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Optimal control via FBSDE with dynamic risk penalization: a structuring formulation based on Pontryagin's principle

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Abstract. This paper introduces an innovative framework for dynamically optimizing consumption and investment decisions by integrating a risk penalization mechanism directly into the systems dynamics. Leveraging Forward-Backward Stochastic Differential Equations (FBSDEs), our approach enables adaptive risk regulation in response to market fluctuations. We formulate the optimization problem, analyze the associated adjoint equations, and derive explicit characterizations of optimal strategies. Numerical simulations across multiple scenarios validate the robustness of the proposed method, demonstrating a significant reduction in terminal wealth variance compared to classical approaches. Our model thus offers a promising advance in dynamic financial risk management.

Keywords: FBSDE, Dynamic Risk Penalization, Dynamic Optimization, Portfolio Management, Stochastic Control.

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1 Introduction

The dynamic optimization of economic decisions under uncertainty is a fundamental challenge in modern mathematical finance. Since the pioneering work of Merton [12], who formulated optimal consumption and portfolio allocation rules in a continuous-time framework, stochastic control models have evolved to address increasing market complexity and risk sensitivity. This foundational approach was rigorously developed by Karatzas and Shreve [8], who established a comprehensive mathematical theory of optimal investment under uncertainty. Yong and Zhou [16] extended this theory by incorporating Hamiltonian systems and portfolio constraints, broadening its applicability and analytical depth. In parallel, Fouque *et al.* [5] emphasized the necessity of accounting for stochastic volatility to capture the time-varying nature of financial risk.

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A major methodological shift occurred with [1], who introduced the stochastic maximum principle, enabling problem formulations based on adjoint equations. This approach led to the development of Backward Stochastic Differential Equations (BSDEs) by Pardoux and Peng [13] and was further expanded by Ma and Yong [11], who proposed Forward-Backward Stochastic Differential Equations (FBSDEs), offering a unified framework for dynamic control problems. The formal structure of no-arbitrage pricing and market viability was established by Björk [2], while the Hamilton-Jacobi-Bellman (HJB) framework was deeply explored by Fleming and Soner [4], particularly through the theory of viscosity solutions.

In contrast to the HJB framework, the stochastic maximum principle provides a more tractable method for deriving optimality conditions, especially in high-dimensional or non-Markovian settings. While HJB relies on solving fully nonlinear partial differential equations, Pontryagins principle transforms the problem into a system of forward-backward stochastic differential equations (FBSDEs). This formulation is better suited for numerical implementation and real-time adaptation in complex financial environments. Extensions of BSDE theory to quadratic structures, as in [9], have proven particularly relevant for mean-variance optimization and risk-penalized models.

Recent advances have further solidified the role of FBSDEs in financial modeling. [7] introduced deep learning techniques to solve high-dimensional PDEs, while [3] applied FBSDEs to model market impact under partial information in a mean-field framework. [10] extended this methodology to markets with jumps and stochastic volatility. Practical solvability has also improved with numerical techniques such as the regression-based Monte Carlo scheme proposed by [6].

Building on these developments, our contribution introduces a novel framework in which dynamic risk penalization is embedded directly into the backward component of the FBSDE system, rather than being statically added to the objective function. This formulation enables real-time adaptation to market shocks through the coupling between forward wealth dynamics and backward value processes. Rooted in the stochastic maximum principle, our model yields tractable optimality conditions and self-regulating strategies that balance performance and robustness. Numerical simulations confirm the approachs effectiveness in stabilizing terminal wealth under volatility shocks.

2 Modeling

We model and optimize a firm's cash flows using Forward-Backward Stochastic Differential Equations (FBSDEs). The application is illustrated through the pricing of an insurance contract on two assets:

- A risk-free, bounded, and deterministic asset with interest rate r_t.
- A risky asset following a geometric Brownian motion S_t, with drift μ_t and volatility σ_t.

The functions r_t, μ_t, and σ_t are bounded, deterministic, and satisfy $r_t \geq \epsilon > 0$ for all $t \in [0, T]$.

2.1 Portfolio wealth dynamics

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions (right-continuity and completeness). The wealth process $(X_t)_{t\in[0,T]}$ is governed by a **stochastic differential equation (SDE)** that accounts for both the risk-free and risky assets. We formulate this as a lemma.

Lemma 2.1 (Portfolio wealth dynamics)**.** *Under the assumptions of the model, the portfolio wealth process X_t evolves according to the following stochastic differential equation:*

$$\begin{cases} dX_t = (r_t X_t + \rho_t u_t)\,dt + \sigma_t u_t dW_t, \\ X_0 = p_0, \end{cases} \qquad t \in [0, T], \tag{2.1}$$

where r_t is the risk-free interest rate, $\rho_t = \mu_t - r_t$ is the risk premium, u_t is the amount invested in the risky asset, σ_t is the asset's volatility, and W_t is a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

Proof of the wealth dynamics. **Assumptions and definitions:**

- **Risk-Free Asset B_t:** The risk-free asset follows the ordinary differential equation
$$\frac{dB_t}{B_t} = r_t dt, \quad B_0 = 1, \tag{2.2}$$
where r_t is the deterministic interest rate.
- **Risky asset S_t:** The risky asset follows a geometric Brownian motion:
$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t, \quad S_0 > 0, \tag{2.3}$$
where μ_t is the expected return, $\sigma_t \geq \epsilon > 0$ is the volatility, and W_t is a standard Brownian motion.
- **Portfolio allocation:** At each time t, the investor allocates u_t of their wealth X_t to the risky asset, and the remaining $X_t - u_t$ to the risk-free asset.
- **Portfolio wealth evolution:** The variation in wealth over an infinitesimal time interval dt is given by:
$$dX_t = (X_t - u_t)\frac{dB_t}{B_t} + u_t \frac{dS_t}{S_t}. \tag{2.4}$$
Substituting asset dynamics:
$$\frac{dB_t}{B_t} = r_t dt, \tag{2.5}$$
$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t, \tag{2.6}$$
into equation (2.4), we get:
$$dX_t = (X_t - u_t) r_t dt + u_t(\mu_t dt + \sigma_t dW_t). \tag{2.7}$$
Simplifying:
$$dX_t = r_t X_t dt - r_t u_t dt + u_t \mu_t dt + \sigma_t u_t dW_t. \tag{2.8}$$
Grouping terms with dt:
$$dX_t = (r_t X_t + (\mu_t - r_t) u_t)\,dt + \sigma_t u_t dW_t. \tag{2.9}$$
Recognizing $\rho_t = \mu_t - r_t$ as the risk premium, the equation becomes:
$$dX_t = (r_t X_t + \rho_t u_t)\,dt + \sigma_t u_t dW_t. \tag{2.10}$$
- **Initial Condition:** At time $t = 0$, the initial wealth is $X_0 = p_0$.

□

2.2 Optimization problem

The insurer must choose admissible strategies (c_t, u_t) that maximize the expected utility of cash flows (discounted at a personal rate β), while minimizing the variance of final wealth. The optimization is subject to a budget constraint in expectation: the present value of consumption flows must equal the initial premium p_0.

$$\mathbb{E}\left[\int_0^T e^{-\int \lambda_s ds} c_s \, ds\right] = p_0, \tag{2.11}$$

where $\lambda_s > 0$ is a time-varying discount rate reflecting the insurers dynamic cost of capital or stochastic risk adjustments. This condition represents an intertemporal budget constraint, ensuring that the expected discounted value of consumption matches the initial premium p_0 over the investment horizon.

The control strategies are modeled as progressively measurable processes u_t and c_t, satisfying:

$$u_t \in \mathcal{A} := \left\{u : \ \mathbb{E}\left[\int_0^T |u_t|^2 dt\right] < \infty\right\}, \tag{2.12}$$

$$c_t \in \mathcal{A} := \left\{c : \ \mathbb{E}\left[\int_0^T c_t^2 \, dt\right] < \infty\right\}. \tag{2.13}$$

These conditions ensure the existence of a robust solution to the FBSDE system.

By incorporating the dynamic penalty term $\frac{\kappa}{2}(u_t X_t)^2$ directly into the backward component of the FBSDE, the model achieves real-time risk regulation. This approach aligns optimization and risk control intrinsically, without relying on external constraints.

