \frac{y_{n+1}}{y_n} x_{n-k},\ n = 0, 1, \ldots, \tag{2.6}$$

from this equation, we obtain that

$$x_{(k+1)n+l+1} = x_{-k+l} \prod_{r=0}^{n} \left(\frac{y_{(k+1)r+l+1}}{y_{(k+1)r+l}} \right), \, l = 0, ..., k. \tag{2.7}$$

Using (2.4) and (2.5), from(2.7), we get

$$x_{(k+1)n+l+1} = x_{-k+l} \prod_{r=0}^{n} \left(\frac{\left(\prod_{i=0}^{(k+1)r+l} \alpha_i\right) x_0 x_{-1} ... x_{-k} + \sum_{j=0}^{(k+1)r+l} \left(\prod_{i=j+1}^{(k+1)r+l} \alpha_i\right) \beta_j}{\left(\prod_{i=0}^{(k+1)r+l-1} \alpha_i\right) x_0 x_{-1} ... x_{-k} + \sum_{j=0}^{(k+1)r+l-1} \left(\prod_{i=j+1}^{(k+1)r+l-1} \alpha_i\right) \beta_j} \right), \, l = 0, ..., k. \tag{2.8}$$

Now, if $\alpha_n = \alpha$, $\beta_n = \beta$, then by replacing in (2.8), we get

$$x_{(k+1)n+l+1} = x_{-k+l} \left(\prod_{r=0}^{n} \frac{\alpha^{(k+1)r+l+1} x_0 x_{-1} ... x_{-k} + \left(\frac{\alpha^{(k+1)r+l+1} - 1}{\alpha - 1}\right) \beta}{\alpha^{(k+1)r+l} x_0 x_{-1} ... x_{-k} + \left(\frac{\alpha^{(k+1)r+l} - 1}{\alpha - 1}\right) \beta} \right), \, l = 0, ..., k \tag{2.9}$$

if $\alpha \neq 1$, and

$$x_{(k+1)n+l+1} = x_{-k+l} \prod_{r=0}^{n} \left(\frac{x_0 x_{-1} ... x_{-k} + ((k+1)r + l + 1)\, \beta}{x_0 x_{-1} ... x_{-k} + ((k+1)r + l)\, \beta} \right), \, l = 0, ..., k \tag{2.10}$$

if $\alpha = 1$. □

The following result concerns the periodicity of well-defined solutions of the autonomous equation.

Corollary 2.3. *Assume that $\alpha_n = \alpha$, $\beta_n = \beta$, $\alpha \neq 1$, and $x_0 x_{-1} \cdots x_{-k} = \frac{\beta}{1-\alpha}$. Then every well-defined solution of the equation (1.3) is periodic with period $k+1$.*

Proof. Replacing $x_0 x_{-1} \cdots x_{-k} = \frac{\beta}{1-\alpha}$ in (2.9), we obtain that for $n = 0, 1, \cdots$,

$$x_{(k+1)n+l+1} = x_{-k+l}, \, l = 0, \ldots, k,$$

which implies that the solution is periodic with period $k+1$ and takes the form

$$x_{-k}, x_{-k+1}, \ldots, x_0, x_{-k}, x_{-k+1}, \ldots, x_0, \cdots.$$

□

3 The system $x_{n+1} = \frac{a_1 y_{n-1} z_{n-1}}{a x_{n-1} + b y_{n-1} + c z_{n-1}}$, $y_{n+1} = \frac{a_2 x_{n-1} z_{n-1}}{a x_{n-1} + b y_{n-1} + c z_{n-1}}$, $z_{n+1} = \frac{a_3 x_{n-1} y_{n-1}}{a x_{n-1} + b y_{n-1} + c z_{n-1}}$

In this part, we are interested in solving the three-dimensional system of second-order difference equations given in (1.4). We begin with the following definition.

Definition 3.1. A well-defined solution $(x_n, y_n, z_n)_{n \geq -1}$ of System (1.4) is a solution such that:

$$ax + by_{n-1} + cz_{n-1} \neq 0, \, n \in \mathbb{N}_0.$$

Lemma 3.2. *Let $(x_n, y_n, z_n)_{n \geq -1}$ be a well-defined solution of System (1.4). Then,*

$$x_n \cdot y_n \cdot z_n \neq 0, \, n = -1, 0, \cdots.$$

Proof. Assume, for example, that there exists $n_0 \geq -1$ such that $x_{n_0} = 0$. It is straightforward to see that $y_{n_0+2} = 0$ and $z_{n_0+2} = 0$, leading to $x_{n_0+4} = 0$, $y_{n_0+4} = 0$, and $z_{n_0+4} = 0$. From these equalities, we obtain $x_{n_0+6} = \frac{0}{0}$, $y_{n_0+6} = \frac{0}{0}$, and $z_{n_0+6} = \frac{0}{0}$. This implies that the terms $x_{n_0+6}, y_{n_0+6}, z_{n_0+6}$ are undefined. □

