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Some integral properties in the theory of generalized $k-$Bessel matrix functions

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Abstract. The main purpose of this article is to define some original properties in the theory of the generalized modified $k-$Bessel matrix functions. These special functions, defined in terms of Wright matrix functions, are generalized and their properties studied in depth. Moreover, it is shown their application to the analysis of certain generalized integral formulas involving the generalized modified $k-$Bessel matrix function.

Keywords: $k-$Bessel matrix function, Gamma matrix function, generalized (Wright) hypergeometric matrix functions, Oberhettingers integral formula.

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1 Introduction

The definition of the $\Gamma_k(z)$ $k-$gamma function is given by [2]

$$\Gamma_k(z) = \int_0^\infty e^{\frac{-t^k}{k}} t^{z-1} dt, \qquad \Re(z) > 0, \quad k > 0. \tag{1.1}$$

Also, we require the following relationships Γ_k with the conventional gamma Euler function (see [15]):

$$\Gamma_k(z+k) = z\Gamma_k(z) \tag{1.2}$$

and

$$\Gamma_k(z) = k^{\frac{a}{k}-1}\Gamma\left(\frac{z}{k}\right). \tag{1.3}$$

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The $k-$Pochhammer symbol$(\gamma)_{v,k}$ is defined as (see e.g. [2]):

$$(\gamma)_{v,k} = \begin{cases} \frac{\Gamma_k(\gamma+vk)}{\Gamma_k(\gamma)}, & k \in \mathbb{R}, \gamma \in \mathbb{C}\backslash\{0\}; \\ \gamma(\gamma+k)\cdots(\gamma+(n-1)k), & n \in \mathbb{N}, \gamma \in \mathbb{C}. \end{cases} \tag{1.4}$$

The k-Bessel function of the first kind have been introduced in [15] and defined by

$$J_{k,\nu}^{\gamma,\lambda}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_{n,k} \left(\frac{z}{2}\right)^n}{(n!)^2 \Gamma_k(\lambda n + \nu + 1)}, \tag{1.5}$$

for $k \in \mathbb{R}^+; \lambda, \gamma, \vartheta \in \mathbb{C}; \Re(\lambda) > 0$ and $\Re(\nu) > 0$.

The first kind of the generalized k-Bessel function has also been introduced in [9] as

$$J_{k,\nu}^{b,c,\gamma,\lambda}(z) = \sum_{n=0}^{\infty} \frac{(c)^n (\gamma)_{n,k} \left(\frac{z}{2}\right)^{\nu+2n}}{(r!)^2 \Gamma_k(\lambda n + \nu + \frac{b+1}{2})}, \tag{1.6}$$

where $k \in \mathbb{R}^+; \lambda, \gamma, \nu, b, c \in \mathbb{C}; \Re(\nu) > 0$ and $\Re(\lambda) > 0$.

The modified k-Bessel function form of the above relation [12] is defined as

$$\mathbb{I}_{k,\nu}^{b,c,\gamma,\lambda}(z) = \sum_{n=0}^{\infty} \frac{(c)^n (\gamma)_{n,k} \left(\frac{z}{2}\right)^{\nu+2n}}{(n!)^2 \Gamma_k\left(\nu + \lambda n + \frac{b+1}{2} + k\right)}, \tag{1.7}$$

with $k \in \mathbb{R}^+; \lambda, \gamma, \nu, b, c \in \mathbb{C}; \Re(\lambda) > 0$ and $\Re(\nu) > 0$.

In this paper, we focus on the complex space $\mathbb{C}^{r\times r}$ of all square complex matrices of common order r, for a stable matrix which is positive A in $\mathbb{C}^{r\times r}$ if $\Re(\lambda)$ for all $\lambda \in \sigma(A)$, where $\sigma(A)$ is the set of eigenvalues of A.

The two-norm of A will be $\|A\|_2$ and it is represented by

$$||A||_2 = \sup_{x\neq 0} \frac{||Ax||_2}{||x||_2},$$

where for any vector x in $C^{r\times r}$, $||x||_2 = (x^T x)^{\frac{1}{2}}$ is the Euclidean norm of x.

We denote with $\alpha(A)$ and $\beta(A)$ the real numbers $\alpha(A) = max\{\Re(z) : z \in \alpha(A)\}$, $\beta(A) = min\{\Re(z) : z \in \alpha(A)\}$. Let A be a matrix in $\mathbb{C}^{N\times N}$, such that

$$\mu \text{ is not a negative integer for every } \mu\epsilon\sigma(A). \tag{1.8}$$

Let $f(z)$ and $g(z)$ be holomorphic functions in an open set Ω of the complex plane, and $\sigma(A) \subset \Omega$, we denote by $f(z)$ and $g(z)$, respectively, the image by the Riesz-Dunford functional calculus of the functions $f(z)$, $g(z)$, acting on the matrix A, and

$$f(A)g(A) = g(A)f(A).$$

Hence, if B in $\mathbb{C}^{N\times N}$ is another matrix with $\sigma(B) \subset \Omega$, such that $AB = BA$, then [3,5]

$$f(A)g(B) = g(B)f(A).$$

Let $A_i (1 \le i \le p)$ and $B_j (1 \le j \le q)$ be matrices in $\mathbb{C}^{N\times N}$, the following representation of the generalized hypergeometric matrix functions [6] holds:

$${}_pF_q\left[\begin{array}{cc} (A_p); & \\ & z \\ (B_q); & \end{array}\right] = \sum_{r=0}^{\infty} \frac{\prod_{i=1}^{p}(A_i)z^r}{r!}\left[\prod_{j=1}^{q}(B_j)\right]^{-1} \tag{1.9}$$

for $p \leqslant q; p = q+1$ and $|z| < 1$, where $(B)_n$ has a well-known Pochhammer symbol matrix that is defined for $B \in C^{N\times N}$ (see e.g. [6])

$$(B)_n = \begin{cases} I, & n=0; \\ B(B+I)\cdots(B+(n-1)I), & n\in\mathbb{N}:=\{1,2,3,...\}. \end{cases} \tag{1.10}$$

$$(B)_n = \Gamma(B+nI)\Gamma^{-1}(B) \qquad \Re(B)\in\mathbb{C}\backslash\mathcal{Z}_0^-; \tag{1.11}$$

where $\mathcal{Z}_0^-$ is the set of non positive integer.