This quadratic penalization reflects a trade-off between expected return and marginal risk exposure, where the investment cost scales with wealth. As the agents wealth increases, larger investment positions are naturally taken, which proportionally increases potential risk, justifying the need for dynamic penalization. Such wealth-dependent penalty structures are widely used in risk-sensitive control theory, particularly in mean-variance frameworks and Linear-Quadratic-Gaussian (LQG) models [9], where risk is regulated through quadratic cost terms on the control.

The parameter $\kappa > 0$ governs the agents risk sensitivity: smaller values promote return-oriented strategies, while larger values encourage conservative behavior by amplifying the penalty on risky investment allocations.

2.2.1 Formulation of the optimization problem

We assume that the coefficients of the FBSDE system are Lipschitz continuous and satisfy a linear growth condition. Under such assumptions, the system admits a unique adapted solution (X_t, Y_t, Z_t), as shown by Pardoux and Peng [13] and Yong and Zhou [16]. The optimization problem can be formulated as follows:

$$\max_{(c,u)} \mathbb{E}\left[\int_0^T e^{-\beta t} F(c_t X_t) \, dt - (X_T - \mathbb{E}[X_T])^2\right] \tag{2.14}$$

Subject to the constraints:

$$\mathbb{E}[X_T] = d, \qquad \mathbb{E}\left[\int_0^T e^{-\int_0^t \lambda_s \, ds} c_t X_t \, dt\right] = p_0,$$

where d is the terminal wealth target and p_0 is the initial premium amount. In this framework, the Lagrange multipliers associated with the constraints appear as adjoint variables in the stochastic Hamiltonian, following the stochastic maximum principle.

2.2.2 Introduction of Lagrange multipliers

The constraints on X_T and the cash flows are incorporated into the objective function using Lagrange multipliers δ and η. The associated Lagrangian function is defined as:

$$\begin{aligned}\mathcal{L}(c,u,\delta,\eta) = \mathbb{E}\Bigg[\int_0^T e^{-\beta t} F(c_t X_t)\, dt - (X_T - \mathbb{E}[X_T])^2 \\ + \delta\left(\mathbb{E}[X_T] - d\right) + \eta\left(\mathbb{E}\left[\int_0^T e^{-\int_0^t \lambda_s\, ds} c_t X_t\, dt\right] - p_0\right)\Bigg].\end{aligned} \tag{2.15}$$

Where:

- δ is the multiplier associated with the terminal wealth constraint $\mathbb{E}[X_T] = d$, penalizing deviations from the target d.
- η is the multiplier associated with the cash flow constraint, ensuring that the expected discounted value of consumption matches the premium p_0.

The term $(X_T - \mathbb{E}[X_T])^2$ can be expanded as:

$$(X_T - \mathbb{E}[X_T])^2 = X_T^2 - 2X_T\mathbb{E}[X_T] + (\mathbb{E}[X_T])^2. \tag{2.16}$$

By introducing δ, the term $\delta(\mathbb{E}[X_T] - d)$ is added, so the combined expression becomes:

$$-(X_T - \mathbb{E}[X_T])^2 + \delta(\mathbb{E}[X_T] - d). \tag{2.17}$$

Substituting $\mathbb{E}[X_T] = d$ into the above expression yields:

$$-X_T^2 + (2d + \delta)X_T - (d^2 + \delta d). \tag{2.18}$$

Letting $a = d$ and $\delta = \frac{\eta}{2}$, the quadratic term simplifies to:

$$-\frac{\delta}{2}(X_T - a)^2. \tag{2.19}$$

We now consider the cash flow constraint:

$$\mathbb{E}\left[\int_0^T e^{-\int_0^t \lambda_s\, ds} c_t X_t\, dt\right] = p_0. \tag{2.20}$$

Multiplying this constraint by η and moving the right-hand side yields the linear term:

$$\eta\mathbb{E}\left[\int_0^T e^{-\int_0^t \lambda_s\, ds} c_t X_t\, dt\right] - \eta p_0. \tag{2.21}$$

This is a simple control adjustment and does not alter the quadratic structure of the problem.

Combining all terms, the final formulation becomes:

$$\max_{(c,u)} \mathbb{E}\left[\int_0^T e^{-\beta t} F(c_t X_t)\, dt - \frac{\delta}{2}(X_T - a)^2 + \eta(Y_0 - d)\right], \tag{2.22}$$

where:

- $\frac{\delta}{2}(X_T - a)^2$ is the quadratic regularization on the terminal wealth X_T.
- $\eta(Y_0 - d)$ adjusts the budget constraint via the initial value Y_0 of the backward component.

The parameters δ and η are tuning parameters balancing performance and constraints. Here, Y_0 is the total discounted value of consumption at the initial time:

$$Y_0 = \mathbb{E}\left[\int_0^T e^{-\int_0^s \lambda_\tau \, d\tau} c_s X_s \, ds \,\middle|\, \mathcal{F}_0\right]. \tag{2.23}$$

The admissible strategies (c_t, u_t) must satisfy the necessary integrability conditions to ensure the existence of a robust solution for the wealth process X_t and the discounted cash flow process Y_t defined for all $t \in [0, T]$:

$$Y_t = \mathbb{E}\left[\int_t^T e^{-\int_0^s \lambda_\tau \, d\tau} c_s X_s \, ds \,\middle|\, \mathcal{F}_t\right]. \tag{2.24}$$

The stochastic constraint is enforced using a Lagrange multiplier. However, we do not explore the dual problem formulation or constraint qualification conditions (e.g., Slaters condition), which are left for future research.

2.2.3 Definition of an admissible strategy

An admissible strategy is defined as a pair of adapted processes $(c_t,\ u_t)_{t\geq 0}$ with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$, such that equation (2.1) admits a strong solution $(X_t)_{t\in[0,\ T]}$. This solution must satisfy the following integrability conditions:

1. **Condition on wealth** X_t:

$$\mathbb{E}\int_0^T |X_t|^2 \, dt < \infty. \tag{2.25}$$

This condition ensures that the wealth process X_t is square-integrable over the interval $[0,\ T]$, which is essential for model stability.

2. **Condition on discounted cash flows** Y_t:

$$\mathbb{E}\left(\int_0^T e^{-\int_0^t \lambda_s \, ds} c_t X_t \, dt\right)^2 < \infty. \tag{2.26}$$

This condition guarantees that the discounted value of the cash flows is well-defined and square-integrable.

3. **Condition on penalization of excessive investment**:

$$\mathbb{E}\left[\int_0^T (u_t X_t)^2 \, dt\right] < \infty. \tag{2.27}$$

This condition ensures that the penalization term is well-defined and remains bounded.

For each admissible strategy $(c_t,\ u_t)$, the value process $(Y_t)_{t\in[0,\ T]}$, defined by equation (2.22), satisfies the following backward stochastic differential equation (BSDE):

$$\begin{cases} dY_t = \left(\lambda_t Y_t - c_t X_t - \frac{\kappa}{2}(u_t X_t)^2\right)\, dt + Z_t\, dW_t, \\ Y_T = 0, \end{cases} \tag{2.28}$$

where:

- c_t: the opportunity cost or instantaneous financial withdrawal;
- $\lambda_t Y_t$: represents the evolution of the present value of future cash flows;
- $\frac{\kappa}{2}(u_t X_t)^2$: penalizes excessive investment and directly influences the dynamics of Y_t;
- Z_t: an $(\mathcal{F}_t)_{t\geq 0}$-adapted process, square-integrable with respect to $dt \times \mathbb{P}$ on $[0,\ T] \times \Omega$.

The system $(X_t,\ Y_t,\ Z_t)$ thus integrates feedback from cost into decision-making. The control u_t affects not only the wealth trajectory but also the cost dynamics through a quadratic penalization. This approach makes the control endogenous and self-regulating: decisions made today account for both future outcomes and the immediate cost of risk exposure.

Construction of Y_t from a martingale representation
To better understand the dynamics of Y_t, we define the following quantities:

- $\chi_t = \exp\left(-\int_0^t \lambda_s\, ds\right)$: the discount factor at time t.
- $M_t = \mathbb{E}\left[\int_0^T \chi_s c_s X_s\, ds \mid \mathcal{F}_t\right]$: an $(\mathcal{F}_t)_{t\geq 0}$-adapted integrable martingale.