We now proceed to solve the system. From (1.4), we get:

$$\frac{x_{n+1}}{y_{n+1}} = \frac{a_1 y_{n-1}}{a_2 x_{n-1}}, \ \frac{y_{n+1}}{z_{n+1}} = \frac{a_2 z_{n-1}}{a_3 y_{n-1}}, \ \frac{z_{n+1}}{x_{n+1}} = \frac{a_3 x_{n-1}}{a_1 z_{n-1}}. \tag{3.1}$$

Let

$$u_n = \frac{x_n}{y_n}, \ v_n = \frac{y_n}{z_n}, \ w_n = \frac{z_n}{x_n}, \ n \in \mathbb{N}_0. \tag{3.2}$$

Substituting (3.2) into (3.1), we obtain the following three independent equations:

$$u_{n+1} = \frac{a_1}{a_2 u_{n-1}}, \ v_{n+1} = \frac{a_2}{a_3 v_{n-1}}, \ w_{n+1} = \frac{a_3}{a_1 w_{n-1}}, \ n \in \mathbb{N}_0.$$

It is easy to verify that the sequences $(u_n)_{n\geq -1}$, $(v_n)_{n\geq -1}$, and $(w_n)_{n\geq -1}$ are periodic with a period of four, and for $n \in \mathbb{N}_0$, we have:

$$u_{4n-1} = u_{-1}, \ u_{4n} = u_0, \ u_{4n+1} = \frac{a_1}{a_2 u_{-1}}, \ u_{4n+2} = \frac{a_1}{a_2 u_0}, \tag{3.3}$$

$$v_{4n-1} = v_{-1}, \ v_{4n} = v_0, \ v_{4n+1} = \frac{a_2}{a_3 v_{-1}}, \ v_{4n+2} = \frac{a_2}{a_3 v_0}, \tag{3.4}$$

$$w_{4n-1} = w_{-1}, \ w_{4n} = w_0, \ w_{4n+1} = \frac{a_3}{a_1 w_{-1}}, \ w_{4n+2} = \frac{a_3}{a_1 w_0}. \tag{3.5}$$

Now, using equations in (1.4) and the change of variable in (3.2), we get for $n = 0, 1, \cdots,$

$$x_{n+1} = \frac{a_1 w_{n-1}}{a u_{n-1} + b + c u_{n-1} w_{n-1}} x_{n-1}, \tag{3.6}$$

$$y_{n+1} = \frac{a_2 u_{n-1}}{a u_{n-1} v_{n-1} + b v_{n-1} + c} y_{n-1}, \tag{3.7}$$

$$z_{n+1} = \frac{a_3 v_{n-1}}{a + b w_{n-1} v_{n-1} + c w_{n-1}} z_{n-1}. \tag{3.8}$$

Using the formulas of the sequences $(u_n)_{n\geq -1}$, $(v_n)_{n\geq -1}$, and $(w_n)_{n\geq -1}$ given in (3.3)-(3.5), it follows from (3.6)-(3.8) that for $n \in \mathbb{N}_0$, we have

- $$x_{4n+1} = \alpha_0 x_{4n-1}, \ \alpha_0 = \frac{a_1 w_{-1}}{a u_{-1} + c u_{-1} w_{-1} + b}. \tag{3.9}$$

- $$x_{4n+2} = \alpha_1 x_{4n}, \ \alpha_1 = \frac{a_1 w_0}{a u_0 + c u_0 w_0 + b}. \tag{3.10}$$

- $$x_{4n+3} = \alpha_2 x_{4n+1},$$

  using (3.9), we get

$$x_{4n+3} = \alpha_0 \alpha_2 x_{4n-1}, \ \alpha_2 = \frac{a_2 a_3 u_{-1}}{a a_1 w_{-1} + b a_2 w_{-1} u_{-1} + c a_3}. \tag{3.11}$$

- $$x_{4n+4} = \alpha_3 x_{4n+2},$$

  and by (3.10), we obtain

  $$x_{4n+4} = \alpha_1 \alpha_3 x_{4n},\ \alpha_3 = \frac{a_2 a_3 u_0}{a a_1 w_0 + b a_2 w_0 u_0 + c a_3}. \tag{3.12}$$