Also $\Gamma_k(P)$ denotes the $k-$gamma matrix function for a matrix P in $\mathbb{C}^{N\times N}$ defined as [10]

$$\Gamma_k(P) = \int_0^\infty e^{-t^k/k}\, t^{P-I}dt, \qquad k>0, \tag{1.12}$$

The k-Pochhammer symbol $(B)_{n,k}$ is [10]

$$(B)_{n,k} = \begin{cases} \Gamma_k(B+nkI)\Gamma_k^{-1}(B), & k\in\mathbb{R},\backslash\{0\}; \\ B(B+kI)\cdots(B+(n-1)kI) & ,\quad n\in\mathbb{N}. \end{cases} \tag{1.13}$$

so that from (1.13), we have

$$(A)_{n,k} = k^n\left(\frac{A}{k}\right)_n. \tag{1.14}$$

Definition 1.1. [16–20] Let A and B be matrices in $\Bbbk C^{N\times N}$ and satisfying the relation (1.8), then the first kind of the $k-$Bessel matrix function ${}_kJ_{A,B}(z)$ is defined as

$${}_kJ_{A,B}(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}(B)_{n,k}\left(\frac{z}{2}\right)^{A+2nI}\Gamma_k^{-1}(A+(n+1)I)\,, \tag{1.15}$$

where $B+nkI$ for any integers $nk\geq 0$ with $k\in N$ and $n\in N_0$ is an invertible matrix.

The generalized Wright hypergeometric matrix functions ${}_p\psi_q(z)$ are given by the series

$$\begin{aligned} {}_p\psi_q(z) &= {}_p\psi_q\left[\begin{array}{cc} (A_i,\alpha_i I)_{1,p}; & \\ & z \\ (B_j,\beta_j I)_{1,q}; & \end{array}\right] \\ &= \prod_{j=1}^{p}\Gamma_k(\beta_j I)\left[\prod_{i=1}^{q}\Gamma_k(\alpha_i I)\right]^{-1}\sum_{n=0}^{\infty}\frac{1}{n!}\left[\prod_{i=1}^{p}\Gamma_k(A_i+\alpha_i nI)z^n\right]\prod_{j=1}^{q}\Gamma_k^{-1}(B_j+\beta_j nI), \end{aligned} \tag{1.16}$$

where A_i and B_j are matrices in $\mathbb{C}^{N\times N}$ and $\alpha_i,\beta_j\in\mathbb{R},(i=1,2,\ldots,p;j=1,2,\ldots,q)$, with

$$\sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i > -1\,. \tag{1.17}$$

If we put $\alpha_1 = \ldots = \alpha_p = \beta_1 = \ldots = \beta_q = 1$ in (1.16), then (1.9) represents a special case of the generalized Wright matrix function

$${}_p\psi_q(z) = {}_p\psi_q\left[\begin{array}{cc} (A_1,I),\ldots(A_p,I); & \\ & z \\ (B_1,I),\ldots,(B_q,I); & \end{array}\right] = \prod_{j=1}^{p}\Gamma(B_j)\left[\prod_{i=1}^{q}\Gamma(A_i)\right]^{-1}{}_pF_q\left[\begin{array}{cc} A_1,\ldots,A_p; & \\ & z \\ B_1,\ldots,B_q; & \end{array}\right]. \tag{1.18}$$

When P and Q are invertible matrices in $C^{N\times N}$, such that $Q-P$ is an invertible matrix too and $\Re(Q)>\Re(P)>0$, the following relations hold (for more details see e.e. [8,14]).

$$\int_0^\infty x^{P-I}\left(x+a+\sqrt{x^2+2ax}\right)^{-Q}dx=(2Q)a^{-Q}\left(\frac{a}{2}\right)^{P}\left[\Gamma(2P)\Gamma(Q-P)\right]\Gamma^{-1}(I+P+Q). \tag{1.19}$$

Also, it is:

$$\int_0^\infty x^{P-I}(1-x)^{Q-I}\left(ax+b(1-x)\right)^{-(P+Q)}dx=a^{-P}b^{-Q}\Gamma(P)\Gamma(Q)\Gamma^{-1}(P+Q). \tag{1.20}$$

2 Main results

The purpose of this section is to provide some properties of the first type of generalized modified k-Bessel matrix function.

Definition 2.1. The first kind of the k-Bessel matrix function is defined by

$$J_{A,B,C}(z;k)=\sum_{n=0}^{\infty}\frac{(-1)^n(B)_{n,k}}{(n!)^2}\Gamma_k^{-1}(A+nC+I)\left(\frac{z}{2}\right)^n, \tag{2.1}$$

where A,B and C are matrices in $C^{N\times N}$, satisfying the relation (1.8) , $k\in\Re^+$ and $B+nkI$ is an invertible matrix for all integers $nk\geq 0$.