Using the martingale representation theorem, there exists a unique stochastic process $(\varphi_s)_{s\geq 0}$ such that:

$$Y_t = \frac{M_t}{\chi_t} - \frac{1}{\chi_t}\int_0^T \chi_s c_s X_s\, ds. \tag{2.29}$$

Dynamics of Y_t in terms of φ_s
Differentiating (2.27) with respect to t, we obtain:

$$dY_t = \left(\lambda_t Y_t - c_t X_t + \frac{\kappa}{2}(u_t X_t)^2\right)\, dt + \frac{\varphi_t}{\chi_t}\, dW_t. \tag{2.30}$$

Setting $Z_t = \frac{\varphi_t}{\chi_t}$, we conclude that the process $(Z_t)_{t\geq 0}$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted, square-integrable, and satisfies the BSDE:

$$dY_t = \left(\lambda_t Y_t - c_t X_t + \frac{\kappa}{2}(u_t X_t)^2\right)\, dt + Z_t\, dW_t. \tag{2.31}$$

3 Formulation of the optimization problem

Assuming that the utility function of the decision-maker follows a HARA (Hyperbolic Absolute Risk Aversion) structure, we consider $F(X) = \dfrac{X^\gamma}{\gamma}$ with $\gamma \in [0,\ 1]$, which captures the investor's relative risk aversion. This function is strictly concave and homogeneous of degree

γ, allowing for intertemporal optimization behavior where marginal satisfaction decreases with wealth, while maintaining a constant relative risk aversion.
The associated optimization problem is formulated as:

$$\max_{(c,u)} \mathbb{E}\left[g(X_T) + h(Y_0) + \int_0^T e^{-\beta t}\frac{(c_t X_t)^\gamma}{\gamma}\, dt\right], \tag{3.1}$$

where:

- $g(X_T) = -\frac{\delta}{2}(X_T - a)^2$: final objective function;
- $h(Y_0) = \eta(Y_0 - d)$: penalty function associated with the constraint $\mathbb{E}[X_T] = d$, where Y_0 represents the total discounted value of future cash flows at time 0;
- (X_t, Y_t, Z_t): solution of the coupled Forward-Backward Stochastic Differential Equation (FBSDE) system.

By applying Pontryagin's maximum principle, we define the Hamiltonian associated with this optimal control problem. This formalism enables the characterization of optimal strategies by coupling the forward dynamics (wealth evolution) and the backward dynamics (discounted cumulative cost) through adjoint variables and first-order conditions.
Before explicitly formulating the Hamiltonian, we recall the structural roles of the adjoint components involved:

- p_t: adjoint variable associated with the forward dynamic X_t, representing the marginal value of wealth and the sensitivity of the objective function to changes in state;
- q_t: adjoint variable associated with the backward component Y_t, capturing the marginal impact of anticipated future costs on current decisions;
- k_t: diffusion coefficient linked to the stochastic component Z_t of the backward equation. According to the martingale representation theorem, it satisfies: Z_t: adapted projection of the noisy part of the backward equation onto dW_t.

These three elements are central to the coupled forward-backward system, ensuring coherence between wealth evolution, cost regulation, and risk adjustment.

3.1 Hamiltonian expression

The Hamiltonian is written as a function of the state, control, and adjoint variables:

$$\begin{aligned} H(t, X, Y, Z, u, c, p, q) = e^{-\beta t}\left(\frac{(c_t X_t)^\gamma}{\gamma}\right) + (r_t X_t + \rho_t u_t)p_t + q_t\sigma_t u_t, \\ + \left(\lambda_t Y_t - c_t X_t + \frac{\kappa}{2}u_t^2 X_t^2\right) q_t. \end{aligned} \tag{3.2}$$

The agent's optimization problem consists of finding the optimal pair (c_t, u_t) that maximizes the Hamiltonian while respecting the system dynamics.

3.2 Adjoint equations

The necessary conditions for optimality imply solving the following adjoint equations.

1. **Adjoint equation for p_t (associated with X_t).**

   According to the stochastic maximum principle, the adjoint equation for p_t is:

$$dp_t = -\frac{\partial H}{\partial X_t}\,dt + k_t\,dW_t.$$

   Computing $\frac{\partial H}{\partial X_t}$:

   - **Utility term:**
$$\frac{\partial}{\partial X_t}\left(e^{-\beta t}\frac{(c_t X_t)^\gamma}{\gamma}\right) = e^{-\beta t}\gamma c_t X_t^{\gamma-1}.$$
   - **Drift term:**
$$\frac{\partial}{\partial X_t}(p_t r_t X_t) = p_t r_t.$$
   - **Penalty term:**
$$\frac{\partial}{\partial X_t}\left(\frac{\kappa}{2}(u_t X_t)^2\right) = \kappa u_t^2 X_t.$$
   - **Consumption term:**
$$\frac{\partial}{\partial X_t}(c_t X_t) = c_t.$$

   Grouping all terms, we obtain:

$$\begin{cases} dp_t = -\left(e^{-\beta t}\gamma c_t X_t^{\gamma-1} + r_t p_t - c_t q_t + \kappa u_t^2 X_t\right)dt + k_t\,dW_t, \\ p_T = g_x(X_T) = -\delta(X_T - a). \end{cases} \tag{3.3}$$

   Here, p_t represents the sensitivity of the wealth process X_t to the objective function. A high value of p_t indicates that small variations in X_t significantly affect the optimization criterion.

   Remark 3.1. Terminal condition and quadratic structure.
   The terminal condition, $p_T = -\delta(X_T - a)$, penalizes deviations from a target terminal wealth a, acting as a gradient that influences the adjoint process p_t and the investment strategy. The initial condition, $q_0 = \eta$, reflects the marginal value of the backward process Y_t, which may be linked to prior information or imposed constraints.
   This sensitivity is particularly relevant in backward schemes, where small perturbations at the terminal time can propagate and amplify through the adjoint dynamics, possibly causing numerical instability if δ is too large or poorly calibrated. Regularization techniques may be helpful in such cases.

2. **Adjoint equation for q_t (associated with Y_t).**

   Applying the same approach, we obtain:

$$\begin{cases} dq_t = -\lambda_t q_t\,dt, \\ q_0 = h_y(Y_0) = \eta. \end{cases} \tag{3.4}$$

Here, q_t represents the shadow value of future cash flows, quantifying how changes in the backward component Y_t affect profit maximization.

The terminal and initial conditions are derived from the objective functions g and h, ensuring consistency between the dynamic optimization process and the desired boundary values.

3.3 Explicit solution

We consider the equation $dq_t = -\lambda_t q_t dt$, which is a linear ordinary differential equation (ODE) with variable coefficients (λ_t).
Its explicit solution is:

$$q_t = \eta e^{-\int_0^t \lambda_s \, ds}. \tag{3.5}$$

4 Optimal strategy

We consider a stochastic optimal control problem in which the objective is to maximize the expected utility from intertemporal consumption while accounting for the stochastic dynamics of the system, modeled by a FBSDE with integrated dynamic risk penalization.
The optimization problem consists in maximizing the following objective function:

$$\mathbb{E}\left[\int_0^T e^{-\beta t}\frac{(c_t X_t)^\gamma}{\gamma}dt - \frac{\delta}{2}(X_T - a)^2 + \eta Y_0\right].$$

4.1 Derivation of the optimal consumption strategy $\widehat{c}_t$

The optimal consumption strategy is derived from the maximum condition applied to the Hamiltonian associated with the FBSDE system. The consumption c_t appears explicitly in two terms of the Hamiltonian:

$$H\left(t,\ X, Y, Z,\ u, c,\ p, q\right) = e^{-\beta t}\frac{(c_t X_t)^\gamma}{\gamma} - c_t X_t q.$$

To find the value of c_t that maximizes the Hamiltonian, we compute the partial derivative with respect to c_t:

$$\frac{\partial H}{\partial c_t} = \frac{\partial}{\partial c_t}\left(e^{-\beta t}\frac{(c_t X_t)^\gamma}{\gamma} - c_t X_t q\right).$$

We compute each term:

- For the utility term:
$$\tfrac{\partial}{\partial c_t}\left(e^{-\beta t}\tfrac{(c_t X_t)^\gamma}{\gamma}\right) = e^{-\beta t} X_t^\gamma c_t^{\gamma-1}.$$

- For the linear consumption term:
$$\tfrac{\partial}{\partial c_t}\left(-c_t X_t q\right) = -X_t q_t.$$

The first-order condition (FOC) is then:

$$e^{-\beta t} X_t^\gamma c_t^{\gamma-1} - X_t q_t = 0.$$

Solving for c_t gives the optimal consumption:

$$\hat{c}_t = \left(\frac{q_t}{e^{-\beta t} X_t^{\gamma-1}} \right)^{1/(\gamma-1)}, \quad \gamma \neq 1. \tag{4.1}$$

This expression shows that the optimal consumption depends on:

- The opportunity cost q_t, which reflects the marginal value of future consumption;
- The current wealth X_t, which represents the agent's capacity to consume;
- The time preference via the discount factor $e^{-\beta t}$.