- $$y_{4n+1} = \beta_0 y_{4n-1},\ \beta_0 = \frac{a_2 u_{-1}}{a u_{-1} v_{-1} + b v_{-1} + c}, \tag{3.13}$$

- $$y_{4n+2} = \beta_1 y_{4n},\ \beta_1 = \frac{a_2 u_0}{a u_0 v_0 + b v_0 + c}, \tag{3.14}$$

- $$y_{4n+3} = \beta_2 y_{4n+1},$$

  and using (3.13), we get

  $$y_{4n+3} = \beta_0 \beta_2 y_{4n-1},\ \beta_2 = \frac{a_1 a_3 v_{-1}}{c a_3 v_{-1} u_{-1} + b a_2 u_{-1} + a a_1}. \tag{3.15}$$

- $$y_{4n+4} = \beta_3 y_{4n+2},$$

  using (3.14), we obtain

  $$y_{4n+4} = \beta_1 \beta_3 y_{4n},\ \beta_3 = \frac{a_1 a_3 v_0}{c a_3 v_0 u_0 + b a_2 u_0 + a a_1}.$$

- $$z_{4n+1} = \gamma_0 z_{4n-1},\ \gamma_0 = \frac{a_3 v_{-1}}{a + b w_{-1} v_{-1} + c w_{-1}}. \tag{3.16}$$

- $$z_{4n+2} = \gamma_1 z_{4n},\ \gamma_1 = \frac{a_3 v_0}{a + b w_0 v_0 + c w_0}. \tag{3.17}$$

- $$z_{4n+3} = \gamma_2 z_{4n+1},$$

  using (3.16), we get

  $$z_{4n+3} = \gamma_0 \gamma_2 z_{4n-1},\ \gamma_2 = \frac{a_2 a_1 w_{-1}}{a a_1 v_{-1} w_{-1} + c a_3 v_{-1} + b a_2}. \tag{3.18}$$

- $$z_{4n+4} = \gamma_3 z_{4n+2},$$

  and by (3.17), we obtain

  $$z_{4n+4} = \gamma_1 \gamma_3 z_{4n},\ \gamma_3 = \frac{a_2 a_1 w_0}{a a_1 v_0 w_0 + c a_3 v_0 + b a_2}. \tag{3.19}$$

To obtain the forms of the sequences $(x_n)_{n\geq -1}$, $(y_n)_{n\geq -1}$, and $(z_n)_{n\geq -1}$, we need to solve the equations (3.9)-(3.19).
Let

$$X_n^1 = x_{4n},\ n \in \mathbb{N}_0,$$

then we have $X_0^1 = x_0$, and the equation (3.12) becomes

$$X_{n+1}^1 = \alpha_1\alpha_3 X_n^1,$$

which is a first-order linear (homogeneous) difference equation. Its solution is given by

$$X_n^1 = (\alpha_1\alpha_3)^n X_0^1,\ n \in \mathbb{N}_0.$$

This implies that

$$x_{4n} = \left(\frac{a_1a_2a_3u_0w_0}{((a+cw_0)u_0+b)\,((aa_1+ba_2u_0)w_0+ca_3)}\right)^n x_0.$$

Finally, using the fact that

$$u_0 = \frac{x_0}{y_0},\ v_0 = \frac{y_0}{z_0},\ w_0 = \frac{z_0}{x_0}, \tag{3.20}$$

we get

$$x_{4n} = \left(\frac{a_1a_2a_3x_0y_0z_0}{(ax_0+by_0+cz_0)\,((aa_1y_0+ba_2x_0)z_0+ca_3x_0z_0)}\right)^n x_0. \tag{3.21}$$

Now, let

$$X_n^2 = x_{4n-1},\ n \in \mathbb{N}_0,$$

then we have $X_0^2 = x_{-1}$, and the equation (3.11) becomes

$$X_{n+1}^2 = \alpha_0\alpha_2 X_n^2,\ n \in \mathbb{N}_0.$$

This is a first-order linear (homogeneous) difference equation, and its solution is given by

$$X_n^2 = (\alpha_0\alpha_2)^n X_0^2,\ n \in \mathbb{N}_0.$$

This implies that

$$x_{4n-1} = \left(\frac{a_1a_2a_3u_{-1}w_{-1}}{((a+cw_{-1})u_{-1}+b)\,((aa_1+ba_2u_{-1})w_{-1}+ca_3)}\right)^n x_{-1}.$$