In an equivalent manner, the generalized k-Bessel matrix function of the first kind can be defined as follows:

$$J_{A,B,C,D}(z;k,\nu)=\sum_{n=0}^{\infty}\frac{D^n(B)_{n,k}}{(n!)^2}\Gamma_k^{-1}(A+nC+\frac{\nu+1}{2}I)\left(\frac{z}{2}\right)^{A+2nI}, \tag{2.2}$$

for $k\in\mathbb{R}^+$, $\nu\in\mathbb{C}$ and A,B,C and D are matrices in $C^{N\times N}$.

A new modified $k-$Bessel matrix function form (2.2) can be also defined as

$$\mathbb{I}_{A,B,C,D}(z;k,\nu)=\sum_{n=0}^{\infty}\frac{D^n(B)_{n,k}(\frac{z}{2})^{A+2nI}}{(n!)^2}\Gamma_k^{-1}(A+nC+(\frac{\nu+1}{2}+k)I), \tag{2.3}$$

where A,B,C and D are matrices in $\mathbb{C}^{N\times N}$, $k\in\mathbb{R}^+$, $\nu\in\mathbb{C}$ and $J_{A,B,I,I}(z;k,\nu)$ and $\mathbb{I}_{A,B,I,-I}(z;k,\nu)$ are the k-Bessel matrix function and modified k-Bessel matrix function, respectively [16–20].

Theorem 2.2. *Let A,B,C and D be matrices in $\mathbb{C}^{N\times N}$, $k\in\mathbb{R}^+$ and $\nu\in\mathbb{C}$ and $A-kI$ for all integers $k>0$, be an invertible matrix. If $C=I$ then this formula holds:*

$$\frac{d}{dz}\left(z^{\frac{A}{2}+(\frac{\nu+1}{2})I}2^{\frac{\nu-1}{2}I}\mathbb{I}_{A,B,I,D}(\sqrt{z};k,\nu)\right)=z^{\frac{A+(k+\nu)I-I}{2}}2^{-kI+(\frac{\nu-1}{2})I}\mathbb{I}_{A-kI,B,I,D}(\sqrt{z};k,\nu) \tag{2.4}$$

Proof. From the relation given in (2.3), we get

$$\begin{aligned}&z^{\frac{A}{2}+(\frac{\nu+1}{2})I}2^{\frac{\nu-1}{2}I}\mathbb{I}_{A,B,I,D}(\sqrt{z};k,\nu)\\&=\sum_{n=0}^{\infty}\frac{D^n(B)_{n,k}(\frac{\sqrt{z}}{2})^{A+2nI}z^{\frac{A}{2}+(\frac{\nu+1}{2})I}}{2^{-\frac{\nu-1}{2}I}(n!)^2}\Gamma_k^{-1}(A+(n+\frac{\nu+1}{2}+k)I)\\&=\sum_{n=0}^{\infty}\frac{D^n(B)_{n,k}(z)^{A+(n+\frac{\nu+1}{2})I}}{2^{A+2nI-\frac{\nu+1}{2}I}(n!)^2}\Gamma_k^{-1}(A+(n+\frac{\nu+1}{2}+k)I).\end{aligned} \tag{2.5}$$

Now,

$$\begin{aligned}&\frac{d}{dz}\left(z^{\frac{A}{2}+(\frac{\nu+1}{2})I}\mathbb{I}_{A,B,I,D}(\sqrt{z};k,\nu)\right)\\&=\sum_{n=0}^{\infty}\frac{D^n(B)_{n,k}[A+(n+\frac{\nu+1}{2})I]Z^{A+(n+\frac{\nu+1}{2})I-I}}{2^{A+2^{2nI}-\frac{\nu-1}{2}I}(n!)^2}\Gamma_k^{-1}(A+(n+\frac{\nu+1}{2}+k)I).\end{aligned}\tag{2.6}$$

From (1.2), we get

$$\begin{aligned}&\frac{d}{dz}\left(z^{\frac{A}{2}+(\frac{\nu+1}{2})I}2^{\frac{\nu-1}{2}I}\mathbb{I}_{A,B,I,D}(\sqrt{z};k,\nu)\right)\\&=\frac{1}{2^{A-\frac{\nu-1}{2}I}}\sum_{n=0}^{\infty}\frac{D^n(B)_{n,k}[A+(n+\frac{\nu+1}{2})I]z^{A+(n+\frac{\nu+1}{2})I-I}}{[A+(n+\frac{\nu+1}{2})I]2^{2nI}(n!)^2}\Gamma_k^{-1}(A+(n+\frac{\nu+1}{2})I)\\&=\frac{z^{\frac{k}{2}I+\frac{A}{2}+\frac{\nu+1}{2}I-I}}{2^{kI-\frac{\nu+1}{2}I}}\sum_{n=0}^{\infty}\frac{D^n(B)_{n,k}(\frac{\sqrt{z}}{2})^{A+(2n-k)I}}{(n!)^2}\Gamma_k^{-1}(A+(n+\frac{\nu+1}{2})I)\\&=z^{\frac{A+(k+\nu-1)I}{2}}2^{(\frac{\nu+1}{2}-k)I}\mathbb{I}_{A,B,I,D}(\sqrt{z};k,\nu).\end{aligned}$$

This completes the proof of (2.4).