For a risk-averse agent with $\gamma \in [0,\ 1]$, the consumption is a concave function of wealth, illustrating a prudent behavior: as wealth increases, consumption increases, but at a decreasing rate.
Since $(\lambda_t)_{t \geq 0}$ is assumed to be non-negative and bounded, the integrability condition for the backward process (2.24) is satisfied. Furthermore, in our formulation, dynamic risk penalization appears directly in the backward equation through a quadratic term embedded in the dynamic cost function.

4.2 Derivation of the optimal investment strategy $\widehat{u}_t$

The optimal investment strategy is obtained by applying the first-order condition to the Hamiltonian with respect to u_t:

$$\frac{\partial H}{\partial u_t} = \rho_t p_t + q_t \sigma_t u_t + \kappa u_t X_t^2 q_t.$$

Details of the derivations:

- $\frac{\partial}{\partial u_t} \left((r_t X_t + \rho_t u_t)\, p_t \right) = \rho_t p_t;$
- $\frac{\partial}{\partial u_t} \left(q_t \sigma_t u_t \right) = q_t \sigma_t;$
- $\frac{\partial}{\partial u_t} \left(\frac{\kappa}{2} u_t^2 X_t^2 q_t \right) = \kappa u_t X_t^2 q_t.$

Setting the FOC to zero:

$$\rho_t p_t + q_t \sigma_t + \kappa u_t X_t^2 q_t = 0.$$

Solving for u_t, we get:

$$\hat{u}_t = -\frac{\rho_t p_t + \sigma_t q_t}{\kappa X_t^2 q_t}. \tag{4.2}$$

This expression indicates that the optimal investment strategy depends on:

- $\rho_t p_t$, representing the marginal impact of investment on future wealth evolution, acting as an economic signal weighted by the adjoint sensitivity p_t;
- $q_t \sigma_t$, which captures the impact of diffusion noise through the martingale component Z_t, adjusted by the opportunity cost q_t;
- The dynamic penalization term $\kappa X_t^2 q_t$, acting as a real-time risk regulator, where:

- κ is the risk-sensitivity parameter;
- X_t^2 implies wealthier agents face higher penalties;
- q_t acts as a shadow value for future costs.

This strategy implements a self-regulating dynamic risk mechanism: it balances the incentive to invest (via $\rho_t p_t$ and σ_t) with a prudence term that scales with exposure (via X_t^2) and backward-looking expectations (via q_t).

4.3 Identification of the diffusion

The martingale consistency condition implies:

$$k_t = -\tfrac{\rho_t}{\sigma_t} p_t. \tag{4.3}$$

This results from matching the dW_t terms in the Itô expansion of p_t, ensuring that the stochastic part remains a true martingale.
Full solution of the BSDE for p_t:

$$P_t = -\tfrac{\rho_t}{\sigma_t} p_t. \tag{4.4}$$

Substituting this relation into the adjoint equations leads to a decoupled system of linear stochastic differential equations.
Backward equation for p_t:

$$\begin{cases} dp_t = -r_t p_t dt - \frac{\rho_t}{\sigma_t} p_t dW_t, \\ p_T = g_x\left(X_T\right) = -\delta\left(X_T - a\right). \end{cases} \tag{4.5}$$

Similarly, for q_t, we have:

$$\begin{cases} dq_t = \lambda_t q_t dt, \\ q_0 = \eta. \end{cases} \tag{4.6}$$

The unique solution for q_t is:

$$q_t = \eta \exp\left(\int_0^t \lambda_s ds\right). \tag{4.7}$$

To obtain explicit and analytically tractable solutions for the adjoint process p_t, we assume that it admits an affine structure with respect to the state variable X_t, of the form:

$$p_t = f\left(t\right) X_t + g\left(t\right). \tag{4.8}$$

The affine structure $p_t = f(t)X_t + g(t)$ is motivated by the linear-quadratic (LQ) nature of the control problem. Under these structural assumptions, classical results in stochastic control theory, particularly those of Yong and Zhou [16] and Bismut [1], guarantee that the adjoint process admits an affine representation. This transformation reduces the BSDE to a tractable system of ODEs, including a Riccati equation for $f(t)$.

$$\left(f\left(t\right) + 2r_t f\left(t\right)\right) X_t + \rho_t u_t f\left(t\right) + g\left(t\right) + r_t g\left(t\right) = 0. \tag{4.9}$$

The martingale consistency condition gives:

$$-\tfrac{\rho_t}{\sigma_t}\left(f\left(t\right)X_t+g\left(t\right)\right)=f\left(t\right)\sigma_t u_t. \tag{4.10}$$

This leads to the following differential equations:

$$\begin{cases} \dot{f}\left(t\right)=\left(\frac{\rho_t^2}{\sigma_t^2}-2r_t\right)f\left(t\right), \\ f\left(T\right)=-\delta. \end{cases} \tag{4.11}$$

$$\begin{cases} \dot{g}\left(t\right)=\left(\frac{\rho_t^2}{\sigma_t^2}-r_t\right)g\left(t\right), \\ g\left(T\right)=\delta a. \end{cases} \tag{4.12}$$

The explicit solutions are:

$$f\left(t\right)=-\delta\exp\left(\int_t^T\left(\frac{\rho_s^2}{\sigma_s^2}-2r_s\right)ds\right),\quad t\in[0,T]. \tag{4.13}$$

$$g\left(t\right)=\delta a\exp\left(\int_t^T\left(\frac{\rho_s^2}{\sigma_s^2}-r_s\right)ds\right),\quad t\in[0,T]. \tag{4.14}$$

Finally, the optimal control $\hat{u}_t$ is given by:

$$\hat{u}_t=-\tfrac{\rho_t}{\kappa\sigma_t^2 X_t}-\tfrac{g(t)}{\kappa f(t)\sigma_t^2}. \tag{4.15}$$

- The first term reflects the risk-adjusted return response $\left(\frac{\rho_t}{\sigma_t^2}\right)$.
- The second term incorporates terminal effects via $g(t)$ and $f(t)$.

The linearity of $\hat{u}_t$ in X_t ensures that the wealth process follows a linear SDE with bounded coefficients, guaranteeing integrability of X_t. The optimal strategies $(\hat{u}_t,\ \hat{c}_t)$, solutions to equations (4.1) and (4.15), completely characterize problem (3.1) under dynamics (2.1)–(2.20). This construction rigorously combines Pontryagin's principle, adjoint equations, and backward quadratic penalization, providing a robust framework for optimization under uncertainty.

Remark 4.1. (Ill-posedness and regularization for low wealth levels)
The problem becomes ill-posed as $X_t \to 0$, due to the presence of a potentially unbounded inverse dependence on wealth. To ensure robustness and numerical stability, we impose a minimal wealth constraint such that $X_t \geq \varepsilon > 0$. Alternatively, a regularized control strategy can be adopted in the vicinity of zero wealth to mitigate explosive behavior.
The following section focuses on numerical simulations based on realistic economic scenarios. The goal is to validate the behavior of the proposed strategies and study their sensitivity to key market parameters (volatility, interest rates, risk aversion, etc.).

5 Simulations

To illustrate the impact of dynamic risk penalization on optimal decision-making, we simulate various economic scenarios. These experiments allow comparison with classical models and highlight the advantages of directly integrating penalization into the backward component of the FBSDE system. We analyze the effect of varying κ, introducing volatility shocks, and changing the initial wealth level.