Finally, using the fact that

$$u_{-1} = \frac{x_{-1}}{y_{-1}},\ v_{-1} = \frac{y_{-1}}{z_{-1}},\ w_{-1} = \frac{z_{-1}}{x_{-1}}, \tag{3.22}$$

we get

$$x_{4n-1} = \left(\frac{a_1a_2a_3x_{-1}y_{-1}z_{-1}}{(ax_{-1}+by_{-1}+cz_{-1})\,((aa_1y_{-1}+ba_2x_{-1})z_{-1}+ca_3x_{-1}z_{-1})}\right)^n x_{-1}. \tag{3.23}$$

From (3.9), (3.22), and (3.23), we get

$$x_{4n+1} = \left(\frac{a_1y_{-1}z_{-1}}{ax_{-1}+by_{-1}+cz_{-1}}\right)\left(\frac{a_1a_2a_3x_{-1}y_{-1}z_{-1}}{(ax_{-1}+by_{-1}+cz_{-1})\,((aa_1y_{-1}+ba_2x_{-1})z_{-1}+ca_3x_{-1}z_{-1})}\right)^n. \tag{3.24}$$

Finally, from (3.10), (3.20), and (3.21), we get

$$x_{4n+2} = \left(\frac{a_1 y_0 z_0}{ax_0 + by_0 + cz_0}\right)\left(\frac{a_1 a_2 a_3 x_0 y_0 z_0}{(ax_0 + by_0 + cz_0)\left((aa_1 y_0 + ba_2 x_0)z_0 + ca_3 x_0 z_0\right)}\right)^n. \tag{3.25}$$

Following the same steps as in the formulas for the sequences $(x_n)_{n\geq -1}$, we obtain

$$y_{4n-1} = \left(\frac{a_1 a_2 a_3 x_{-1} y_{-1} z_{-1}}{(ax_{-1} + by_{-1} + cz_{-1})\left((aa_1 y_{-1} + ba_2 x_{-1})z_{-1} + ca_3 x_{-1} y_{-1}\right)}\right)^n y_{-1}, \tag{3.26}$$

$$y_{4n} = \left(\frac{a_1 a_2 a_3 x_0 y_0 z_0}{(ax_0 + by_0 + cz_0)\left((aa_1 y_0 + ba_2 x_0)z_0 + ca_3 x_0 y_0\right)}\right)^n y_0, \tag{3.27}$$

$$y_{4n+1} = \left(\frac{a_2 x_{-1} z_{-1}}{ax_{-1} + by_{-1} + cz_{-1}}\right)\left(\frac{a_1 a_2 a_3 x_{-1} y_{-1} z_{-1}}{(ax_{-1} + by_{-1} + cz_{-1})\left((aa_1 y_{-1} + ba_2 x_{-1})z_{-1} + ca_3 x_{-1} y_{-1}\right)}\right)^n, \tag{3.28}$$

$$y_{4n+2} = \left(\frac{a_2 x_0 z_0}{ax_0 + by_0 + cz_0}\right)\left(\frac{a_1 a_2 a_3 x_0 y_0 z_0}{(ax_0 + by_0 + cz_0)\left((aa_1 y_0 + ba_2 x_0)z_0 + ca_3 x_0 y_0\right)}\right)^n, \tag{3.29}$$

$$z_{4n-1} = \left(\frac{a_1 a_2 a_3 x_{-1} y_{-1} z_{-1}}{(ax_{-1} + by_{-1} + cz_{-1})\left((aa_1 y_{-1} + ba_2 x_{-1})z_{-1} + ca_3 x_{-1} y_{-1}\right)}\right)^n z_{-1}, \tag{3.30}$$

$$z_{4n} = \left(\frac{a_1 a_2 a_3 x_0 y_0 z_0}{(ax_0 + by_0 + cz_0)\left((aa_1 y_0 + ba_2 x_0)z_0 + ca_3 x_0 y_0\right)}\right)^n, \tag{3.31}$$

$$z_{4n+1} = \left(\frac{a_3 x_{-1} y_{-1}}{ax_{-1} + by_{-1} + cz_{-1}}\right)\left(\frac{a_1 a_2 a_3 x_{-1} y_{-1} z_{-1}}{(ax_{-1} + by_{-1} + cz_{-1})\left((aa_1 y_{-1} + ba_2 x_{-1})z_{-1} + ca_3 x_{-1} y_{-1}\right)}\right)^n, \tag{3.32}$$

$$z_{4n+2} = \left(\frac{a_3 x_0 y_0}{ax_0 + by_0 + cz_0}\right)\left(\frac{a_1 a_2 a_3 x_0 y_0 z_0}{(ax_0 + by_0 + cz_0)\left((aa_1 y_0 + ba_2 x_0)z_0 + ca_3 x_0 y_0\right)}\right)^n. \tag{3.33}$$

In summary, and after some rearrangement, the formulas for the solutions of System (1.4) are given in the following result:

Theorem 3.3. *Let $(x_n, y_n, z_n)_{n\geq -1}$ be a solution for System (1.4). Then, for $n = 0, 1, \cdots$, we have*

$$x_{4n-1} = x_{-1}^{n+1}\left(\frac{a_1 a_2 a_3 y_{-1} z_{-1}}{(ax_{-1} + by_{-1} + cz_{-1})\left((aa_1 y_{-1} + ba_2 x_{-1})z_{-1} + ca_3 x_{-1} z_{-1}\right)}\right)^n,$$

$$x_{4n} = x_0^{n+1}\left(\frac{a_1 a_2 a_3 x_0 y_0 z_0}{(ax_0 + by_0 + cz_0)\left((aa_1 y_0 + ba_2 x_0)z_0 + ca_3 x_0 z_0\right)}\right)^n,$$

$$x_{4n+1} = \left(\frac{a_2 a_3 x_{-1}}{(aa_1 y_{-1} + ba_2 x_{-1})z_{-1} + ca_3 x_{-1} z_{-1}}\right)^n \left(\frac{a_1 y_{-1} z_{-1}}{ax_{-1} + by_{-1} + cz_{-1}}\right)^{n+1},$$

$$x_{4n+2} = \left(\frac{a_2 a_3 x_0}{(aa_1 y_0 + ba_2 x_0)z_0 + ca_3 x_0 z_0}\right)^n \left(\frac{a_1 y_0 z_0}{ax_0 + by_0 + cz_0}\right)^{n+1},$$

$$y_{4n-1} = y_{-1}^{n+1}\left(\frac{a_1 a_2 a_3 x_{-1} y_{-1} z_{-1}}{(ax_{-1} + by_{-1} + cz_{-1})\left((aa_1 y_{-1} + ba_2 x_{-1})z_{-1} + ca_3 x_{-1} y_{-1}\right)}\right)^n,$$

$$y_{4n} = y_0^{n+1}\left(\frac{a_1 a_2 a_3 x_0 y_0 z_0}{(ax_0 + by_0 + cz_0)\left((aa_1 y_0 + ba_2 x_0)z_0 + ca_3 x_0 y_0\right)}\right)^n,$$

$$y_{4n+1} = \left(\frac{a_1 a_3 y_{-1}}{(aa_1 y_{-1} + ba_2 x_{-1})z_{-1} + ca_3 x_{-1} y_{-1}}\right)^n \left(\frac{a_2 x_{-1} z_{-1}}{ax_{-1} + by_{-1} + cz_{-1}}\right)^{n+1},$$

$$y_{4n+2} = \left(\frac{a_1a_3y_0}{(aa_1y_0+ba_2x_0)z_0+ca_3x_0y_0}\right)^n \left(\frac{a_2x_0z_0}{ax_0+by_0+cz_0}\right)^{n+1},$$

$$z_{4n-1} = z_{-1}^{n+1}\left(\frac{a_1a_2a_3x_{-1}y_{-1}z_{-1}}{(ax_{-1}+by_{-1}+cz_{-1})\left((aa_1y_{-1}+ba_2x_{-1})z_{-1}+ca_3x_{-1}y_{-1}\right)}\right)^n,$$

$$z_{4n} = z_0^{n+1}\left(\frac{a_1a_2a_3x_0y_0z_0}{(ax_0+by_0+cz_0)\left((aa_1y_0+ba_2x_0)z_0+ca_3x_0y_0\right)}\right)^n,$$

$$z_{4n+1} = \left(\frac{a_1a_2z_{-1}}{(aa_1y_{-1}+ba_2x_{-1})z_{-1}+ca_3x_{-1}y_{-1}}\right)^n \left(\frac{a_3x_{-1}y_{-1}}{ax_{-1}+by_{-1}+cz_{-1}}\right)^{n+1},$$

$$z_{4n+2} = \left(\frac{a_1a_2z_0}{(aa_1y_0+ba_2x_0)z_0+ca_3x_0y_0}\right)^n \left(\frac{a_3x_0y_0}{ax_0+by_0+cz_0}\right)^{n+1}.$$

Remark 3.4. The formulas for the solutions of the following particular system

$$x_{n+1} = \frac{a_1y_{n-1}z_{n-1}}{x_{n-1}+y_{n-1}+z_{n-1}}, \quad y_{n+1} = \frac{a_2x_{n-1}z_{n-1}}{x_{n-1}+y_{n-1}+z_{n-1}}, \quad z_{n+1} = \frac{a_3x_{n-1}y_{n-1}}{x_{n-1}+y_{n-1}+z_{n-1}},$$

can be obtained from Theorem 3.3 by setting $a = b = c = 1$, and the formulas obtained in this case are the same as those given by Elsayed et al. in Theorem 3 of [8].