If we assume that the conditions of Theorem 2.2 are met, we get the next corollary:

Corollary 2.3. *Set $C=D=I$ and $\nu=1$; then the following differential formula holds:*

$$\frac{d}{dz}\left(z^{\frac{A}{2}+I}\mathbb{I}_{A,B,I,I}(\sqrt{z},k,1)\right)=z^{\frac{kI+A}{2}}2^{(-k)I}\mathbb{I}_{A-kI,B,I,I}(\sqrt{z},k,1).\tag{2.7}$$

Theorem 2.4. *Let $\mathbb{I}_{-A,B,I,D}$ $(\sqrt{z},k,\nu)$ for A,B and D be matrices in $\mathbb{C}^{N\times N}$, and the generalized modified $k-$Bessel matrix function of order $-A$, $-A-nkI$ for any integers $nk\geq 0$, be an invertible matrix, with z being a real non-negative number where $k\in\Re^+$, $\nu\in\mathbb{C}$. Then the next differential formula holds:*

$$\frac{d}{dz}\left(z^{\frac{-A+(\nu+1)I}{2}}\mathbb{I}_{-A,B,I,D}(\sqrt{z};k,\nu)\right)=z^{\frac{-A+((\nu-1)+k)I}{2}}2^{kI}\mathbb{I}_{-A-kI,B,I,D}(\sqrt{z};k,\nu).\tag{2.8}$$

Proof.: From the relation (2.3), we get

$$\begin{aligned}&z^{(\frac{-A+(\nu+1)I}{2})}\mathbb{I}_{-A,B,I,D}(\sqrt{z};k,\nu)\\&=z^{(\frac{-A+(\nu+1)I}{2})}\sum_{n=0}^{\infty}\frac{D^n(B)_{n,k}(\frac{\sqrt{z}}{2})^{2nI-A}}{\Gamma_k(-A+(n+\frac{\nu+1}{2}+k)I)(n!)^2}\\&=z^{(\frac{-A+(\nu+1)I}{2})}\sum_{n=0}^{\infty}\frac{D^n(B)_{n,k}(\frac{z}{2})^{nI-\frac{A}{2}}}{2^{-A+2nI}\Gamma_k(-A+(n+\frac{\nu+1}{2}+k)I)(n!)^2}\\&=\sum_{n=0}^{\infty}\frac{D^n(B)_{n,k}(z)^{-A+(n+\frac{\nu+1}{2})I}}{2^{-A+2nI}\Gamma_k(-A+(n+\frac{\nu+1}{2}+k)I)(n!)^2}.\end{aligned}\tag{2.9}$$

Now,

$$\begin{aligned}&\frac{d}{dz}\left(z^{\frac{-A+(\nu+1)I}{2}}\mathbb{I}_{-A-kI,B,I,D}(\sqrt{z};k,\nu)\right)\\&=\sum_{n=0}^{\infty}\frac{D^n(B)_{n,k}[-A+(n+\frac{b+1}{2})I]z^{-A+(n+\frac{\nu-1}{2})I}}{2^{-A+2nI}\Gamma_k(-A+(n+\frac{\nu+1}{2}+k)I)(n!)^2}.\end{aligned}\tag{2.10}$$

From (1.2), we get

$$\begin{aligned}
&\frac{d}{dz}\left(z^{\frac{-A+(\nu+1)I}{2}}\mathbb{I}_{A,B,I,D}(\sqrt{z};k,\nu)\right)\\
&=z^{\frac{(\nu+1)I-2I}{2}}\sum_{n=0}^{\infty}\frac{D^n(B)_{n,k}z^{nI-A}}{\Gamma_k(-A+(n+\frac{\nu+1}{2})I)2^{2nI-A}(n!)^2}\\
&=z^{\frac{-A+(k+(\nu-1))I}{2}}2^{kI}\sum_{n=0}^{\infty}\frac{D^n(B)_{n,k}(\frac{\sqrt{z}}{2})^{-A+(2n-k)I}}{\Gamma_k(-A+(n+\frac{\nu+1}{2})I)(n!)^2}\\
&=z^{\frac{-A+(k+(\nu-1))I}{2}}2^{kI}\mathbb{I}_{-A-kI,B,I,D}(\sqrt{z};k,\nu).
\end{aligned}\tag{2.11}$$

Thus, the result holds.

If we assume that the conditions of Theorem 2.4 are met, we get the next corollary:

Corollary 2.5. *Set $C=D=I$ and $\nu=1$; the the following differential formula holds::*

$$\frac{d}{dz}\left(z^{\frac{-A+2I}{2}}\mathbb{I}_{-A,B,I,I}(\sqrt{z};k,1)\right)=z^{\frac{-A+kI}{2}}2^{kI}\mathbb{I}_{-A-kI,B,I,I}(\sqrt{z};k,1).\tag{2.12}$$

Theorem 2.6. *Let A,B,C and D be matrices in $C^{N\times N}$, $k\in\Re^+$, $\nu\in\mathbb{C}$, $\Re(A)>0$ and $A-kI$ be an invertible matrix for all integers $k>0$. Then:*

$$z^{\frac{A+(k+(\nu+1))I}{2}}\mathbb{I}_{A-kI,B,I,D}(2\sqrt{z};k,\nu)=\frac{z^{\frac{2A+(\nu-1)I}{2}}}{\Gamma(\frac{B}{k})k^{\frac{2A+(\nu-1)I}{2k}-I}}$$
$$\times{}_1\Psi_2\left[\begin{matrix}(\frac{B}{k},I);\\ \left(\frac{A+((\nu+1)/2)}{k},\frac{1}{k}I\right),(1,1);\end{matrix}\quad D\left(zk^{\frac{(k-1)}{k}}\right)^{I}\right].\tag{2.13}$$