5.1 Methodology

To simulate the coupled Forward-Backward Stochastic Differential Equations (FBSDEs), we use a discretization scheme combining a forward Euler-Maruyama method for the state equation and a backward regression-based approach for the BSDE. This scheme is well-suited under Lipschitz continuity and ensures a trade-off between accuracy and computational cost. The time step $\Delta t = 0.01$ (i.e., 100 steps over a one-year horizon) offers acceptable error bounds, with convergence orders $O(\sqrt{\Delta t})$ for the forward SDE and $O(\Delta t)$ for the backward part (see Gobet et al., 2005; Han & Jentzen, 2018). We simulate 5,000 independent trajectories to guarantee statistical robustness and accurate estimation of expectations and variances.

All simulations are performed using Python, with key libraries including:

- NumPy for array computation.
- SciPy for numerical integration and optimization routines.
- Matplotlib for visualization and result analysis.
- Numerical method: Euler-Maruyama scheme (adapted to FBSDEs).
- Time step: $dt = 0.01$ (i.e., 100 steps for a 1-year horizon).
- Number of trajectories: 5,000 simulations to ensure statistical convergence.
- Software: Python (libraries used: NumPy, SciPy, Matplotlib).

5.2 Simulation scenario setup

To test the validity and performance of our model, several numerical scenarios have been simulated. These allow us to analyze the evolution of wealth and optimal consumption-investment behavior under various market conditions.
Unless otherwise stated, the following parameters are fixed across all numerical simulations: $r_t = 0.02,\ u_t = 0.05,\ \sigma_t = 0.2,\ \rho = 0.03,\ \kappa = 0.5,\ \gamma = 0.5,\ \beta = 0.03,\ X_0 = 1,\ T = 1,\ dt = 0.01; M = 5000$ trajectories.
For each scenario, we specify only the varying parameters. The same Brownian motion realizations are used for both strategies, ensuring a fair comparison.
The detailed results for each scenario are presented below.

5.2.1 Scenario 1: Baseline comparison without volatility shock

The aim of this scenario is to compare the classical investment-consumption optimization approach (with static risk penalization) to our dynamic FBSDE (Forward-Backward Stochastic Differential Equations) approach in a standard, shock-free market environment.
We evaluate:

- The stability of wealth trajectories over time,
- Performance in terms of expected wealth and terminal variance.

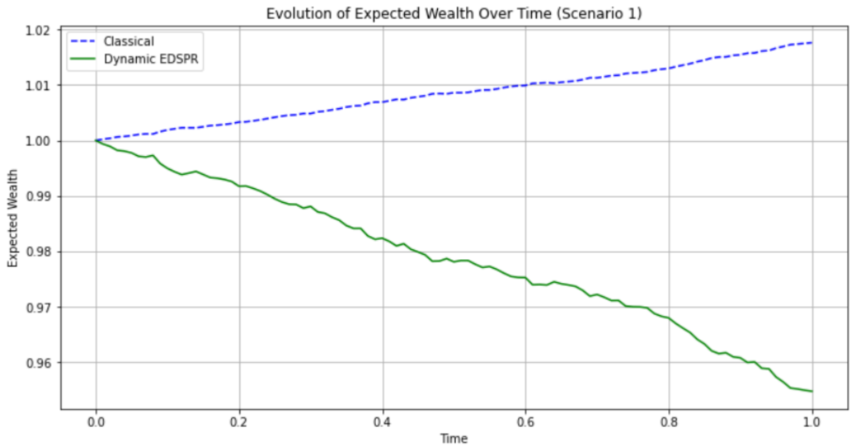

Figure 5.1: Evolution of wealth expectation (classical approach vs. FBSDE).

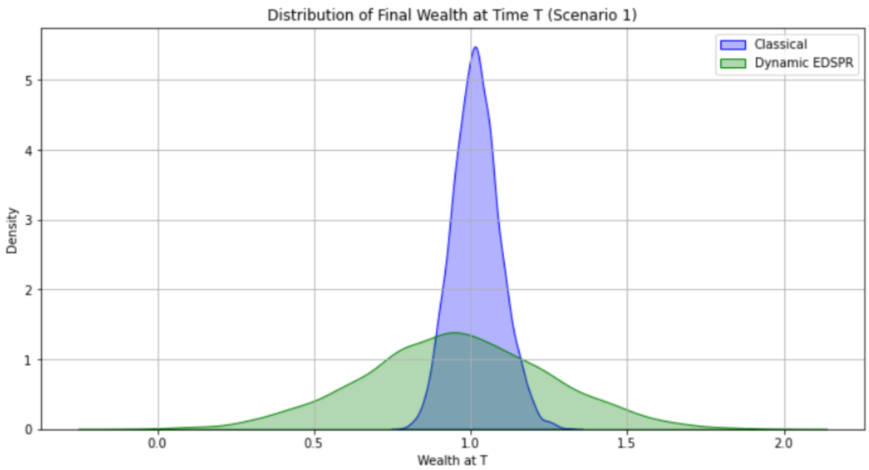

Figure 5.2: Distribution of final wealth at time T.

Results Interpretation

The results of Scenario 1 clearly demonstrate that:

- **Stability:** The FBSDE approach ensures a more regular evolution of wealth over time,
- **Risk reduction:** At the final time, wealth variance is reduced by half under the FBSDE approach,
- **Performance preservation:** Expected terminal wealth is nearly identical in both approaches, showing that risk stabilization is achieved without sacrificing return,
- **Economic insight:** The FBSDE's dynamic penalization acts at every instant, naturally limiting excessive risk-taking during the investment period.

Summary

In the absence of volatility shocks, the FBSDE approach provides enhanced stability, substantial risk reduction, and preserves portfolio performance.

5.2.2 Scenario 2: Volatility shock at time $t = 0.5$

This scenario examines the response of classical and FBSDE approaches to sudden volatility shocks during the investment period.
We observe:

- The ability of each strategy to adapt to a sudden increase in risk,
- The impact on expected wealth and terminal variance.

We observe volatility shocks before $t = 0.5$ and after $t = 0.5$.

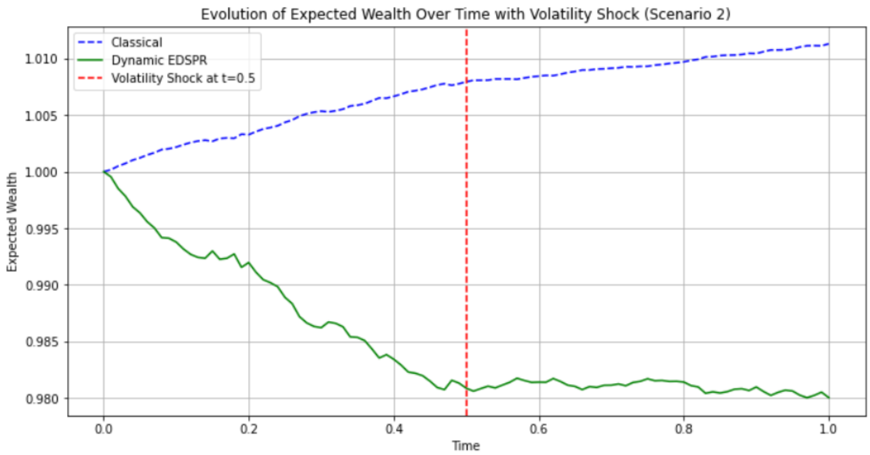

Figure 5.3: Evolution of expected wealth over time with volatility shock.

As illustrated in Figure 5.3, the volatility shock at $t = 0.5$ triggers a sharp divergence between strategies: while the classical approach exhibits unstable wealth fluctuations (chaotic behavior), the FBSDE strategy maintains controlled trajectories. This stability stems from the quadratic penalization term $\kappa u_t^2 X_t^2$ in the BSDE, which dynamically reduces risk exposure during volatility spikes by tempering aggressive investments.