4 The system $x_{n+1} = \frac{a_1y_nz_n}{ax_n+by_n+cz_n}$, $y_{n+1} = \frac{a_2x_nz_n}{ax_n+by_n+cz_n}$, $z_{n+1} = \frac{a_3x_ny_n}{ax_n+by_n+cz_n}$.

In this part, we explicitly solve the three-dimensional system of first-order difference equations defined by (1.6).
For System (1.6), a solution $(x_n, y_n, z_n)_{n\geq 0}$ is said to be well-defined if $ax_n + by_n + cz_n \neq 0$, for $n \in \mathbb{N}_0$. Additionally, it is easy to see that for every well-defined solution $(x_n, y_n, z_n)_{n\geq 0}$ of System (1.6), we have

$$x_n \cdot y_n \cdot z_n \neq 0, \quad n = 0, 1, \ldots.$$

To solve System (1.6), we will proceed similarly to the method used for System (1.4). From (1.6), we get the following relations:

$$\frac{x_{n+1}}{y_{n+1}} = \frac{a_1y_n}{a_2x_n}, \quad \frac{y_{n+1}}{z_{n+1}} = \frac{a_2z_n}{a_3y_n}, \quad \frac{z_{n+1}}{x_{n+1}} = \frac{a_3x_n}{a_1z_n}. \tag{4.1}$$

Let

$$u_n = \frac{x_n}{y_n}, \quad v_n = \frac{y_n}{z_n}, \quad w_n = \frac{z_n}{x_n}, \quad n \in \mathbb{N}_0. \tag{4.2}$$

From (4.1) and (4.2), we obtain the following three independent equations:

$$u_{n+1} = \frac{a_1}{a_2u_n}, \quad v_{n+1} = \frac{a_2}{a_3v_n}, \quad w_{n+1} = \frac{a_3}{a_1w_n}, \quad n \in \mathbb{N}_0.$$

It is easy to see that the sequences $(u_n)_{n\geq 0}$, $(v_n)_{n\geq 0}$, and $(w_n)_{n\geq 0}$ are periodic with period two, and their terms are given for $n \in \mathbb{N}_0$ by

$$u_{2n} = u_0, \quad u_{2n+1} = \frac{a_1}{a_2u_0}, \tag{4.3}$$

$$v_{2n} = v_0, \quad v_{2n+1} = \frac{a_2}{a_3v_0}, \tag{4.4}$$

$$w_{2n} = w_0, \quad w_{2n+1} = \frac{a_3}{a_1 w_0}. \tag{4.5}$$

Now, using the equations in (1.6) and the change of variables in (4.2), we obtain the following equations for $n = 0, 1, \dots$:

$$x_{n+1} = \frac{a_1 w_n}{a u_n + b + c u_n w_n} x_n, \tag{4.6}$$

$$y_{n+1} = \frac{a_2 u_n}{a u_n v_n + b v_n + c} y_n, \tag{4.7}$$

$$z_{n+1} = \frac{a_3 v_n}{a + b w_n v_n + c w_n} z_n. \tag{4.8}$$

Using the formulas for the sequences $(u_n)_{n\geq 0}$, $(v_n)_{n\geq 0}$, and $(w_n)_{n\geq 0}$ given in (4.3)-(4.5), it follows from (4.6)-(4.8) that for $n \in \mathbb{N}_0$, we have:

- $$x_{2n+1} = \alpha_0 x_{2n}, \quad \alpha_0 = \frac{a_1 w_0}{a u_0 + c u_0 w_0 + b}. \tag{4.9}$$
- $$x_{2n+2} = \alpha_1 \alpha_0 x_{2n}, \quad \alpha_1 = \frac{a_2 a_3 u_0}{a a_1 w_0 + b a_2 w_0 u_0 + c a_3}. \tag{4.10}$$
- $$y_{2n+1} = \beta_0 y_{2n}, \quad \beta_0 = \frac{a_2 u_0}{a u_0 v_0 + b v_0 + c}. \tag{4.11}$$
- $$y_{2n+2} = \beta_1 \beta_0 y_{2n}, \quad \beta_1 = \frac{a_1 a_3 v_0}{c a_3 v_0 u_0 + b a_2 u_0 + a a_1}. \tag{4.12}$$
- $$z_{2n+1} = \gamma_0 z_{2n}, \quad \gamma_0 = \frac{a_3 v_0}{a + b w_0 v_0 + c w_0}. \tag{4.13}$$
- $$z_{2n+2} = \gamma_1 \gamma_0 z_{2n}, \quad \gamma_1 = \frac{a_2 a_1 w_0}{a a_1 v_0 w_0 + c a_3 v_0 + b a_2}. \tag{4.14}$$