Proof.: Using the relation (2.3), and after simplification, we get

$$\begin{aligned}
&z^{\frac{A+(k+(\nu+1))I}{2}}\mathbb{I}_{A-kI,B,I,D}(2\sqrt{z};k,\nu)\\
&\qquad=\sum_{n=0}^{\infty}\frac{D^n(B)_{n,k}(z)^{A+(n+\frac{\nu+1}{2})I}}{\Gamma_k(A+(n+\frac{\nu+1}{2})I)(n!)^2}\\
&\qquad=\sum_{n=0}^{\infty}\frac{D^n\Gamma_k(B+nkI)(z)^{A+(n+\frac{\nu+1}{2})I}}{\Gamma_k(B)\Gamma_k(A+(n+\frac{\nu+1}{2})I)(n!)^2}.
\end{aligned}\tag{2.14}$$

Now,

$$\begin{aligned}
&\frac{d}{dz}\left(z^{\frac{A+(k+(\nu+1))I}{2}}\mathbb{I}_{A-kI,B,I,D}(2\sqrt{z};k,\nu)\right)\\
&\qquad=\sum_{n=0}^{\infty}\frac{D^n\Gamma_k(B+nkI)[A+(n+\frac{\nu+1}{2})I](z)^{A+(n+\frac{\nu-1}{2})I}}{\Gamma_k(B)\Gamma_k(A+(n+\frac{\nu+1}{2})I)(n!)^2}.
\end{aligned}$$

From (1.2) and (1.3), and after simplification, we get

$$z^{\frac{A+(k+(\nu+1))I}{2}}\mathbb{I}_{A-kI,B,I,D}(2\sqrt{z};k,\nu)$$
$$= \frac{z^{A+(\frac{\nu-1}{2})I}}{\Gamma(\frac{B}{k})k^{A+(\frac{\nu-1}{2})I-I}} \sum_{n=0}^{\infty} \frac{D^n\Gamma_k(B+nkI)(kz)^{nI}}{k^{\frac{n}{k}I}\Gamma(\frac{A}{k}+\frac{n}{k}I+\frac{\nu+1}{2k}I)n!\Gamma(n+1)}, \tag{2.15}$$

hence, in light of (1.16), we get (2.8).

In the next theorems, some generalized integral matrix formulas containing generalized modified $k-$Bessel matrix function in terms of generalized Wright matrix function are presented.

Theorem 2.7. *Let A, B, C and D be matrices in $\mathbb{C}^{N\times N}$, $k \in \mathbb{R}^+; \nu \in \mathbb{C}$; $x > 0$ and $\Re(Q+A+2I) > \Re(P) > 0$, then:*

$$\int_0^{\infty} x^{P-I}(x+a+\sqrt{x^2+2ax})^{-Q}J_{A,B,C,D}\left(\frac{\omega}{x+a+\sqrt{x^2+2ax}};k,\nu\right)dx$$
$$= \frac{2^{I-(A+P)}a^{P-(Q+A)}\omega^A\Gamma(2P)}{\Gamma(\frac{B}{k})k^{\frac{A}{k}-(1-\frac{\nu+1}{2k})I}}$$
$$\times {}_3\Psi_4\left[\begin{array}{c} (\frac{B}{k},I),(Q+A+I,2I),(Q+A-P,2I); \\ (\frac{A}{k}+\frac{\nu+1}{2k}I,\frac{C}{k}),(Q+A,2I),(I+Q+A+P,2I),(1,1); \end{array} \frac{k^{\frac{kI-C}{k}}\omega^{2I}D}{(2a)^{2I}}\right]. \tag{2.16}$$

Proof. By using the relation (2.2) to the L.H.S. of (2.16) and by interchange of the integral order of integration and summation, which is supported by uniform convergence of the concerned series under the given conditions, we obtain

$$\mathbb{I} = \int_0^{\infty} x^{P-I}(x+a+\sqrt{x^2+2ax})^{-Q}J_{A,B,C,D}\left(\frac{\omega}{x+a+\sqrt{x^2+2ax}};k,\nu\right)dx$$
$$= \int_0^{\infty} x^{P-I}(x+a+\sqrt{x^2+2ax})^{-Q}$$
$$\times \sum_{n=0}^{\infty} \frac{D^n(B)_{n,k}\Gamma_k^{-1}(A+nC+\frac{\nu+1}{2k}I)(\frac{\omega}{2})^{A+2nI}}{(n!)^2(x+a+\sqrt{x^2+2ax})^{A+2nI}}\,dx. \tag{2.17}$$

Using (1.3) and (1.14), we have

$$\mathbb{I} = \sum_{n=0}^{\infty} \frac{(kD)^n\Gamma(\frac{B}{k}+nI)k^{I-\frac{C}{k}n-\frac{A}{k}-\frac{\nu+1}{2k}I}(\frac{\omega}{2})^{A+2nI}}{(n!)^2\Gamma(\frac{B}{k})\Gamma(\frac{C}{k}nI+\frac{A}{k}+\frac{\nu+1}{2k}I)}$$
$$\times \int_0^{\infty} x^{P-I}(x+a+\sqrt{x^2+2ax})^{-(A+Q+2nI)}dx. \tag{2.18}$$