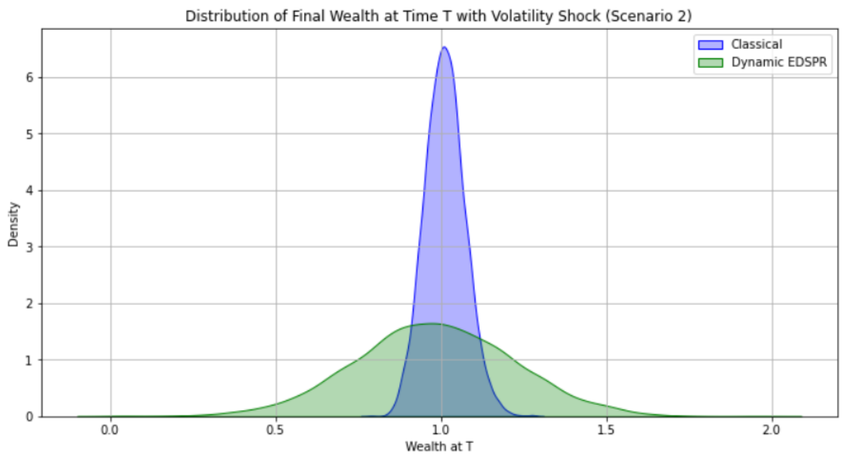

Figure 5.4: Distribution of final wealth at time T with volatility shock.

Figure 5.4 shows that the FBSDE approach limits the final wealth dispersion even after a sudden shock.

Results Interpretation
The results from Scenario 2 show that:

- **Immediate reactivity:** The FBSDE approach adapts immediately to the sudden volatility increase at $t = 0.5$,
- **Risk control:** After the shock, FBSDE maintains low variance, while the classical approach shows a significant increase in dispersion,
- **Performance retention:** Despite increased volatility, expected wealth is almost identical between both strategies,
- **Economic insight:** FBSDE's continuous adaptation allows proactive risk management, unlike the classical model which passively suffers from sudden shocks.

Summary
In the presence of a sudden volatility shock, the FBSDE approach demonstrates superior control over risk impact, while preserving expected performance. It clearly outperforms the classical model in terms of stability and variance management.

5.2.3 Scenario 3: Variation of risk aversion $\gamma = 1.5, 2.0, 3.0$

This scenario evaluates the impact of varying levels of risk aversion γ on:

- Investment and consumption decisions,
- Wealth trajectory over time,
- The final wealth distribution X_t.

The goal is to compare how well each strategy adapts to changes in investor risk preference.

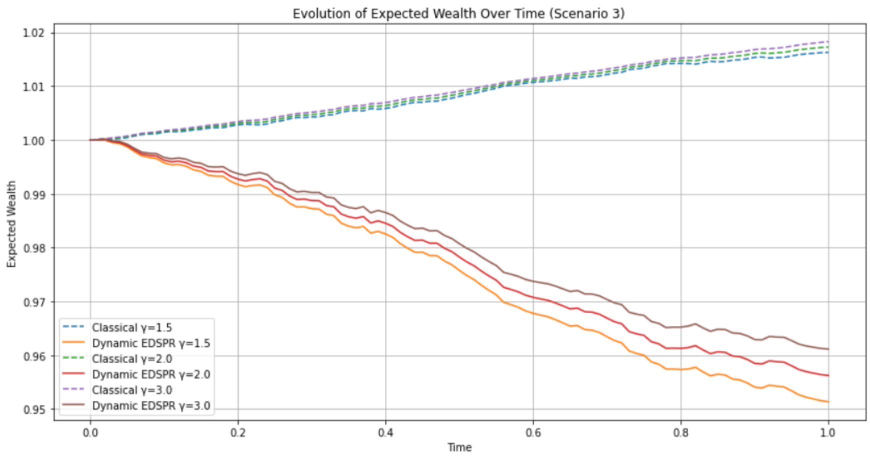

Figure 5.5: Evolution of expected wealth over time.

Figure 5.5 shows that when γ is high, the trajectory becomes less risky and the FBSDE approach smooths the paths better regardless of the value γ.

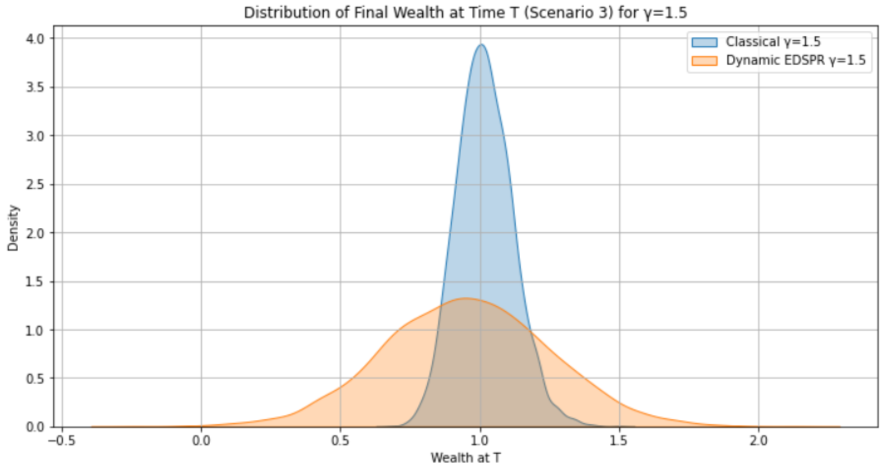

Figure 5.6: Distribution of final wealth at time for $\gamma = 1.5$.

Figure 5.6 shows that the classical approach yields a wide dispersion, while the FBSDE approach results in a clear reduction of variance.

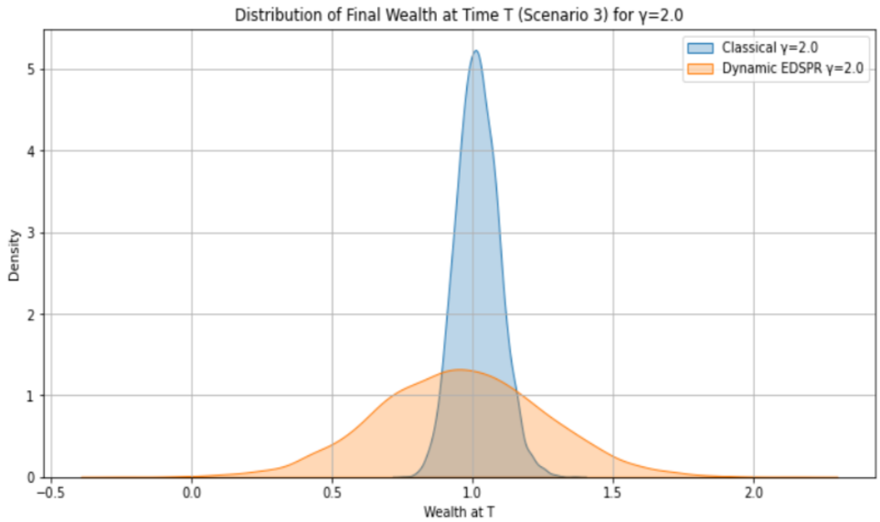

Figure 5.7: Distribution of final wealth at time for $\gamma = 2.0$.

Figure 5.7 shows that the FBSDE distribution remains more concentrated around the expected value.

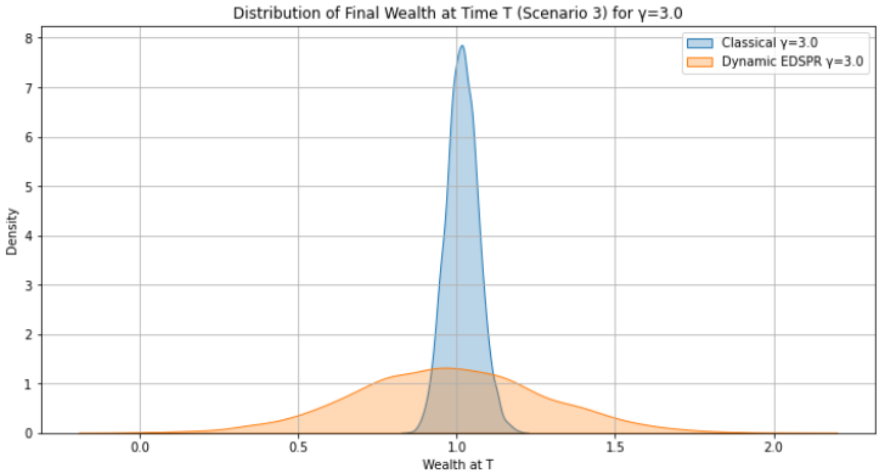

Figure 5.8: Distribution of final wealth at time for $\gamma = 3.0$.