To obtain the formulas for the terms of the sequences $(x)_{n\geq 0}$, $(y)_{n\geq 0}$, and $(z)_{n\geq 0}$, we need to solve the equations (4.9)-(4.14).
Let

$$X_n = x_{2n}, \quad n \in \mathbb{N}_0,$$

then we have $X_0 = x_0$, and the equation (4.10) becomes

$$X_{n+1} = \alpha_0 \alpha_1 X_n,$$

which is a first-order linear (homogeneous) difference equation. Its solution is given by

$$X_n = (\alpha_0 \alpha_1)^n X_0, \quad n \in \mathbb{N}_0,$$

which implies that

$$x_{2n} = \left(\frac{a_1 a_2 a_3 u_0 w_0}{(a u_0 + c u_0 w_0 + b)(a a_1 w_0 + b a_2 u_0 w_0 + c a_3)} \right)^n x_0.$$

Finally, using the fact that

$$u_0 = \frac{x_0}{y_0}, \quad v_0 = \frac{y_0}{z_0}, \quad w_0 = \frac{z_0}{x_0}, \tag{4.15}$$

we get

$$x_{2n} = x_0^{n+1}\left(\frac{a_1a_2a_3y_0z_0}{(ax_0+by_0+cz_0)(aa_1y_0z_0+ba_2x_0z_0+ca_3x_0y_0)}\right)^n. \tag{4.16}$$

From (4.9), we have $x_{2n+1} = \alpha_0 x_{2n}$. Using (4.16) and (4.15), we get

$$x_{2n+1} = x_0^n\left(\frac{a_1y_0z_0}{ax_0+by_0+cz_0}\right)\left(\frac{a_1a_2a_3y_0z_0}{(ax_0+by_0+cz_0)(aa_1y_0z_0+ba_2x_0z_0+ca_3x_0y_0)}\right)^n. \tag{4.17}$$

Following the same steps as for the sequences $(x_n)_{n\geq 0}$, we obtain the following formulas for the sequences $(y_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$:

$$y_{2n} = y_0^{n+1}\left(\frac{a_1a_2a_3x_0z_0}{(ax_0+by_0+cz_0)(aa_1y_0z_0+ba_2x_0z_0+ca_3x_0y_0)}\right)^n, \tag{4.18}$$

$$y_{2n+1} = y_0^n\left(\frac{a_2x_0z_0}{ax_0+by_0+cz_0}\right)\left(\frac{a_1a_2a_3x_0z_0}{(ax_0+by_0+cz_0)(aa_1y_0z_0+ba_2x_0z_0+ca_3x_0y_0)}\right)^n, \tag{4.19}$$

$$z_{2n} = z_0^{n+1}\left[\frac{a_1a_2a_3x_0y_0}{(ax_0+by_0+cz_0)(aa_1y_0z_0+ba_2x_0z_0+ca_3x_0y_0)}\right]^n, \tag{4.20}$$

$$z_{2n+1} = z_0^n\left(\frac{a_2x_0y_0}{ax_0+by_0+cz_0}\right)\left(\frac{a_1a_2a_3x_0y_0}{(ax_0+by_0+cz_0)(aa_1y_0z_0+ba_2x_0z_0+ca_3x_0y_0)}\right)^n. \tag{4.21}$$

5 Conclusion and open problem.

In the present work, we have first explicitly solved the higher-order difference equation (1.3) in both the non-autonomous and autonomous cases. A condition for the existence of periodic solutions in the case of constant coefficients was provided. Secondly, we have also solved in closed form the systems of difference equations defined by (1.4) and (1.6). Notably, the results we obtained for System (1.4) explain and extend those of Elsayed et al. in [8]. Finally, for interested readers, we propose the following open problem:
Solve explicitly the following higher-order system of difference equations defined by

$$x_{n+1} = \frac{a_1y_{n-k}z_{n-k}}{ax_{n-k}+by_{n-k}+cz_{n-k}},\ y_{n+1} = \frac{a_2x_{n-k}z_{n-k}}{ax_{n-k}+by_{n-k}+cz_{n-k}},\ z_{n+1} = \frac{a_3x_{n-k}y_{n-k}}{ax_{n-k}+by_{n-k}+cz_{n-k}},$$

where $n \in \mathbb{N}_0$, $k \in \mathbb{N}_2$, the parameters a, b, c, a_1, a_2, a_3 are real numbers, and the initial values $x_{-k}, \ldots, x_0$, $y_{-k}, \ldots, y_0$, $z_{-k}, \ldots, z_0$ are non-zero real numbers.