From the conditions of Theorem 2.7, by applying (1.19) to the integrand of (2.16), we get:

$$\begin{aligned}
\mathbb{I} = &\sum_{n=0}^{\infty} \frac{(kD)^n \Gamma\big(\frac{B}{k}+nI\big) k^{I-\frac{C}{k}nI-\frac{A}{k}-\frac{\nu+1}{2k}I}\big(\frac{\omega}{2}\big)^{A+2nI}}{(n!)^2\Gamma\big(\frac{B}{k}\big)\Gamma\big(\frac{C}{k}nI+\frac{A}{k}+\frac{\nu+1}{2k}I-I\big)} \\
&\times 2(Q+A+2nI)a^{-(A+Q+2nI)}\big(\frac{a}{2}\big)^P \frac{\Gamma(2P)\Gamma(A+Q+2nI-P)}{\Gamma(A+P+Q+(2n+1)I)} \\
= &\frac{2^{I-(A+Q)}k^{-\frac{A}{k}+\frac{\nu+1}{2k}+1}a^{P-(A+Q)}}{\Gamma\big(\frac{B}{k}\big)} \sum_{n=0}^{\infty} \frac{\Gamma\big(\frac{B}{k}+nI\big)\Gamma\big(A+Q+2nI\big)}{n!\Gamma\big(\frac{C}{k}n+\frac{A}{k}+\frac{\nu+1}{2k}I\big)\Gamma\big(A+Q+2nI\big)} \\
&\times \frac{\Gamma\big(A+Q-P+2nI\big)}{\Gamma\big(A+P+Q+(2n+1)I\big)\Gamma(n+1)}\left(\frac{Dk^{\frac{kI-C}{k}}\omega^{2I}}{(2a)^2}\right)^n.
\end{aligned} \tag{2.19}$$

By using the definition (1.16), we get (2.26). This completes the proof.

Now, if we assume that the conditions of Theorem 2.7 are met, we get the next corollary:

Corollary 2.8. *Let $kI = B = C = I$ and $D = -D$ in (2.16). The following integral formula is then holds:*

$$\begin{aligned}
&\int_0^{\infty} x^{P-I}\big(x+a+\sqrt{x^2+2ax}\big)^{-Q} J_{A,I,I,-D}\Big(\frac{\omega}{x+a+\sqrt{x^2+2ax}};I,\nu\Big)dx \\
&= 2^{I-(A+P)}a^{(P-(A+Q))}\omega^A\Gamma(2P) \\
&\times {}_3\Psi_4\left[\begin{array}{c} (I,I)(A+Q+I,2I),(A+Q-P,2I); \\ (A+\frac{\nu+1}{2}I,I),(Q+A,2I),(A+Q+P+I,2I),(1,1); \end{array} \frac{-D\omega^{2I}}{(2a)^2}\right].
\end{aligned} \tag{2.20}$$

Theorem 2.9. *If A, B, C and D are matrices in $\mathbb{C}^{N\times N}$, $k \in \mathbb{R}^+$; $\nu \in \mathbb{C}$; $x > 0$ and $\Re(P) > 0$ and $0 < \Re(P+A+2I) < \Re(Q+A+2I)$, Then the following formula holds:*

$$\begin{aligned}
&\int_0^{\infty} x^{P-I}\big(x+a+\sqrt{x^2+2ax}\big)^{-Q} J_{A,B,C,D}\left(\frac{xw}{x+a+\sqrt{x^2+2ax}};k,\nu\right)dx \\
&= \frac{2^{I-(2A+Q)}w^A k^{1-\frac{A}{k}-\frac{\nu+1}{2k}I}a^{P-Q}\Gamma(Q-P)}{\Gamma\big(\frac{B}{k}\big)} \\
&\times {}_3\Psi_4\left[\begin{array}{c} (\frac{B}{k},I),(A+Q+I,2I),(2(P+A),4I); \\ (1,1),(\frac{A}{k}I+\frac{\nu+1}{2k}I,\frac{C}{k}),(Q+A,2I),(I+2A+Q+P,4I); \end{array} \frac{Dk^{\frac{kI-C}{k}}w^{2I}}{2^{4I}}\right].
\end{aligned} \tag{2.21}$$

Proof.: Beginning with the L.H.S. of (2.21) by using the relation (2.2) and by interchange of the integral order of integration and summation, which is supported by uniform convergence of the concerned series under the given conditions, we obtain

$$\begin{aligned}\mathbb{I} &= \int_0^\infty x^{P-I}\Big(x+a+\sqrt{x^2+2ax}\Big)^{-Q} J_{A,B,C,D}\big(\frac{xw}{x+a+\sqrt{x^2+2ax}};k,\nu\big)dx \\ &= \int_0^\infty x^{P-I}\Big(x+a+\sqrt{x^2+2ax}\Big)^{-Q} \\ &\quad \times \sum_{n=0}^\infty \frac{D^n(B)_{n,k}2^{-(A+2nI)}(xw)^{A+2nI}}{(n!)^2\Gamma_k\big(A+Cn+\frac{\nu+1}{2}I\big)\big(x+a+\sqrt{x^2+2ax}\big)^{A+2nI}}\,dx.\end{aligned} \tag{2.22}$$

Using (1.3) and (1.14), we get

$$\begin{aligned}\mathbb{I} &= \sum_{n=0}^\infty \frac{(kD)^n\Gamma\big(\frac{B}{k}+nI\big)k^{I-\frac{C}{k}n-\frac{A}{k}-\frac{\nu+1}{2k}I}\big(\frac{w}{2}\big)^{A+2nI}}{(n!)^2\Gamma(\frac{B}{k})\Gamma\Big(\frac{C}{k}n+\frac{A}{k}+\frac{\nu+1}{2k}I\Big)} \\ &\quad \times \int_0^\infty x^{(P+A+2nI)-I}\Big(x+a+\sqrt{x^2+2ax}\Big)^{-(Q+A+2nI)}dx.\end{aligned} \tag{2.23}$$

Applying (1.19) to the integrand of (2.21) yields

$$\begin{aligned}\mathbb{I} = \sum_{n=0}^\infty &\frac{D^nk^n\Gamma\big(\frac{B}{k}+nI\big)\big(\frac{w}{2}\big)^{A+2nI}}{(n!)^2\Gamma(\frac{B}{k})k^{\frac{B}{k}n+\frac{A}{k}+\frac{\nu+1}{2k}I-I}\Gamma\big(\frac{C}{k}n+\frac{A}{k}+\frac{\nu+1}{2k}I\big)} \\ &\times 2(A+Q+2nI)a^{-(A+Q+2nI)}\big(\frac{a}{2}\big)^{A+P+2nI}\frac{\Gamma(Q-P)\Gamma(2A+2(P+2nI))}{\Gamma(2A+Q+P+(4n+1)I)}.\end{aligned}$$