Figure 5.8 shows that the FBSDE strategy consistently exhibits very low variance, whereas the classical approach shows slight improvement but remains less stable.

Results Interpretation

The results from Scenario 3 reveal that:

- **Effect of γ:** Higher risk aversion leads to more conservative strategies and naturally reduces final wealth variance,
- **Advantage of FBSDE:** The dynamic FBSDE approach consistently reduces wealth dispersion, even for risk-seeking investors (low γ),
- **Dynamic adaptation:** While the classical approach adjusts weakly to changes in γ, FBSDE reacts quickly and effectively, tailoring the strategy to the investor's risk profile.

Summary

The FBSDE model offers superior adaptability to varying levels of risk aversion, ensuring stable wealth trajectories regardless of the investor's risk profile.

5.2.4 Scenario 4: Variation of initial wealth $X_0 = 1.0, 5.0, 10.0$

The aim of this scenario is to assess the impact of initial wealth X_0 on:

- Wealth trajectory evolution,
- Final risk levels,
- Comparison between classical and FBSDE strategies.

We test several values of X_0 to evaluate each strategy's ability to adapt to initial market conditions.

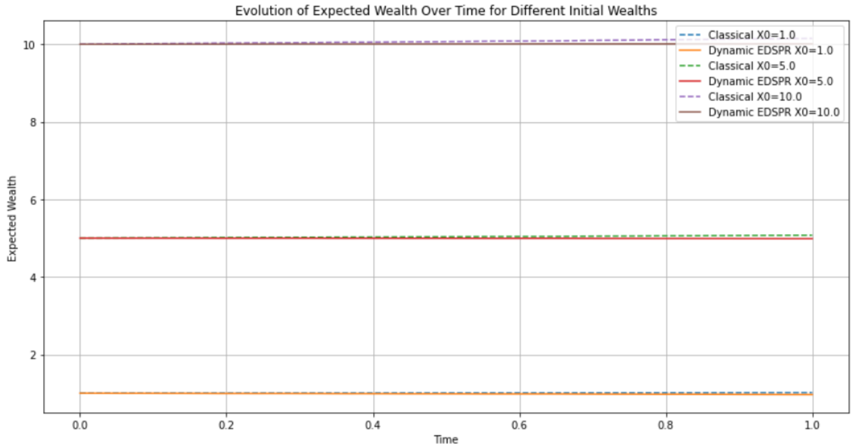

Figure 5.9: Evolution of expected wealth over time for different initial wealths.

Figure 5.9 shows that higher values of X_0 lead to enhanced stability and more effective growth control in the FBSDE dynamics.

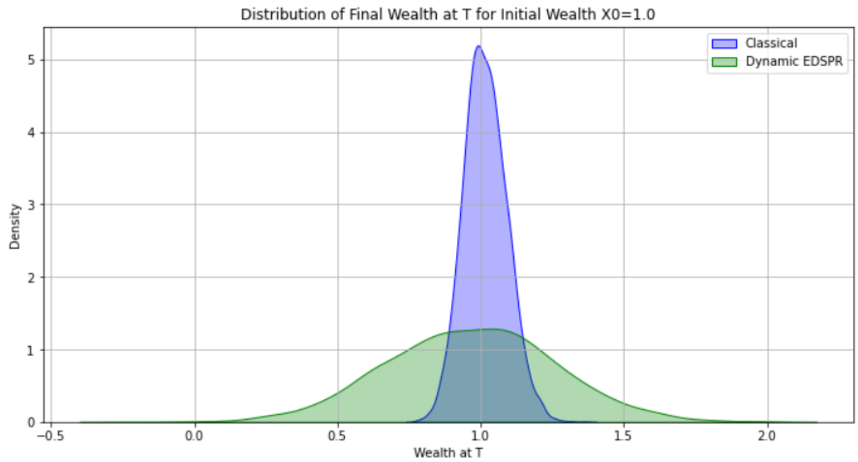

Figure 5.10: Distribution of final wealth at T for initial $X_0 = 1.0$.

Figure 5.10 shows that with $X_0 = 1.0$, both approaches exhibit variance, but the FBSDE strategy shows more concentrated results.

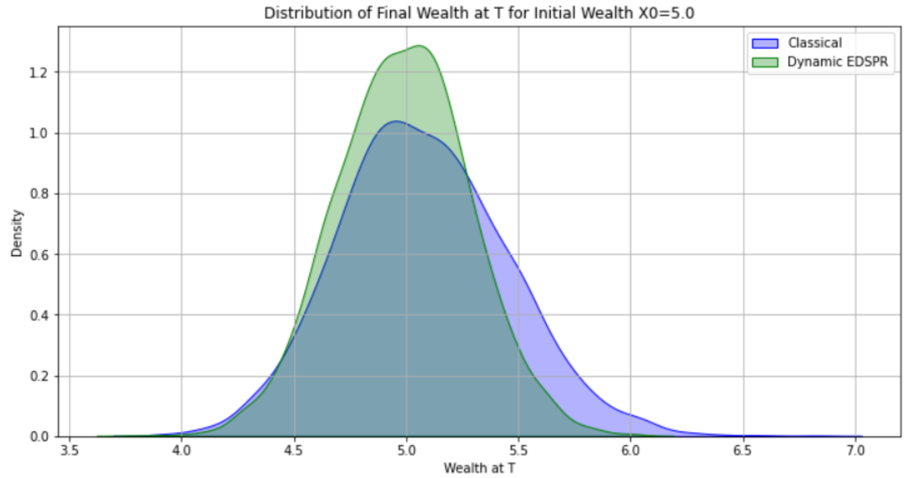

Figure 5.11: Distribution of final wealth at T for initial $X_0 = 5.0$.

The gap widens, and the FBSDE approach significantly reduces wealth dispersion.

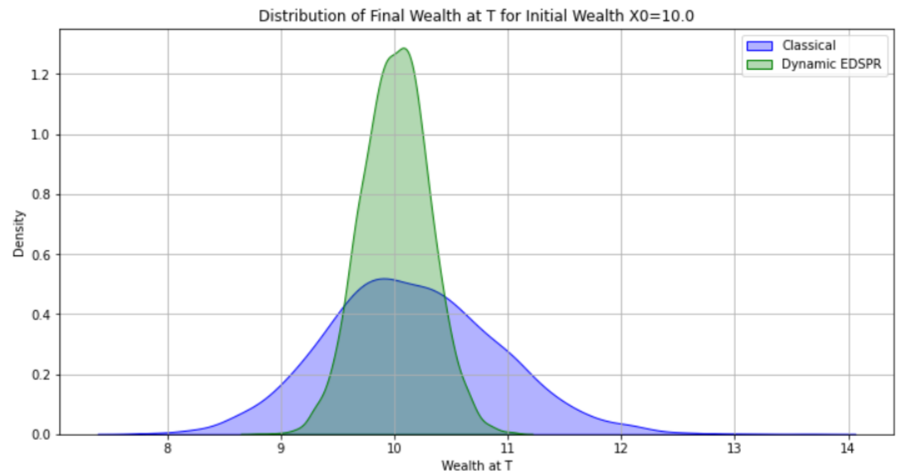

Figure 5.12: Distribution of final wealth at T for initial $X_0 = 10.0$.

Protection is maximized, with the FBSDE distribution nearly twice as tight as the classical one.

Results Interpretation

The findings show that:

- **Adaptability to initial wealth:** FBSDE automatically adjusts the investment strategy to the initial wealth level,
- **Stability with large portfolios:** Even for high X_0, FBSDE controls terminal variance better than the classical approach, where deviations increase significantly,
- **Robustness:** FBSDE offers natural protection for high-wealth investors by scaling risk exposure proportionally to capital.

Summary

As initial wealth increases, the FBSDE strategy maintains stronger control over risk and ensures more stable final wealth than the classical approach, which shows growing dispersion.

5.2.5 Scenario 5: Variation of the penalization coefficient $\kappa = 0.1, 0.5, 1.0$

This scenario studies how different levels of the risk penalization coefficient κ affect:

- Optimal investment strategies,
- Wealth evolution X_t,
- Final portfolio stability.

We compare both strategies across different penalization intensities.