Declarations

Availability of data and materials

Data sharing not applicable to this article.

Funding

Not applicable.

Conflict of interest

The authors have no conflicts of interest to declare.

Acknowledgements

The authors want to thank the two referees for their remarks and suggestions.

References

[1] R. Abo-Zeid, *On the solutions of a higher order difference equation*, Georgian Mathematical Journal, **27**(2) (2020), 165–175. DOI

[2] R. Abo-Zeid, *Behavior of solutions of a second order rational difference equation*, Mathematica Moravica, **23**(1) (2019), 11–25. DOI

[3] Y. Akrour, N. Touafek and Y. Halim, *On a system of difference equations of second order solved in closed form*, Miskolc Mathematical Notes, **20**(2) (2019), 701–717. DOI

[4] Y. Akrour, N. Touafek and Y. Halim *On a system of difference equations of third order solved in closed form*, Journal of Innovative Applied Mathematics and Computational Sciences, **1**(1) (2021), 1–15. DOI

[5] K.N. Alharbi, and E.M. Elsayed, *The expressions and behavior of solutions for nonlinear systems of rational difference equations*, Journal of Innovative Applied Mathematics and Computational Sciences. 2(1) (2022), 78–91. DOI

[6] M. B. Almatrafi and M. M. Alzubaidi, *The solution and dynamic behaviour of some difference equations of seventh order*, Journal of Applied Nonlinear Dynamics, **10**(4) (2021), 709–719. DOI

[7] S. Elaydi, *An Introduction to Difference Equations*, Undergraduate Texts in Mathematics, Vol. 1, Springer-Verlag, New York, 2005. URL

[8] E. M. Elsayed, A. Alshareef and F. Alzahrani, *Qualitative behavior and solution of a system of three-dimensional rational difference equations*, Mathematical Methods in the Applied Sciences, **45**(9) (2022), 5456–5470. DOI

[9] M. Gümüş and R. Abo-Zeid, *An explicit formula and forbidden set for a higher order difference equation*, Journal of Applied Mathematics and Computing, **63** (2020), 133–142. DOI

[10] N. Haddad, N. Touafek and J. F. T. Rabago, *Well-defined solutions of a system of difference equations*, Journal of Applied Mathematics and Computing, **56**(1-2) (2018), 439–458. DOI

[11] S. Kaouache, M. Feckan, Y. Halim and A. Khelifa, *Theoretical analysis of higher-order system of difference equations with generalized balancing numbers*, Mathematica Slovaca, **74**(3) (2024), 691–702. DOI

[12] D. Karakaya, Y. Yazlik and M. Kara, *On a solvable system of difference equations of sixth-order*, Miskolc Mathematical Notes, **24**(3) (2023), 1405–1426. DOI

[13] M. Kara and Y. Yazlik, *On eight solvable systems of difference equations in terms of generalized Padovan sequences*, Miskolc Mathematical Notes, **22**(2) (2021), 695–708. DOI

[14] M. Kara and Y. Yazlik, *Solvability of a system of non-linear difference equations of higher order*, Turkish Journal of Mathematics, **43**(3) (2019), 1533–1565. DOI

[15] M. Kara and Y. Yazlik, *On the solutions of three-dimensional system of difference equations via recursive relations of order two and Applications*, Journal of Applied Analysis and Computation, **12**(2) (2022), 736–753. DOI

[16] M. Kara and Y. Yazlik, *Solvable three-dimensional system of higher-order nonlinear difference equations*, Filomat, **36**(10) (2022), 3449–3469. DOI

[17] M. Kara and Y. Yazlik, *On a solvable system of rational difference equations of higher order*, Turkish Journal of Mathematics, **46**(2), SI-1 (2022), 587–611. DOI

[18] A. Khelifa, Y. Halim, A. Bouchair and M. Berkal, *On a system of three difference equations of higher order solved in terms of Lucas and Fibonacci numbers*, Mathematica Slovaca, **70**(3) (2020), 641–656. DOI

[19] E. Magnucka-Blanzdi and J. Popenda, *On the asymptotic behavior of a rational system of difference equations*, Journal of Difference Equations and Applications, **5** (1999), 272–286. DOI

[20] D. T. Tollu, Y. Yazlik and N. Taskara, *On fourteen solvable systems of difference equations*, Applied Mathematics and Computation, **233** (2014), 310–319. DOI

[21] N. Touafek and E. M. Elsayed, *On a second order rational systems of difference equations*, Hokkaido Mathematical Journal, **44**(1) (2015), 29–45. DOI