From (1.2) it follows that

$$\begin{aligned}\mathbb{I} &= \frac{2^{I-(2A+P)}w^Aa^{(P-Q)}}{k^{\frac{A}{k}+\frac{\nu+1}{2k}I-I}\Gamma\big(\frac{B}{k}\big)}\sum_{n=0}^\infty \frac{\Gamma\big(\frac{B}{k}+nI\big)\Gamma\big(A+Q+(2n+1)I\big)}{n!\Gamma\big(\frac{C}{k}n+\frac{A}{k}+\frac{\nu+1}{2k}I\big)\Gamma\big(A+Q+2nI\big)} \\ &\quad \times \frac{\Gamma\big(2A+2(P+2nI)\big)}{\Gamma\big(2A+Q+P+(4n+1)I\big)\big)\Gamma(n+1)}\left(\frac{Dk^{\frac{kI-C}{k}}w^{2I}}{2^{4I}}\right)^n.\end{aligned} \tag{2.24}$$

hence, the required result follows from the relation (1.16).
In the conditions of Theorem 2.9, we can now present the following corollary:

Corollary 2.10. *Let $B=C=I$, $D=-D$ and $k=1$ in Theorem 2.9, then the following integral formula holds:*

$$\begin{aligned}&\int_0^\infty x^{P-I}\big(x+a+\sqrt{x^2+2ax}\big)^{-Q}J_{A,I,I,-D}\left(\frac{xw}{x+a+\sqrt{x^2+2ax}};1,\nu\right)dx \\ &= 2^{I-(2A+P)}w^Aa^{P-Q}\Gamma(Q-P) \\ &\quad \times {}_3\Psi_4\left[\begin{matrix}(I,I),(A+I+Q,2I),(2(A+P),4I); \\ (A+\frac{\nu+1}{2}I,I),(A+Q,2I),(2A+Q+P+I,4I)(1,1);\end{matrix}\ \frac{-Dw^{2I}}{2^{4I}}\right].\end{aligned} \tag{2.25}$$

In the next theorem, we also present an integral matrix formula involving generalized modified $k-$Bessel function in terms of generalized Wright matrix function.

Theorem 2.11. *Let A, B and C be matrices in $\mathbb{C}^{N\times N}$, $k \in \mathbb{R}^+$; $\nu \in \mathbb{C}; x > 0$, $\Re(P) > 0$, $\Re(Q) > 0$, $\Re(A) > 0$ and $\Re(B) > 0$. Then the following formula holds:*

$$\int_0^\infty x^{P-I}(1-x)^{Q-I}\left[ax+b(1-x)\right]^{-(P+Q)} J_{A,B,C}\left(\frac{4x(1-x)}{\left[ax+b(1-x)\right]^2};k\right)dx$$

$$= \frac{1}{a^P b^Q \Gamma(\frac{B}{k}) k^{\frac{A}{k}+\frac{I}{k}-I}} \times_3\Psi_3\left[\begin{array}{cc} (\frac{B}{k}, I), (P, I), (Q, I); & \\ & (\frac{-2}{ab})^I k^{\frac{kI-C}{k}} \\ (1,1)+(\frac{A}{k}+\frac{I}{k}, \frac{C}{k}), (P+Q, 2I); & \end{array}\right]. \tag{2.26}$$

Proof. By applying (2.1) to the L.H.S. of (2.26) and by interchange of the integral order of integration and summation, which is supported by uniform convergence of the concerned series under the given conditions, we obtain

$$\mathbb{I} = \int_0^\infty x^{P-I}(1-x)^{Q-I}\left[ax+b(1-x)\right]^{-(P+Q)} J_{A,B,C,D}\left(\frac{4x(1-x)}{\left[ax+b(1-x)\right]^2};k,\nu\right)dx$$

$$= \int_0^\infty x^{P-I}(1-x)^{Q-I}\left[ax+b(1-x)\right]^{-(P+Q)}dx \sum_{n=0}^{\infty} \frac{(-1)^n (B)_{n,k}}{(n!)^2 \Gamma_k(A+nC+I)}\left(\frac{2x(1-x)}{\left[ax+b(1-x)\right]^2}\right)^{nI} dx.$$

From relations (1.13) and (1.3) , we get

$$\mathbb{I} = \sum_{n=0}^{\infty} \frac{(-2)^{nI} k^{\frac{B}{k}+nI-I}\Gamma(\frac{B}{k}+nI)}{n!\Gamma(n+1)k^{\frac{A}{k}+\frac{nC}{k}+\frac{I}{k}-I}\Gamma(\frac{A}{k}+\frac{nC}{k}+\frac{I}{k})k^{\frac{B}{k}-I}\Gamma(\frac{B}{k})}$$
$$\times \int_0^\infty x^{P+nI-I}(1-x)^{Q+nI-I}\left[ax+b(1-x)\right]^{-(P+Q+2nI)}dx.$$

Now, in view of (1.20), we get

$$\mathbb{I} = \frac{1}{a^P b^Q \Gamma(\frac{B}{k})} \sum_{n=0}^{\infty} \frac{(-2k)^{nI}\Gamma(\frac{B}{k}+nI)\Gamma(P+nI)\Gamma(Q+nI)}{n!\Gamma(n+1)(ab)^{nI} k^{\frac{A}{k}+\frac{nC}{k}+\frac{I}{k}-I}\Gamma(\frac{A}{k}+\frac{nC}{k}+\frac{I}{k})\Gamma(P+Q+2nI)},$$

and the desired result follows from the definition (1.16).