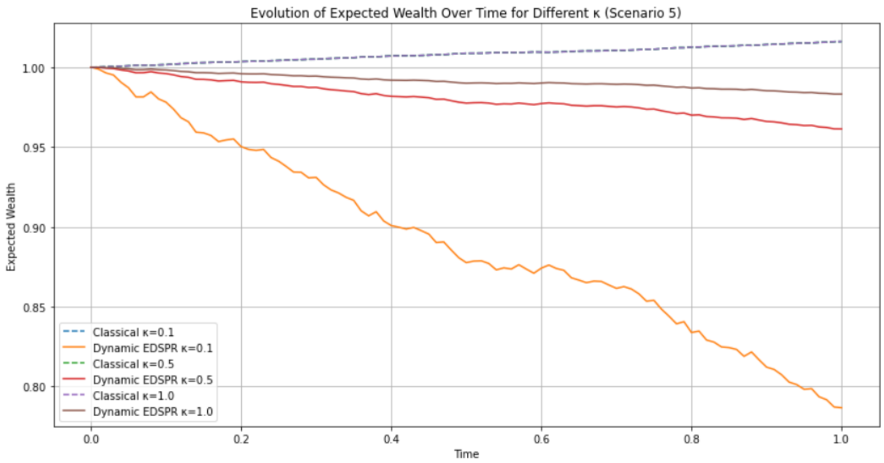

Figure 5.13: Evolution of expected wealth over time for different κ.

The higher the κ, the more stable and cautious the FBSDE trajectories become, whereas the classical model remains less responsive to changes in κ.

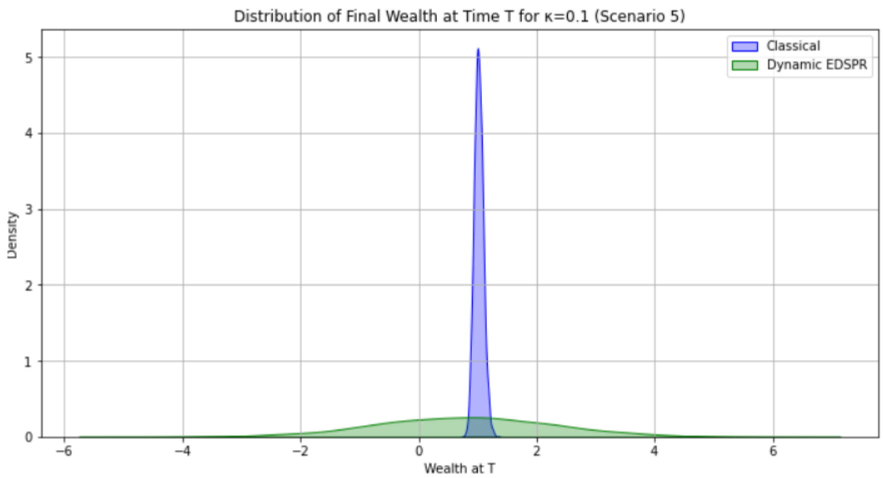

Figure 5.14: Distribution of final wealth at time for $\kappa = 0.1$.

Figure 5.14 shows that the classical approach is very volatile, while FBSDE already reduces dispersion even with a low penalization.

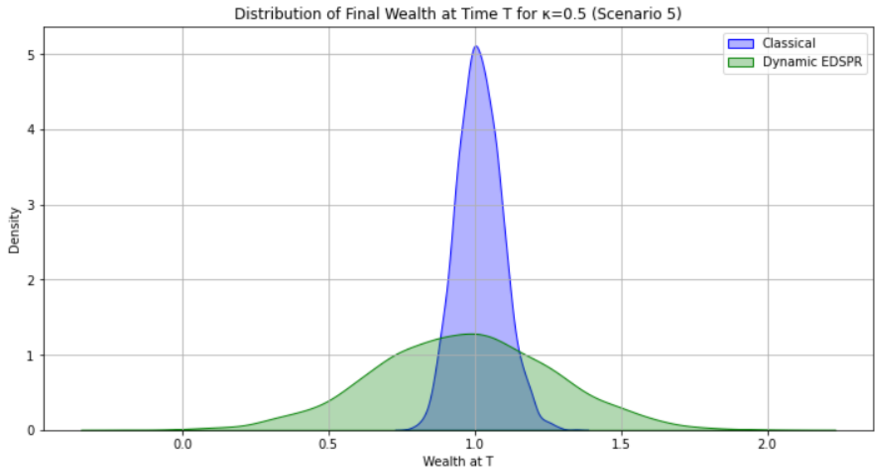

Figure 5.15: Distribution of final wealth at time for $\kappa = 0.5$.

The gap between the two strategies grows: FBSDE leads to a more concentrated final wealth distribution.

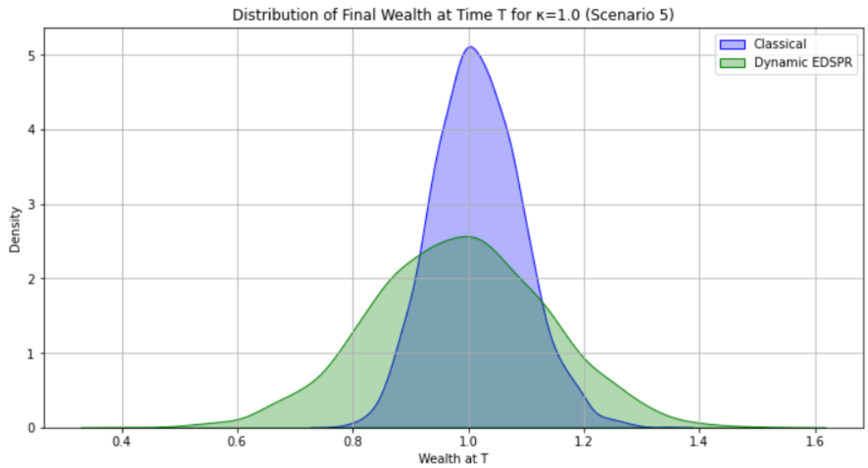

Figure 5.16: Distribution of final wealth at time for $\kappa = 1.0$.

With high penalization, the FBSDE strategy yields a tightly concentrated final distribution, whereas the classical approach remains more spread out and riskier.

Results Interpretation

The results from Scenario 5 show that:

- **Effect of κ:** Lower κ leads to aggressive strategies and higher variance, while higher κ produces more conservative strategies and significantly reduces variance,
- **FBSDE advantage:** FBSDE immediately adjusts investment based on the penalization level and better stabilizes wealth, even with low κ,
- **Performance stability:** Expected wealth remains comparable between both approaches.

Summary

The FBSDE approach naturally adapts to changes in risk penalization intensity κ, ensuring superior risk control and portfolio stability without sacrificing performance.

5.3 Comparative summary with classical models

Compared to classical frameworks such as Merton's model [12] and HJB-based formulations [4,8], our FBSDE approach introduces several key innovations:

- Risk penalization is dynamic and endogenous (via $\kappa/2(u_t X_t)^2$) rather than static and externally imposed,
- The model is built on the stochastic maximum principle [1], avoiding the analytical complexity of solving HJB PDEs in high dimensions,
- The coupling between forward (wealth) and backward (cost) components ensures real-time feedback and adaptation to volatility,
- Numerical schemes such as regression-based Monte Carlo [6] and deep learning methods [7] enhance scalability and precision.

In short, this framework unifies decision-making and risk management, yielding self-regulating strategies that outperform classical models under dynamic market conditions.

6 General conclusion and perspectives

This study introduces a dynamic portfolio optimization model using FBSDEs with embedded risk penalization. The approach enables real-time adjustment to market volatility by integrating a quadratic penalty directly into the backward component, leading to self-regulating investment strategies. The model offers practical value for financial institutions by reducing terminal wealth variance without sacrificing expected returns. It avoids the complexity of HJB equations and remains tractable in high-dimensional settings.

The framework assumes deterministic coefficients and does not account for transaction costs, liquidity constraints, or partial information. It may also become unstable when wealth approaches zero, requiring regularization. Extensions could include: stochastic coefficients, incomplete markets, learning-based solvers, and integration of market frictions to enhance realism and applicability.

Declarations

Availability of data and materials

Data sharing not applicable to this article.

Funding

Not applicable.

Author's contributions

All authors contributed equally to this work. They have reviewed and approved the final manuscript.

Conflict of interest

The authors declare that they have no conflicts of interest related to this work.

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