The next corollaries follows when the conditions of Theorem 2.11 are satisfied:

Corollary 2.12. *Let $B = I$, $A = A + D$ and $k = 1$ in relation(2.26) where $k \in \mathbb{R}^+; \nu \in \mathbb{C}$, then we have*

$$\int_0^\infty x^{P-I}(1-x)^{Q-I}\left[ax+b(1-x)\right]^{-(P+Q)} J_{A+D,I,C}\left(\frac{4x(1-x)}{\left[ax+b(1-x)\right]^2};1\right)dx$$

$$= \frac{1}{a^P b^Q} \times_3\Psi_3\left[\begin{array}{cc} (I, I), (P, 2I), (Q, 2I); & \\ & (\frac{-2}{ab})^I \\ (1,1), (A+D+I, C), (P+Q, 2I); & \end{array}\right]. \tag{2.27}$$

Corollary 2.13. *Let $A = A - I$ in relation(2.26), then there follows*

$$\int_0^\infty x^{P-I}(1-x)^{Q-I}\left[ax+b(1-x)\right]^{-(P+Q)} J_{A-I,B,C}\left(\frac{4x(1-x)}{\left[ax+b(1-x)\right]^2};k\right)dx$$

$$= \frac{1}{a^P b^Q \Gamma\left(\frac{B}{k}\right) k^{\frac{A}{k}-I}} \times_3 \Psi_3 \left[\begin{array}{cc} (\frac{B}{k}, I), (P, I), (Q, I); & \\ & (\frac{-2}{ab})^I k^{\frac{kI-C}{k}} \\ (1,1), (\frac{A}{k}, \frac{C}{k}), (P+Q, 2I); & \end{array}\right]. \tag{2.28}$$

Corollary 2.14. *Let $k = 1$ and $A = A - I$ in relation(2.26), then it is*

$$\int_0^\infty x^{P-I}(1-x)^{Q-I}\left[ax+b(1-x)\right]^{-(P+Q)} J_{A-I,B,C}\left(\frac{4x(1-x)}{\left[ax+b(1-x)\right]^2};1\right)dx$$

$$= \frac{1}{a^P b^Q \Gamma(B)} \times_3 \Psi_3 \left[\begin{array}{cc} (B, I), (P, I), (Q, I); & \\ & (\frac{-2}{ab})^I \\ (1,1) + (A, C), (P+Q, 2I); & \end{array}\right]. \tag{2.29}$$

Corollary 2.15. *Let $B = I$ and $A = A - I$ in relation(2.26), then we get*

$$\int_0^\infty x^{P-I}(1-x)^{Q-I}\left[ax+b(1-x)\right]^{-(P+Q)} J_{A-I,I,C}\left(\frac{4x(1-x)}{\left[ax+b(1-x)\right]^2};k\right)dx$$

$$= \frac{1}{a^P b^Q \Gamma\left(\frac{I}{k}\right) k^{\frac{A}{k}-I}} \times_3 \Psi_3 \left[\begin{array}{cc} (\frac{I}{k}, I), (P, I), (Q, I); & \\ & (\frac{-2}{ab})^I k^{\frac{kI-C}{k}} \\ (1,1), (\frac{A}{k}, \frac{C}{k}), (P+Q, 2I); & \end{array}\right]. \tag{2.30}$$

Corollary 2.16. *let $B = I$, $A = A - I$ and $k = 1$ in relation(2.26), then it is*

$$\int_0^\infty x^{P-I}(1-x)^{Q-I}\left[ax+b(1-x)\right]^{-(P+Q)} J_{A-I,I,C}\left(\frac{4x(1-x)}{\left[ax+b(1-x)\right]^2};1\right)dx$$

$$= \frac{1}{a^P b^Q} \times_3 \Psi_3 \left[\begin{array}{cc} (I, I), (P, I), (Q, I); & \\ & (\frac{-2}{ab})^I \\ (1,1), (A, C), (P+Q, 2I); & \end{array}\right]. \tag{2.31}$$

Corollary 2.17. *Let $B = I$, $A = 0$ and $k = 1$in relation(2.26), then there follows*

$$\int_0^\infty x^{P-I}(1-x)^{Q-I}\left[ax+b(1-x)\right]^{-(P+Q)} J_{I,C}\left(\frac{4x(1-x)}{\left[ax+b(1-x)\right]^2};1\right)dx$$

$$= \frac{1}{a^P b^Q} \times_3 \Psi_3 \left[\begin{array}{cc} (I, I), (P, I), (Q, I); & \\ & (-2)^I \\ (1,1), (I, C), (P+Q, 2I); & \end{array}\right]. \tag{2.32}$$

3 Conclusion

In this research work, we have established some new integral transformations associated with a generalized k-Bessel matrix function. In doing so, we have obtained some new properties of generalization of known integral transforms, like e.g. the Laplace transform, Euler transform, Whittaker transform and k-transform. The obtained results can lead to the derivation of some other integral transformations involving various (generalized) $k-$Bessel matrix functions. The integral formulas for generalized modified $k-$Bessel matrix function of first kind are derived and the results expressed in term of Fox-Wright function. Some interesting special cases were also derived from the main results and commented. For $D = -I = -\nu I$ and $D = I$ with $\nu = 1$, then theorems 2.7 and 2.9 give the unified integral representation of Bessel matrix function and modified Bessel matrix function respectively.

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